Local Maxima and Minima

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When is \( f'(x) > 0 \)? when is \( f'(x) < 0 \)?
What derivatives tell us about a function and its graph

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- The graph of $f'(x)$ tells us some information of the graph of $f(x)$.
- When is $f'(x) > 0$? when is $f'(x) > 0$?

Local Maxima and Minima
Local maxima and minima

Definition

Suppose $p$ is a point in the domain of $f(x)$:

- $f$ has a **local minimum** at $p$ if $f(p)$ is less than or equal to the values of $f$ for points near $p$. 
Local maxima and minima

**Definition**

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- $f$ has a **local minimum** at $p$ if $f(p)$ is less than or equal to the values of $f$ for points near $p$.
- $f$ has a **local maximum** at $p$ if $f(p)$ is greater than or equal to the values of $f$ for points near $p$. 
How do we detect a local maximum or minimum

Definition (Critical point)

For any function \( f \), a point \( p \) in the domain of \( f \), where \( f'(p) = 0 \) or \( f'(x) \) is undefined is called a **critical point** of the function. In addition, the point \((p, f(p))\) on the graph of \( f \) is also called a critical point (of the graph). A **critical value** of \( f \) is the value, \( f(p) \), of the function at a critical point, \( p \).

- At a critical point where \( f'(p) = 0 \), the tangent line to the graph at \( p \) is horizontal.
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For any function $f$, a point $p$ in the domain of $f$, where $f'(p) = 0$ or $f'(x)$ is undefined is called a critical point of the function. In addition, the point $(p, f(p))$ on the graph of $f$ is also called a critical point (of the graph). A critical value of $f$ is the value, $f(p)$, of the function at a critical point, $p$.

- At a critical point where $f'(p) = 0$, the tangent line to the graph at $p$ is horizontal.
- At a critical point where $f'(p)$ is undefined, there is no horizontal tangent– there is either a vertical tangent or no tangent at all.
Example

Local Maxima and Minima

\[ y = x^2 + 4x + 14 \]
Example

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The graph represents the function $x^3 + x + 5$.
How do we detect a local maximum or minimum

- The critical points divide the domain of $f$ into intervals on which the sign of the derivative remains the same.
How do we detect a local maximum or minimum?

- The critical points divide the domain of $f$ into intervals on which the sign of the derivative remains the same.
- Therefore, if $f$ is defined on the interval between two successive critical points, its graph cannot change direction on that interval, it is either going up or it is going down.
How do we detect a local maximum or minimum

- The critical points divide the domain of $f$ into intervals on which the sign of the derivative remains the same.
- Therefore, if $f$ is defined on the interval between two successive critical points, its graph cannot change direction on that interval, it is either going up or it is going down.
- If a function, continuous on an interval (its domain), has local maximum or minimum at $p$, then $p$ is a critical point or an endpoint of the interval.
Suppose \( p \) is a critical point of a continuous function \( f \). Then, as we go from left to right:
- If \( f \) changes from decreasing to increasing at \( p \), then \( f \) has a local minimum at \( p \).
- If \( f \) changes from increasing to decreasing at \( p \), then \( f \) has a local maximum at \( p \).
Suppose \( p \) is a critical point of a continuous function \( f \). Then, as we go from left to right:

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First Derivative Test for Local Maxima and Minima

Suppose \( p \) is a critical point of a continuous function \( f \). Then, as we go from left to right:

- If \( f' \) changes from negative to positive at \( p \), then \( f \) has a local minimum at \( p \).
- If \( f' \) changes from positive to negative at \( p \), then \( f \) has a local maximum at \( p \).
Suppose \( p \) is a critical point of a continuous function \( f \), and \( f'(p) = 0 \).

- If \( f \) is concave up at \( p \), then \( f \) has a local minimum at \( p \).

Equivalent to

- If \( f''(p) > 0 \), then \( f \) has a local minimum at \( p \).
Suppose $p$ is a critical point of a continuous function $f$, and $f'(p) = 0$.

- If $f$ is concave up at $p$, then $f$ has a local minimum at $p$.
- If $f$ is concave down at $p$, then $f$ has a local maximum at $p$.

Equivalent to

- If $f''(p) > 0$, then $f$ has a local minimum at $p$.
- If $f''(p) < 0$, then $f$ has a local maximum at $p$. 

Local Maxima and Minima
(a) Graph a function $f$ with the following properties:
- $f(x)$ has critical point at $x = -2$ and $x = 3$.
- $f'(x)$ is positive on the left of $-2$ and on the right of 3.
- $f'(x)$ is negative between $-2$ and 3.

(b) Identify the critical points as local maxima, local maxima, or neither.
Find the local maxima and local minima of 
\[ f(x) = x^3 - 6x^2 + 9x + 40. \]
Example

Local Maxima and Minima
Example

If $a$ and $b$ are nonzero constants, find the domain and all critical points of

$$f(x) = \frac{ax^2}{x - b}.$$
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\[ \frac{2x^2}{x-4} \]
Example

The value of an investment at time $t$ is given by $S(t)$. The rate of change, $S'(t)$, of the value of the investment is shown in the figure.

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1. What is the critical point of \( S(t) \)?
2. Identify each critical point as a local maximum, a local minimum, or neither.
Example

The value of an investment at time $t$ is given by $S(t)$. The rate of change, $S'(t)$, of the value of the investment is shown in the figure.

1. What is the critical point of $S(t)$?
2. Identify each critical point as a local maximum, a local minimum, or neither.
3. Explain the financial significance of each of the critical point.
Let $g(x) = x - ke^x$, where $k$ is a constant. For what values of $k$ does the function $g$ have a critical point? a local maximum? a local minimum?
Example

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\[ x - \exp(x) \]
Example

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