Enumeration of Tilings and Related Problems

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A lattice divides the plane into fundamental regions.

- A tiling of a region is a covering of the region by tiles so that there are no gaps or overlaps.
A lattice divides the plane into **fundamental regions**.

A **tile** is a union of any two fundamental regions sharing an edge.
A lattice divides the plane into fundamental regions.

A tile is a union of any two fundamental regions sharing an edge.

A tiling of a region is a covering of the region by tiles so that there are no gaps or overlaps.
Theorem (MacMahon)

The number of (lozenge) tilings of a semi-regular hexagon $H_{a,b,c}$ of sides $a, b, c, a, b, c$ on the triangular lattice is

$$\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{t=1}^{c} \frac{i + j + t - 1}{t} \cdot \frac{i + j + t - 2}{i + j + t - 1}.$$ 

Figure: The hexagon $H_{4,5,6}$ and one of its tilings.
Classical Results

Theorem (Kasteleyn, Temperley and Fisher)

The number of tilings of a $2m \times 2n$ chessboard equals

$$2^{2mn} \prod_{j=1}^{m} \prod_{k=1}^{n} \left( \cos^2 \left( \frac{j\pi}{2m+1} \right) + \cos^2 \left( \frac{k\pi}{2n+1} \right) \right).$$
Theorem (Elkies, Kuperberg, Larsen and Propp 1991)

The Aztec diamond region of order $n$ has $2^{n(n+1)/2}$ (domino) tilings.

Figure: The Aztec diamond of order 5 and one of its tilings.
A quasi-hexagon
A hexagonal dungeon
A tiled double Aztec rectangle
Dragon Regions

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Enumeration of Tilings and Related Problems
Generalized Aztec Diamond

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Enumeration of Tilings and Related Problems
Quartered Aztec Rectangle and Quartered Hexagon

Transforming a quartered Aztec rectangle into a quartered hexagon
Two generalized fortresses
Prove that a sequence of toric mutations on $dP_3$ quiver gives the \textit{weighted sum} (generating function) of perfect matchings a 6-sided graph.

Our family of graphs generalizes many known ones: including Aztec dragons, Aztec castles, dragon graphs.
Conjecture (Kenyon-Wilson)

Any semicontiguous minor can be expressed as a generating function of domino tilings of some region on the square lattice.
Enumeration of Generalized Plane Partitions

\[
\begin{array}{cccc}
6 & 4 & 2 & 1 \\
3 & 2 & 1 & 0 \\
2 & 1 & 0 & 0 \\
\end{array}
\]
Enumeration of Generalized Plane Partitions

\[ \sum \text{volume of the stack} = ? \]

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Enumeration of Tilings and Related Problems
I can use techniques in enumeration of tilings to prove the following theorem.

Theorem (Kamioka)

\[
\sum_{\pi} q^{t^{\text{tr}(\pi)}} \prod_{k=1}^{\pi_{1,1}} \left(\frac{(q^{n-k+1}; q)_{D_k(\pi)}}{(tq^{n-k+1}; q)_{D_k(\pi)}}\right) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - tq^{i+j+k-1}}{1 - tq^{i+j+k-2}},
\]

where the sum is taken over all plane partitions \(\pi\) fitting in an \(a \times b \times c\) box, and where \(D_k(\pi)\) is the size of the Durfee square of the \(k\)-cross-section of \(\pi\).
Thank you!