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Distribution of complex transmission eigenvalues for spherically stratified media

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Abstract
In this paper, we employ transformation operators and Levinson’s density formula to study the distribution of interior transmission eigenvalues for a spherically stratified media. In particular, we show that under smoothness condition on the index of refraction that there exist an infinite number of complex eigenvalues and there exist situations when there are no real eigenvalues. We also consider the case when absorption is present and show that under appropriate conditions there exist an infinite number of eigenvalues near the real axis.

Keywords: inverse scattering, complex transmission eigenvalues, spectral theory

(Some figures may appear in colour only in the online journal)

1. Introduction

The transmission eigenvalue problem is a non-selfadjoint eigenvalue problem that appears in inverse scattering theory [13, 14] and has attracted a considerable amount of research interest in recent years [8]. Although the transmission eigenvalue problem is not selfadjoint, and hence there exists the possibility of complex eigenvalues, until recently only real eigenvalues have been considered. However in [1, 6, 19] it was shown that for the case of spherically stratified media, complex eigenvalues do exist.

After the formulation of the interior transmission eigenvalues problem in [13], the first serious investigation of the inverse spectral problem was done by McLaughlin and Polyakov [22]. These authors (see also [14]) considered the case of a spherically stratified medium with (normalized) support \( x : |x| \leq 1 \) and spherically symmetric eigenfunctions, i.e. the eigenvalue problem
\[ w'' + \frac{2}{r} w' + k^2 n(r) w = 0, \quad 0 < r < 1, \quad (1.1) \]

\[ \nu'' + \frac{2}{r} \nu' + k^2 \nu = 0, \quad 0 < r < 1, \]

\[ w(1) = \nu(1), \quad w'(1) = \nu'(1), \]

where \( n(r) > 0 \) and both \( w(0) \) and \( \nu(0) \) must be finite. Setting \( y(r) = rw(r), \quad \gamma_0(r) = rv(r), \) we see from (1.1) that

\[ y'' + k^2 n(r) y = 0, \quad 0 < r < 1, \quad (1.2) \]

\[ \gamma''_0 + k^2 \gamma_0 = 0, \quad 0 < r < 1, \quad (1.3) \]

\[ y(0) = \gamma_0(0) = 0, \quad y(1) = \gamma_0(1), \quad y'(1) = \gamma_0'(1). \quad (1.4) \]

The eigenvalue problem (1.2)–(1.4) is called the transmission eigenvalue problem for a spherically stratified medium and values of \( k \) for which a nontrivial solution of (1.2)–(1.4) exist are called transmission eigenvalues. As shown in [13], and subsequently in many papers and books (see [5, 11, 14, 22]), the eigenvalues are the zeros of the entire function

\[ d(k) := y(1) \cos k - y'(1) \sin k/k. \quad (1.5) \]

The function \( d(k) \) is entire as a function of \( k \) and goes to zero in the order of \( O(1/k) \) as \( k \) goes to infinity along the real line [14].

Let \( \delta := \int_0^1 \sqrt{n(t)} \, dt \). It was shown in [13] (see also [11]) that an infinite number of real transmission eigenvalues exist under the assumptions that \( n(1) \neq 1 \) and \( \delta \neq 1 \). There was the question whether complex eigenvalues could exist. Finally it was shown in [6] that they do exist if \( \delta \) is close to zero. It was shown in [14] that the function \( d(k) \) has the asymptotic expansion

\[ d(k) = \frac{1}{k} \left( B \sin (k\delta) \cos k - C \cos (k\delta) \sin k \right) + O \left( 1/k^2 \right) \]

for \( k \) going to infinity along the real axis where

\[ B := \frac{1}{(n(0)n(1))^{1/4}}, \quad C := \left( \frac{n(1)}{n(0)} \right)^{1/4} \quad \text{and} \quad \delta = \int_0^1 \sqrt{n(t)} \, dt. \quad (1.6) \]

Using this expansion plus the assumptions that both \( \delta \neq 1 \) and \( n(1) \neq 1 \), it was shown in [19] and in [12] that infinitely many complex transmission eigenvalues in fact exist and they lie in a strip parallel to the real axis. Lastly, a recent article [23] of Sylvester has a detailed study on the distribution of transmission eigenvalues when \( n(r) \) is a constant.

Our main goal here is to investigate the cases when one of the parameters \( \delta \) or \( n(1) \) is 1 with the extra assumption that the refractive index \( n \in C^2[0, 1] \). In the case when \( \delta = 1 \) we show that it is possible to have all the eigenvalues being real. If \( n(1) = 1 \), then in general an infinite number of complex eigenvalues are present. However, in contrast to the case when \( n(1) \neq 1 \), these eigenvalues no longer lie in a strip parallel to the real axis. We will also provide an example with all the transmission eigenvalues being complex when both parameters \( \delta \) and \( n(1) \) are 1. Finally we will consider the case when the medium is absorbing and show that under appropriate assumption there are an infinite number of eigenvalues that accumulate near the real axis.

We will always assume that \( n(r) \) is not identically equal to one.
2. Complex transmission eigenvalues

We first recall a classical result due to Levinson.

**Definition 2.1.** The entire function \( f(z) \) is of order \( \rho \) if

\[
\lim_{r \to \infty} \frac{\log M(r)}{\log r} = \rho.
\]

Here \( M(r) \) denotes the maximum modulus of \( f(z) \) on \( |z| = r \).

**Definition 2.2.** The entire function \( f(z) \) of positive order \( \rho = 1 \) is called a function of exponential type \( \tau \) if

\[
\lim_{r \to \infty} \frac{\log M(r)}{r^\rho} = \tau.
\]

One of the important theorems involving entire functions of exponential type is the Paley–Wiener theorem [18].

**Theorem 2.3 (Paley–Wiener).** The entire function \( f(z) \) is of exponential type \( \leq \tau \) and belongs to \( L^2 \) on the real axis if and only if

\[
f(z) = \int_{-\tau}^{\tau} \phi(t) e^{it} \, dt,
\]

for some \( \phi(t) \in L^2(-\tau, \tau) \). \( f(z) \) is of type \( \tau \) if \( \phi(t) \) does not vanish almost everywhere in a neighborhood of \( \tau \) (or of \( -\tau \)).

We say that an entire function belongs to the Paley–Wiener class if it has the representation given in (2.1).

**Corollary 2.4.** Suppose \( f(z) \) and \( g(z) \) are in the Paley–Wiener class of types \( \tau \) and \( \sigma \) respectively. If \( \sigma < \tau \), then the sum \( f(z) + g(z) \) is of type \( \tau \).

To employ the theorem in the next section, we note that a sine transform \( \int_{0}^{\tau} \psi(t) \sin(tz) \, dt \) can be expressed as \( \int_{-\tau}^{\tau} \phi(t) e^{it} \, dt \) for some complex valued \( \phi(t) \) on \([\tau, 0] \) if \( \psi(t) \) is extended onto the interval \([\tau, 0] \) appropriately.

Let \( n_+(r) \) denote the number of zeros of an entire function \( f(z) \) in the right half plane with \( |z| \leq r \). One can also define a corresponding function \( n_-(r) \) for the zeros in the left half plane. Our tool of counting the density of the non-real zeros of the entire function \( d(k) \) is the following extension of a theorem due to Cartwright [18].

**Theorem 2.5.** (Levinson) Let the entire function \( f(k) \) of exponential type be such that

\[
(a) \int_{-\infty}^{\infty} \frac{\log^+|f(x)|}{1 + x^2} \, dx < \infty,
\]

where \( \log^+ \) denotes the principal value of \( \log(\cdot) \).

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and suppose that
\[
\lim_{y \to \pm \infty} \frac{\log |f(iy)|}{|y|} = \tau.
\]
Then
\[
\lim_{r \to \infty} \frac{n^+ (r)}{r} = \frac{\tau}{\pi}.
\] (2.3)

This limit \(\tau/\pi\) will be called the density of all the zeros of \(f(z)\) on the right half plane. We note that in [3], the integrability condition (2.1) is replaced by
\[
\lim_{x \to \infty} \frac{\log |f(x)|}{|x|} = 0.
\]

Redheffer has further comments on this result in his review of Levinson’s Collected Works ([20] pp 11–13).

To apply these two theorems to count the number of non-real zeros of a given entire function, we first establish the following results.

**Corollary 2.6.** Let \(\tau > 0\) be fixed. Suppose a real entire function \(f(z)\) has the form
\[
f(z) := \sin (\tau z + \alpha) + \int_{-\tau}^{\tau} \phi(t)e^{it} \, dt
\]
with \(\alpha\) being a real constant and \(\phi(t)\) a possibly complex valued function continuous on \([-\tau, \tau]\). Then the zeros of \(f(z)\) have density \(\tau/\pi\) on the right half plane.

**Proof.** The function \(|f(x)|\) is bounded on the real axis for \(x\) real. Along the positive imaginary \(y\) axis, i.e. \(z = iy\) with \(y > 0\),
\[
f(iy) = e^{iy} \left( \frac{\sin e^{-2\tau y} - e^{-i\alpha}}{2i} + \int_{-\tau}^{\tau} \phi(t)e^{iy(t-\tau)} \, dt \right).
\]
\[
\log |f(iy)| = \tau y + \log \left| \left( \frac{\sin e^{-2\tau y} - e^{-i\alpha}}{2i} + \int_{-\tau}^{\tau} \phi(t)e^{iy(t-\tau)} \, dt \right) \right|.
\]
Inside the logarithm on the right, the limit is \((ie^{-i\alpha})/2\) as \(y\) goes up to infinity, so we get
\[
\lim_{y \to \infty} \frac{\log |f(iy)|}{y} = \tau.
\]
The proof of the limit along the negative \(y\) axis runs similarly. Thus we have verified condition (b) in theorem 2.5. \(\square\)

**Corollary 2.7.** Let \(f(z)\) be a real entire function in the Paley–Wiener class of type at most \(\tau\). Suppose \(x^2 f(x) = \sin (\tau x) + O(1/x)\) as \(x\) goes to infinity on the real axis. Then \(f(z)\) is of type \(\tau\).

**Proof.** The density of the positive zeros of \(f(z)\) is \(\tau/\pi\). So the type of \(f(z)\) must be at least \(\tau\), so it is equal to \(\tau\). \(\square\)

In the next result, we are setting up conditions to prove the finiteness of the number of complex roots. The assumptions are not the best possible. The number \(\tau\) below is assumed to
be a positive number. The following theorem is a consequence of the Phragmén–Lindelöf maximum principle ([4], theorem 6.2.6).

**Theorem 2.8.** Let \( g(z) \) be a real entire function of exponential type. Suppose  

(i) \( |g(x)| \leq M \quad \forall x \in (-\infty, \infty), \) and  

(ii)  
\[
\lim_{y \to \pm \infty} \frac{\log |g(iy)|}{|y|} \leq \tau.
\]  

Then \( |g(x + iy)| \leq M \cosh (\tau y). \)

For later use, we note that functions of the form \( \int_0^\delta \phi(t) \sin (zt) dt \) and \( \int_0^\delta \phi(t) \sin (zt) dt \) satisfy the assumptions in the theorem with \( \tau = 1 + \delta \) when \( \phi(t) \) is continuous on \([0, \delta]\). The following corollary is basically due to Duffin and Schaeffer [15].

**Corollary 2.9.** Let \( h(z) \) be a real entire function in the Paley–Wiener class of exponential type at most \( \tau \) and \( h(x) = O(1/x) \) when \( x \) is large. Then \( f(z) := \sin(z) + h(z) \) is of type \( \tau \) and has at most a finite number of non-real zeros.

**Proof.** To prove the first part of the corollary, we note that \( f(z) \) is of type at most \( \tau \) based on the property of \( h(z) \). On the real axis, \( h(x) \) goes to 0 as \( x \) goes to infinity. Hence the density of the real zeros on the positive real axis is \( \tau/\pi \). So the density of all the zeros on the right half plane is at least \( \tau/\pi \). Using Levinson’s theorem, we see that the type of \( f(z) \) is \( \tau \).

In [15], Duffin and Schaeffer proved that \( f(z) \) has only real zeros if the entire function \( |h(x)| \leq 1 \) on the real line. Our corollary is a simple modification stating if \( h(x) \) goes to 0 as \( x \) gets big, then all the zeros of \( f(z) \) are real except finitely many may be complex. The proof presented here is an imitation of theirs. To simplify the notation, we let \( \tau = 1 \). The general case follows from dilation.

From the theorem above and with \( z := x + iy \), there exists a real number \( M \) such that when \( |z| \) is large,

\[
|h(x + iy)| \leq M \frac{\cosh (y)}{|z|}.
\]

We set up a symmetric rectangle \( R \) with vertices at \( \pm(n + 1/2)\pi \pm iY \) with \( n \) being a positive integer and \( Y \) a large positive real number. Our aim is to show that \(|f(z) - \sin(z)| = |h(z)| < |\sin(z)|\) for \( z \) on the boundary of \( R \). An application of Rouché’s theorem shows that \( f(z) \) and \( \sin(z) \) have the same number of zeros inside \( R \).

For \( z = x + iy \), \( |\sin(z)|^2 = \sin^2(x) + \sinh^2(y) \) whose value is \( 1 + \sinh^2(y) \) on a vertical side of \( R \). Since

\[
M \cosh (y) < |z| \sqrt{1 + \sinh^2(y)} = |z| |\sin(z)|
\]

when \( \text{Re}(z) \) is large, \(|h(z)| < |\sin(z)|\) there. On a horizontal segment of \( R \), \(|\sin(z)| > |\sinh(y)|\). So

\[
|h(x + iy)| \leq M \frac{\cosh (y)}{|z|} < |\sinh(y)| \leq |\sin(z)|
\]

when \( \text{Im}(z) = Y \) is large enough. Altogether, \(|f(z) - \sin(z)| < |\sin(z)|\) on the four edges of the rectangle. Thus \( f(z) \) and \( \sin(z) \) have the same number of zeros inside \( R \).
When \( \text{Re}(z) = x \) is large, \(|h(x)|\) is small. By recalling that \( f(z) := \sin(z) + h(z) \), we see that all the real zeros of \( f(z) \) are close to that of \( \sin(z) \). So all the zeros of \( f(z) \) are real inside \( \mathcal{R} \) when \( \text{Re}(z) \) is sufficiently large.

3. Existence of complex zeros

As noted earlier, when both parameters \( \delta \neq 1 \) and \( n(1) \neq 1 \), the entire function \( d(k) \) has infinitely many real and non-real zeros. The main theme of this paper is to show that this situation is drastically different when one of these parameters is 1. If both are 1, then it is possible to have all zeros complex.

Our method to locate the zeros of the function \( d(k) \) as a function of the parameters \( n(1) \) and \( \delta := \int_0^1 \sqrt{n(t)} \, dt \) hinges on the Levitan–Gelfand formulation of the Sturm–Liouville problem. We assume that \( n \in C^2[0, 1] \). Using the Liouville transformation

\[
\xi := \int_0^x \frac{t}{\sqrt{n(t)}} \, dt \tag{3.1}
\]

and setting

\[
z(\xi) := n(r)^{1/4}y(r), \quad r = r(\xi), \tag{3.2}
\]

we can rewrite

\[
y'' + k^2 n(r) y = 0, \quad y(0) = 0, \quad y'(0) = 1
\]

as

\[
z'' + \left(k^2 - p(\xi)\right)z = 0
\]

\[
z(0) = 0, \quad z'(0) = (n(0))^{-1/4}. \tag{3.3}
\]

where

\[
p(\xi) = \frac{n''(r)}{4n^2(r)} - \frac{5}{16} \frac{(n'(r))^2}{n^3(r)}. \tag{3.4}
\]

From [17], section 4.4, we can represent \( z(\xi) \) in the form

\[
z(\xi) = \frac{1}{n(0)^{1/4}} \left[ \frac{\sin(k\xi)}{k} + \int_0^\xi K(\xi, t) \frac{\sin(kt)}{k} \, dt \right]. \tag{3.5}
\]

Hence,

\[
z'(\xi) = \frac{1}{n(0)^{1/4}} \left[ \frac{\cos(k\xi) + K(\xi, \xi) \frac{\sin(k\xi)}{k}}{k} + \int_0^\xi K'(\xi, t) \frac{\sin(kt)}{k} \, dt \right]. \tag{3.6}
\]

Here \( K(\xi, t) \) is the unique solution of

\[
K_{\xi t} - K_{tt} - p(\xi)K = 0,
\]

\[
K(\xi, 0) = 0,
\]

\[
K(\xi, \xi) = \frac{1}{2} \int_0^\xi p(\xi) \, d\xi.
\]

This partial differential equation for \( K(\xi, t) \) is defined on the triangular region \( \Delta_\delta := 0 \leq t \leq \xi \leq \delta = \xi(1) \). It is shown in [17], theorem 4.5 and in [21], that \( K(\xi, t) \) can
be constructed in a straightforward manner by the method of successive approximations. It is a $C^2$ function on the closure of $\Delta_o$ if $p(\xi)$ is assumed to be continuous on $[0, \delta]$.

Set $\alpha := n(0)^{1/4}$. From (3.5) and (3.6), we have

$$z(\delta) = \frac{1}{ak} \left[ \sin(k\delta) + \int_0^\delta K(\delta, t) \sin(kt) \, dt \right].$$

(3.8)

$$z'(\delta) = \frac{1}{ak} \left[ k \cos(k\delta) + K(\delta, \delta) \sin(k\delta) + \int_0^\delta K(\delta, t) \sin(kt) \, dt \right].$$

(3.9)

We note that each of these two entire functions is of type $\delta$ as a function of $k$. Since $z(\xi) = n(r)^{1/4}y(r)$ we have that

$$y(1) = \frac{z(\delta)}{n(1)^{1/4}},$$

$$y'(1) = n(1)^{1/4}z'(\delta) - \frac{n'(1)}{4n(1)^{5/4}}y(1).$$

The entire function $d(k) = y(1) \cos(k) - y'(1) \sin(k)/k$ first defined in (1.5) is of type at most $\delta + 1$ and can be rewritten as

$$d(k) = \left[ \frac{\cos(k)}{n(1)^{1/4}} + \frac{n'(1)}{4n(1)^{5/4}} \frac{\sin(k)}{k} \right] z(\delta) - n(1)^{1/4} \frac{\sin(k)}{k} z'(\delta).$$

(3.10)

Before expanding $d(k)$ out, let us perform one integration by parts on $z(\delta)$ to transform it into

$$z'(\delta) = \frac{1}{ak} \left[ \sin(k\delta) - K(\delta, \delta) \frac{\cos(k\delta)}{k} + \int_0^\delta K(\delta, t) \frac{\cos(kt)}{k} \, dt \right].$$

(3.11)

In terms of the Kernel function $K(\xi, t)$, we have

$$d(k) = \left( \frac{\cos(k)}{ak} + \frac{n'(1)}{4an(1)^{5/4}} \frac{\sin(k)}{k^2} \right)$$

$$\times \left( \sin(k\delta) - K(\delta, \delta) \frac{\cos(k\delta)}{k} + \int_0^\delta K(\delta, t) \frac{\cos(kt)}{k} \, dt \right)$$

$$- \frac{n(1)^{1/4}}{ak} \left[ k \cos(k\delta) + K(\delta, \delta) \sin(k\delta) + \int_0^\delta K(\delta, t) \sin(kt) \, dt \right].$$

We multiply both sides above by $an(1)^{1/4}k$ to arrive at

$$an(1)^{1/4}k \ d(k) = \left( \cos(k) + \frac{n'(1)}{4an(1)k} \frac{\sin(k)}{k} \right)$$

$$\times \left( \sin(k\delta) - K(\delta, \delta) \frac{\cos(k\delta)}{k} + \int_0^\delta K(\delta, t) \frac{\cos(kt)}{k} \, dt \right)$$

$$- \frac{n(1)}{k} \left[ k \cos(k\delta) + K(\delta, \delta) \sin(k\delta) + \int_0^\delta K(\delta, t) \sin(kt) \, dt \right].$$

Let $D(k) := a \ n(1)^{1/4}k \ d(k)$. After expanding the right-hand side and collecting terms of similar order of decay as $k$ goes to infinity along the real line, we have the following formulation:
\[ D(k) := \alpha n(1)^{1/4} k \, d(k) = \cos(k) \sin(k\delta) - \sqrt{n(1)} \sin(k) \cos(k\delta) + H(k), \]  
where

\[ H(k) := \left( \frac{n'(1)}{4n(1)} - \sqrt{n(1)} K(\delta, \delta) \right) \frac{\sin(k) \sin(k\delta)}{k} - K(\delta, \delta) \frac{\cos(k) \cos(k\delta)}{k} \]

\[ - \frac{n'(1)}{4n(1)} K(\delta, \delta) \frac{\sin(k) \cos(k\delta)}{k^2} + \frac{\cos(k)}{k} \int_{0}^{\delta} K(\delta, t) \cos(kt) \, dt \]

\[ - \sqrt{n(1)} \frac{\sin(k)}{k} \int_{0}^{\delta} K(\delta, t) \sin(kt) \, dt + \frac{n'(1)}{4n(1)} \frac{\sin(k)}{k^2} \int_{0}^{\delta} K(\delta, t) \cos(kt) \, dt. \]

The function \( kH(k) \) is bounded on the real line and is of exponential type \( \delta \leq 1 + 1 \). The first two terms on the right-hand side of (3.12) can be written as

\[ T(k) := \frac{1 - \sqrt{n(1)}}{2} \sin((\delta + 1)k) + \frac{1 + \sqrt{n(1)}}{2} \sin((\delta - 1)k), \]  
while all the other terms are \( O(1/k) \) for \( k \) large. When both \( n(1) \neq 1 \) and \( \delta \neq 1 \), a case that has been covered in [19], we see that density of the zeros of \( d(k) \) is \( (\delta + 1)/\pi \). In general, there are many situations that infinitely many of them are complex. Interesting patterns of the location of the zeros can be generated by picking an \( n(r) \) with \( \int_{0}^{\delta} \sqrt{n(1)} \, dt \) close to 1 as the example below shows. However the exact conditions to determine the existence of complex eigenvalues are still lacking.

**Example 1.** If \( n(r) = \frac{16}{9(1 + (1-r)^2)}, \) then \( \delta = \pi/3 = 1.0472\cdots \), \( n(1) = 16/9. \) In this example, the number of zeros in \( 0 < \text{Re} \, (k) < 150 \) is 96. So the average density in this strip is 96/150 = 0.64.

The theoretical density of all the zeros is \( \pi = 3.14159 \cdots \). There are only 8 real zeros in this interval lumped inside two small intervals. Figure 1 shows that the positive real eigenvalues do not appear until after \( k > 60. \) It was shown in [12] that if \( n(1) \neq 1 \), then all the zeros would lie inside an infinite horizontal strip. The example here provides a graphical illustration.

**4. Case \( n(1) = 1 \) and \( n'(1) = 0 \)**

Since the refractive index \( n(r) \) is defined to be one for \( r \geq 1 \), a natural assumption is to let \( n(1) = 1 \) and \( n'(1) = 0. \) We intend to show that an infinite number of complex eigenvalues are present under the additional assumptions \( n''(1) \neq 0 \) and \( \delta \neq 1. \) We believe that the result still holds even when \( n''(1) = 0. \) We intend to discuss this case in a future paper.

**Theorem 4.1.** Suppose the refractive index \( n \in C^2[0, 1] \) with \( n(1) = 1, n'(1) = 0 \) and \( \delta \neq 1. \) Then under the extra assumption that \( n''(1) \neq 0 \) the entire function \( d(k) \) has infinitely many non-real zeros and infinitely many real zeros.

**Proof.** With the given parameters \( n(1) = 1 \) and \( n'(1) = 0, \) we have that

\[ D(k) = \sin((\delta - 1)k) - K(\delta, \delta) \frac{\cos((\delta - 1)k)}{k} \]

\[ + \frac{\cos(k)}{k} \int_{0}^{\delta} K(\delta, t) \cos(kt) \, dt - \frac{\sin(k)}{k} \int_{0}^{\delta} K(\delta, t) \sin(kt) \, dt. \]
If we perform an integration by parts on the last two integrals, we see that

\[
D(k) = \sin((\delta - 1)k) - K(\delta, \delta) \frac{\cos((\delta - 1)k)}{k} + K_\xi(\delta, \delta) \cos(k\delta) + K_{\xi_2}(\delta, \delta) \sin(k\delta) \frac{\sin(k\delta)}{k^2} \frac{\cos(k\delta)}{k^2} - \cos(k\delta) \frac{\int_0^\delta K_t(\delta, t) \sin(kt) \, dt}{k^2} - \frac{\sin(k\delta)}{k^2} \int_0^\delta K_{t_2}(\delta, t) \cos(kt) \, dt.
\]

Note that we simplified one of the integrated terms using the fact that \(K_\xi(\delta, 0) = 0\) since \(K(\xi, 0)\) is an entire function of type \(\delta+1\) if the coefficient of \(\sin((\delta + 1)k)\) (which equals to \(K_\xi(\delta, \delta) + K_{\xi_2}(\delta, \delta))/2\) is non-zero. Since

\[
K_\xi(\delta, \delta) = \frac{(\sin((\delta + 1)k) + \sin((\delta - 1)k)) + K_{\xi_2}(\delta, \delta)}{2k^2} - \sin((\delta + 1)k) - \sin((\delta - 1)k)).
\]

According to corollary 2.7, the sum of this expression with the remainder term which is of order \(O(1/k^2)\) is an entire function of type \((\delta + 1)k\) since \(Q(\xi, 0)\) is an entire function of type \((\delta + 1)k\). From (3.4), we see that \(p(\delta) = n^/(1)/4 \) since \(n(1) = 1\) and \(n'(1) = 0\).
In summary, under the given assumptions, the asymptotic expansion of $D(k)$ has the form

$$D(k) = \sin((\delta - 1)k) - \frac{K(\delta, \delta)}{k} \cos((\delta - 1)k)$$

$$+ \frac{K_i - K_r}{2k^2} \sin((\delta - 1)k) + \frac{K_i + K_r}{2k^2} \sin((\delta + 1)k) + O\left(1/k^3\right)$$

(4.1)

with $K(\delta, \delta) = \left(\int_0^\delta p(s)ds\right)/2$ and $(K_i + K_r)/2 = n^*(1)/8$.

If $\delta \neq 1$, we see from corollaries 2.4 and 2.7 that $D(k)$ is of type $\delta + 1$. Since the leading term $\sin((\delta - 1)k)$ generates an infinite set of positive real zeros with density equal to $|1 - \delta|/\pi$ while the density of all the zeros on right half plane equal to $(\delta + 1)/\pi$ we have both infinitely many real and non-real zeros.

As mentioned in example 1, it was proved in [12] that the zeros of $D(k)$ lie in a strip parallel to the real axis if $n(1) \neq 1$. We now show that if $n(1) = 1$, the imaginary parts of the zeros cannot stay bounded as their real parts move to the right.

**Theorem 4.2.** Suppose the refractive index $n \in C^2[0, 1]$ with $n(1) = 1$ and $\delta \neq 1$. If either $n'(1)$ or $n''(1)$ is non-zero, the zeros of $D(k)$ do not lie inside a fixed strip parallel to the real axis.

**Proof.** Recalling from (3.12)-(3.13) with $n(1) = 1$, $D(k) := \alpha k d(k) = \sin((\delta - 1)k) + H(k)$, where $H(k)$ can be written as

$$-\frac{n'(1) \cos((\delta + 1)k)}{8k} - \frac{(K(\delta, \delta) - n'(1)/2) \cos((\delta - 1)k)}{k} + O\left(1/k^3\right).$$

The real entire function $H(k)$ is in the Paley–Wiener class. We express $H(k) = h(k)/k$ with $h(k)$ being an entire function bounded on the real axis. According to theorem 2.8, there is a constant $M$ such that $|h(k)| < M \cosh(\gamma k)$ for $k = x + iy$.

Assume on the contrary that the zeros of $D(k)$ lie in a fixed strip parallel to the real axis. Now consider a rectangular region lying in the strip as in figure 2 with $I_3$ and $I_3'$ intersecting the real axis at $\frac{2m+1}{2\delta - 1}$ for an integer $m$. On the two vertical boundaries, $|\sin((\delta - 1)k)|^2 = \sin^2((\delta - 1)\gamma) + 1$. This value is at least 1. So $|D(k) - \sin((\delta - 1)k)| = |h(k)/k| < M \cosh(\gamma)/|k| < |\sin((\delta - 1)k)|$ for $y$ bounded and $|k|$ large. The inequality also holds on two fixed horizontal boundaries for $|k|$ large. Thus we have proved that $|D(k) - \sin((\delta - 1)k)| < |\sin((\delta - 1)k)|$ on all four sides of the rectangle when $Re(k)$ is large. By Rouché theorem $D(k)$ has the same density of zeros as $\sin((\delta - 1)k)$ inside a rectangle with fixed height. When $I_2$ moves out to infinity, the density of zeros inside this infinite strip is $|\delta - 1|/\pi$. However the density of all the non-real zeros is $(\delta + 1)/\pi$. This shows that the zeros of $D(k)$ cannot lie inside a fixed horizontal strip.

If we let $n(r) = 1/(2 - r)^4$, then $n(1) = 1$, $n'(1) = 4$, $\delta = 1/2$ and the corresponding determinant $d(k)$ is equal to $\sin(k/2)(k - \sin(k))/k^2$. A simple calculation shows that the imaginary parts of the non-real zeros go to infinity as they move out to the right. A numerical example illustrating the case $n(1) = 1$, $n'(1) = 0$ is shown in figure 3 below.

We now give an example to illustrate theorem 4.1.
Example 2. Let
\[ n(r) = \frac{c^2}{(c + (1 - r)^2)^2} \quad \text{or} \quad n(r) = \frac{(1 + c)^4}{(r + c)^2(r - 2 - c)^2} \]
for any \( c > 0 \). \( n(1) = 1 \) and \( n'(1) = 0 \) in either case. The first case has an absolute maximum at \( r = 1 \) while the second case has an absolute minimum there. Different values of \( \delta \neq 1 \) can be obtained by varying the parameter \( c \). These two Halm type functions (see [16], p 357) have a constant \( p(\xi) \) for its Liouville transform so the entire function \( d(k) \) can be computed by hand.

Let \( n(r) = 16/((r + 1)^2(3 - r)^2) \). Its Liouville transform \( p(s) \) is the constant 1/4 while the value of \( \delta = \int_0^1 \sqrt{n(r)} \, dr = \log (3) = 1.0986 \ldots \). So \( K(\delta, \delta) = 1/2 \int_0^{\delta} p(s) \, ds = \frac{\log (3)}{8} \) and from (3.4), \( n''(1) = 4p(\delta) = 1 \).

The density of real zeros here is \( (1 - \log (3))/\pi = 0.0314 \ldots \) while the density of all the zeros is \( (1 + \log (3))/\pi = 0.6680 \ldots \). The distribution of zeros of \( d(k) \) is shown in figure 2.
where it is seen that the number of all zeros counted in the vertical strip \( 0 < \text{Re}(k) \leq 150 \) is 101. So the average is \( 101/150 = 0.673 \ldots \).

The differential equation in (3.3) is simply \( z'' + (k^2 - 1/4)z = 0 \). Let \( A := \sqrt{k^2 - 1/4} \).

Then

\[
D(k) = \frac{k}{A} \sin (A \log (3)) \cos (k) - \cos (A \log (3)) \sin (k)
\]

\[
= \frac{1}{2} \left[ \left( \frac{k}{A} - 1 \right) \sin (A \log (3) + k) + \left( \frac{k}{A} + 1 \right) \sin (A \log (3) - k) \right].
\]

For \( k \) large,

\[
\frac{k}{A} = 1 + \frac{1}{8k^2} + O \left( \frac{1}{k^4} \right),
\]

\[
A \log (3) + k = (\log (3) + 1)k - \frac{\log (3)}{8k} + O \left( \frac{1}{k^3} \right),
\]

\[
A \log (3) - k = (\log (3) - 1)k - \frac{\log (3)}{8k} + O \left( \frac{1}{k^3} \right),
\]

and hence

\[
D(k) = \frac{1}{16k^2} \sin \left( (\log (3) + 1)k - \frac{\log (3)}{8k} \right)
\]

\[
+ \left( 1 + \frac{1}{16k^2} \right) \sin \left( (\log (3) - 1)k - \frac{\log (3)}{8k} \right) + O \left( \frac{1}{k^3} \right).
\]

By expanding the sine terms further, we arrive at the following asymptotic approximation:

\[
D(k) = \sin ((\log (3) - 1)k - \frac{\log (3)}{8k}) \cos ((\log (3) - 1)k)
\]

\[
+ \frac{1}{16k^2} \sin ((\log (3) + 1)k) + \ldots.
\]

This verifies the formula given in (4.1).

**Remark.** We did not consider the case \( n(1) = 1 \) and \( n'(1) \neq 0 \) in the theorem above. However in this case it is quite easy to deduce from (3.12) that

\[
D(k) = \sin \left( (\delta - 1)k + \frac{1}{k} \left( \frac{n'(1)}{4} \sin (k) \sin (\delta k) \right. \right.
\]

\[
- K (\delta, \delta) \cos ((\delta - 1)k)) + O \left( \frac{1}{k^2} \right).
\]

When \( n'(1) \neq 0 \), the density of all the zeros on the right half plane is still \((\delta + 1)/\pi \) and the density of the real zeros is \(|\delta - 1|/\pi \).

5. Case: \( \delta = 1 \)

An investigation of eigenvalues in the case \( \delta = 1 \) gives a number of surprising results. In particular we will show that in this case it is possible to have all real eigenvalues or all complex eigenvalues.
Theorem 5.1. Let the refractive index \( n \in C^2[0, 1] \). Suppose \( \delta = 1 \) and \( n(1) \neq 1 \). Then there are at most finitely many complex transmission eigenvalues.

However if both \( \delta = 1 \) and \( n(1) = 1 \), then it is possible to have only finitely many real eigenvalues.

Proof. The theoretical aspect is pretty straightforward. If \( \delta = 1 \) then from (3.12)--(3.13) \( D(k) \) has the form

\[
D(k) = \frac{1 - \sqrt{n(1)}}{2} \sin(2k) + G(k),
\]

where

\[
G(k) = \left( \frac{n'(1)}{4n(1)} - \sqrt{n(1)} K(1, 1) \right) \frac{\sin^2(k)}{k} - K(1, 1) \frac{\cos^2(k)}{k} + O\left( \frac{1}{k^2} \right)
\]

The term \( \sin(2k) \) dominates the sum for \( D(k) \) when \( k \) is large along the real axis and we see that the density of the real zeros is \( \pi \). The function \( D(k) \) vanishes at the origin, so the term \( G(k) \) has a zero at the origin.

If we multiply the entire equation by \( k \), we see that \( k G(k) \) is an entire function of type two and there are at most finitely many complex roots as shown by corollaries 2.7 and 2.9 (we will show examples below with one where all the roots are real and another with a few complex roots at the beginning and then all real roots afterwards).

If both \( \delta = 1 \) and \( n(1) = 1 \), the expression in (5.1) gives

\[
D(k) = \left( \frac{n'(1)}{4n(1)} \sin^2(k) - K(1, 1) \right) \frac{1}{k} + O\left( \frac{1}{k^2} \right).
\]

So if \( |n'(1)| < 4 |K(1, 1)| \), then \( D(k) \) will be either strictly positive or strictly negative for large \( k \). Hence there are at most finitely many real zeros.

Finally if in addition \( n'(1) = 0 \) then

\[
D(k) = G(k) = - \frac{K(1, 1)}{k} + O\left( \frac{1}{k^2} \right).
\]

Again, there will be only a finite number of real zeros if \( K(1, 1) \neq 0 \). Surprisingly, a simple constraint like \( n'(0) < 0 \) will show that \( K(1, 1) > 0 \) since

\[
2K(1, 1) = \int_0^1 p(\xi) d\xi = \int_0^1 \left( \frac{n^*(x)}{4n(x)^3} - \frac{5}{16} \frac{(n'(x))^2}{n(x)^3} \right) \xi'(x) dx
\]

\[
= \int_0^1 n^*(x) 4n(x)^{3/2} - \frac{5}{16} \frac{n'(x)^2}{n(x)^3} dx
\]

\[
= \frac{n'(0)}{4n(0)^{3/2}} + \int_0^1 \frac{3}{8} \frac{n(x)^2}{n(x)^{3/2}} - \frac{5}{16} \frac{n'(x)^2}{n(x)^{3/2}} dx
\]

\[
= \frac{n'(0)}{4n(0)^{3/2}} + \frac{1}{16} \int_0^1 \frac{n'(x)^2}{n(x)^{5/2}} dx.
\]

□
Example 3a. Let $b$ be any real number outside $[0,1]$. Choose the positive constant $C$ such that $n(r) = C/(r-b)^4$ has $\delta = 1$. The Liouville transform $\rho(\xi)$ is identically zero (hence $K(x,x) \equiv 0$). It follows from simple computation that $d(k)$ is a constant multiple of

$$\sin \left( \frac{k}{k} \right) (k \cos(k) - \sin(k)).$$

In particular, all the roots are real and the same for each $b \notin [0,1]$.

The same situation occurs when $n(r) = C/(r-b)^2$. All the roots are real and the same for any $b \notin [0,1]$.

Remark. We learned of this particular Halm type index from reading Hille’s book [16]. However, examples like the one above have appeared in [2].

Example 3b. If one picks

$$n(r) = \frac{1}{(C(3/2 + r)(3-r))^2} \quad \text{with} \quad C = \frac{2}{9} \log \left( \frac{5}{2} \right),$$

then $\delta = 1$ and $n(1) \neq 1$. There are two pairs of complex roots before the real ones appear (See figure 4).

Example 4. If $\delta = 1$ and $n(1) = 1$, it turns out to be pretty easy to manipulate the Halm type functions to come up with refractive indices whose associated transmission eigenvalues are all complex. In particular since $\xi = \int_0^\ell \sqrt{n(t)} \, dt$, $\xi(1) = 1$ and $\xi'(1) = \sqrt{n(1)} = 1$. For the Halm type functions whose Liouville transforms are constants say $\ell$, with $\Lambda := k^2 - \ell$, the solution $\tau(\xi)$ to (3.3) is a constant multiple of $\sin(\Lambda \xi)$. We are only interested in the location of the eigenvalues, so we may as well assume the constant multiple to be 1. So $y(r) = n(r)^{-1/4} \sin(\Lambda \xi)$ Thus we have
\[ y(1) = \sin (A), \]  
\[ y'(1) = A \cos (A) - \frac{n'(1)}{4} \sin (A). \]  
\[ \text{(5.2)} \]
\[ \text{(5.3)} \]

The determinant \( d(k) \) is a multiple of

\[ \sin (A) \cos (k) - \left( A \cos (A) - \frac{n'(1)}{4} \sin (A) \right) \frac{\sin (k)}{k}. \]

We now let \( c < 0 \) and \( b > 1 \) and define \( n(r) \) by

\[ n(r) = \left( \frac{(1 - c)(1 - b)}{(r - c)(r - b)} \right)^{\frac{3}{2}}. \]

Its Liouville transform is \( p(\xi) = \ell = \frac{(c - b)^2}{4(1 - c)^2(1 - b)^2} \) and

\[ \delta := \int_0^1 \sqrt{n(t)} \, dt = \frac{(1 - c)(1 - b)}{(b - c)} \log \left( \frac{1 - 1/b}{1 - 1/c} \right). \]

For \( \Lambda := \sqrt{k^2 - \ell} \), the solution \( y(r) \) to (1.1) is a constant multiple of

\[ \frac{1}{\sqrt{(r - c)(b - r)}} \sin (A\xi). \]

By taking \( c = -1/2 \) and solving for \( b \) to have \( \delta = 1 \), we get \( b = 1.8083 \). We note that \( n(1) = 1 \). The function \( d(k) \) can be obtained quite easily and, as shown in the proof of theorem 4.1, has the asymptotic form

\[ d(k) = \frac{1}{n(0)^{1/2}k^2} \left( \frac{n'(1)}{4} \sin^2 (k) - K(1, 1) \right) + O \left( 1/k^3 \right). \]

In this example, \( n'(1) = 1.1401 \), so \( n'(1)/4 = 0.2852 \) while \( K(1, 1) = \ell/2 = 0.4531 \). The horizontal line

\[ y = \frac{n'(1)/4 - K(1, 1)}{n(0)^{1/4}} = -0.1449 \]

provides an upper envelope of the curve \( k^2d(k) \). The graph of \( k^2d(k) \) together with the horizontal envelope is shown in figure 5. In particular there exist no real eigenvalues for this choice of \( n(r) \).

6. Transmission eigenvalues for absorbing media

In this last section of our paper we turn our attention to the case when the medium is absorbing, i.e. the index of refraction is complex valued. In this case, we cannot in general expect that real transmission eigenvalues exist (theorem 8.12 of [11]). However we will show that under appropriate assumption there exist an infinite number of transmission eigenvalues that lie arbitrary close to the real axis.

For the case of absorbing media, the interior transmission eigenvalue problem becomes (see [7])

\[ w'' + \frac{2}{r} w' + k^2 \left( c_1(r) + i \frac{\gamma(r)}{k} \right) w = 0, \quad 0 < r < 1, \]  
\[ \text{(6.1)} \]
where $e_0$ and $\gamma_0$ are positive constants. We look for a solution of (6.1)–(6.3) in the form

$$v(r) = c_1 j_0(kn_o r),$$

$$w(r) = c_2 \frac{y(r)}{r},$$

where $n_o = \sqrt{e_0 + \frac{\gamma_0^2}{k^2}}$ (where the branch cut is chosen such that $n_o$ has a positive real part), $j_0$ is a spherical Bessel function of order zero, $y(r)$ is a solution of

$$y'' + k^2 \left( e_1(r) + i \frac{\gamma(r)}{k} \right) y = 0$$

$$y(0) = 0, \quad y'(0) = 1$$

for $0 < r < 1$ and $c_1$ and $c_2$ are constants. Then there exist constants $c_1$ and $c_2$ not both zero, such that (6.4) will be a nontrivial solution of (6.1)–(6.3) provided that

$$d(k) := \text{Det} \begin{pmatrix} y(1) & -\frac{\sin (kn_o)}{k} \\ y'(1) & -n_o \cos (kn_o) \end{pmatrix} = 0.$$
Proof. Assume the contrary. Then we can choose a semi-infinite strip parallel to the real axis such that there are no (complex) eigenvalues in the strip (see figure 2).

Since

$$\frac{\gamma_0}{\sqrt{\varepsilon_0}} = \int_0^1 \frac{\eta_1(\rho)}{\sqrt{\varepsilon_1(\rho)}} \, d\rho,$$

it is seen from [7] that

$$y(r) = \frac{1}{ik} \left[ \frac{1}{\varepsilon_1(0) \varepsilon_1(r)} \right] \sinh \left( \frac{1}{2} \int_0^r \frac{\eta_1(\rho)}{\sqrt{\varepsilon_1(\rho)}} \, d\rho \right) + O\left( \frac{1}{k^2} \right)$$

holds uniformly for $k$ in the strip. Then

$$d(k) = \frac{1}{ik} \left[ \frac{1}{\varepsilon_1(0) \varepsilon_0} \right] \sin \left( \left( \sqrt{\varepsilon_0} - \int_0^1 \frac{\eta_1(\rho)}{\sqrt{\varepsilon_1(\rho)}} \, d\rho \right) k \right) + O\left( \frac{1}{k^2} \right). \quad (6.8)$$

Let $r := \sqrt{\varepsilon_0} - \int_0^1 \frac{\eta_1(\rho)}{\sqrt{\varepsilon_1(\rho)}} \, d\rho$ and $S(k) := \frac{1}{ik[\varepsilon_1(0)\varepsilon_0]^2} \sin (rk)$. Consider the finite strip shown in figure 2 where $I_3$ and $I_4$ intersect with the real axis at $\frac{Qm + 11\pi}{2r}$ for an integer $m$.

We will show that $|d(k) - S(k)| < |S(k)|$ for large enough $|k|$ on the boundary of the rectangular strip. Indeed, let $k = \alpha + i\beta$. Then

$$|S(k)| = \frac{1}{k} \left[ \frac{1}{\varepsilon_1(0) \varepsilon_0} \right] \frac{e^{-\beta r}}{\sqrt{1 - e^{2\beta r}}} \sqrt{\left( 1 - e^{2\beta r} \right)^2 + 4e^{2\beta r} \sin^2(\alpha r)}. $$

Then on the boundaries $I_3$ and $I_4$ we have

$$|S(k)| \geq \frac{1}{k} \left[ \frac{1}{\varepsilon_1(0) \varepsilon_0} \right] \frac{e^{-\beta r}}{\sqrt{1 - e^{2\beta r}}} > |d(k) - S(k)|$$

and on the boundaries $I_1$ and $I_2$

$$|S(k)| > |d(k) - S(k)|$$

for the real part of $k$ sufficiently large. Now we can apply Rouche’s theorem to see that $d(k)$ and $S(k)$ have the same number of zeros in the strip shown in figure 2. By letting the right-hand side of the strip move to infinity, we see that $d(k)$ and $S(k)$ have the same number of zeros in this semi-infinite strip. Note that since $\frac{2\pi}{2r}$ are the zeros of $S(k)$, then $S(k)$ has infinitely many zeros in the strip. Hence $d(k)$ has infinitely many zeros in the strip which is a contradiction to our assumption.

If one picks $\varepsilon_1(r) = \gamma(r) = 1/(r - 2)^4$ and $\varepsilon_0 = 1, \gamma_0 = 1/2$, then numerically one can show that there is a set of complex eigenvalues that get very close to the real axis as the real part of $k$ increases.

Final remarks. After this article was prepared, we were drawn to attention of two articles by LH Chen [9] and [10]. A formula for the density of zeros identical to what we have at the end of theorem 4.2 here was derived in theorem 1.2 in [9]. However we were unclear on some of
the derivations in the two articles, in particular the formula for the derivative in (1.22) in [9] and in (1.35) in [10]. Our result hinges on the behavior of \( n(r) \) at \( r = 1 \) while it is at \( r = 0 \) for Chen’s result (see (1.20) in [9]). In addition, we could not see why the ‘natural’ assumptions \( n(1) = 1 \) and \( n'(1) = 0 \) would lead to the conditions \( p(B) = p'(B) = 0 \) as stated on the line above (1.35) in [10]. The function \( p(\xi) \), though not explicitly defined, seems to be \( q(\xi) \) in equation (1.32) there. This set of conditions may have been used to derive the result stated in (4.53) in the same paper.

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References

[8] Cakoni F and Haddar H 2013 Inverse Problems 29 100201 Special issue on transmission eigenvalues
[19] Leung Y and Colton D 2012 Complex transmission eigenvalues for spherically stratified media Inverse Problems 28 075005


[23] Sylvester J 2013 Transmission eigenvalues in one-dimension Inverse Problems 29 104009