An analysis of the effect of noise in a heterogeneous agent financial market model

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Abstract

Heterogeneous agent models (HAMs) in finance and economics are often characterised by high dimensional nonlinear stochastic differential or difference systems. Because of the complexity of the interaction between the nonlinearities and noise, a commonly used, often called indirect, approach to the study of HAMs combines theoretical analysis of the underlying deterministic skeleton with numerical analysis of the stochastic model. However, it is well known that this indirect approach may not properly characterise the nature of the stochastic model. This paper aims to tackle this issue by developing a direct and analytical approach to the analysis of a stochastic model of speculative price dynamics involving two types of agents, fundamentalists and chartists, and the market price equilibria of which can be characterised by the stationary measures of a stochastic dynamical system. Using the stochastic method of averaging and stochastic bifurcation theory, we show that the stochastic model displays behaviour consistent with that of the underlying deterministic model when the time lag in the formation of price trends used by the chartists is far away from zero. However, when this lag approaches zero, such consistency breaks down.

1. Introduction

Traditional economics and finance theory based on the paradigm of the representative agent with rational expectations has not only been questioned because of the strong assumptions of agent homogeneity and rationality, but has also encountered some difficulties in explaining the market anomalies and stylised facts that show up in many empirical studies, including high trading volume, excess volatility, volatility clustering, long-range dependence, skewness, and excess kurtosis (see Pagan, 1996; Lux, 2009 for a description of the various anomalies and stylised facts). As a result, there has been a rapid growth in the literature on heterogeneous agent models that is well summarised in the recent survey papers by Hommes (2006), LeBaron (2006), Hommes and Wagener (2009) and Chiarella et al. (2009). These models characterise the dynamics of financial asset prices and returns resulting from the interaction of heterogeneous agents having different attitudes to risk and having different expectations about the future evolution of prices. For example, Brock and Hommes (1997, 1998) propose a simple Adaptive Belief System to model economic and financial markets. A key aspect of these models is that they exhibit feedback of expectations. The resulting dynamical system is nonlinear and, as Brock and Hommes (1998) show, capable of generating complex behaviour from local stability to (a)periodic cycles and even chaos.

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By adding noise to the underlying deterministic system and using the simulation approach, many models (see, for example, Hommes, 2002; Chiarella et al., 2006a, 2006b) are able to generate realistic time series. In particular, it has been shown (see for instance Hommes, 2002; He and Li, 2007; Gaunersdorfer et al., 2008; Lux, 2009) that such simple nonlinear adaptive models are capable of capturing important empirically observed features of real financial time series, including fat tails, clustering in volatility and power-law behaviour (in returns). Most of the simple stylised evolutionary adaptive models that one encounters in the literature are analysed within a discrete-time framework and their numerical analysis provides insights into the connection between individual and market behaviour.

One of the most important issues for many heterogeneous agent asset pricing models is the interaction of the behaviour of the heterogeneous agents and the interplay of noise with the underlying nonlinear deterministic market dynamics. Indeed Chiarella et al. (2006b) and He and Li (2007) in their simulations find that these two effects interact in ways which are not yet understood at a theoretical level. The noise can be either fundamental noise or market noise, or both. The commonly used approach (except the stochastic approach developed in Lux, 1995, 1997, 1998), referred to as the indirect approach for convenience, is first to consider the corresponding deterministic “skeleton” of the stochastic model where noise terms are set to zero and to investigate the dynamics of this nonlinear deterministic system by using deterministic stability and bifurcation theory; one then uses simulation methods to examine the interplay of various types of noise with the deterministic dynamics. This approach relies on a combination of simulations and faith that the properties of the deterministic system carry over to the stochastic one. However, it is well known that the dynamics of stochastic systems can be very different from the dynamics of the corresponding deterministic systems, see for instance Mao (1997). Ideally we would like to deal directly with the dynamics of the stochastic systems, but this direct approach can be difficult.

A number of stochastic asset pricing models have been constructed in the heterogeneous agents literature. The earliest one we are aware of is that of Föllmer (1974) who allows agents’ preferences to be random and governed by a law that depends on their interaction with the economic environment. Rheinlaender and Steinkamp (2004) study a one-dimensional continuously randomised version of Zeeman’s (1974) model and show a stochastic stabilisation effect and possible sudden trend reversal. Wenzelburger (2004) develops a stochastic version of the Brock and Hommes model. Brock et al. (2005) study the evolution of a discrete financial market model with many types of agents by focusing on the limiting distribution over types of agents. They show that the evolution can be well described by the large type limit (LTL) and that a simple version of LTL buffeted by noise is able to generate important stylised facts, such as volatility clustering and long memory, observed in real financial data. Föllmer et al. (2005) consider a discrete-time financial market model in which adaptive heterogeneous agents form their demands and switch among different expectations stochastically via a learning procedure. They show that, if the probability that an agent will switch to being a “chartist” is not too high, the limiting distribution of the price process exists, is unique and displays fat tails. Other related works include Hens and Schenk-Hoppe (2005) who analyse portfolio selection rules in incomplete markets where the wealth shares of investors are described by a discrete random dynamical system, Böhm and Chiarella (2005) who consider the dynamics of a general explicit random price process of many assets in an economy with overlapping generations of heterogeneous consumers forming optimal portfolios, Böhm and Wenzelburger (2005) who provide a simulation analysis of the empirical performance of portfolios in a competitive financial market with heterogeneous investors and show that the empirical performance measure may be misleading. In particular, by assuming that agent demand is derived from intertemporal optimisation and agents are allowed to switch between strategies, Horst and Wenzelburger (2008) develop a discrete-time stochastic model and show that the limiting distributions may be either unimodal or bimodal, exhibiting a bifurcation-type phenomenon. Most of the cited papers focus on the existence and uniqueness of limiting distributions of discrete time models. For continuous time models, we refer to the one-dimensional continuously randomised version of Zeeman’s (1974) stock market model studied by Rheinlaender and Steinkamp (2004) and the work of Horst and Rothe (2008) who examine the impact of time lags in continuous time heterogeneous agent models.

In this paper, we extend the continuous-time deterministic models of speculative price dynamics of Beja and Goldman (1980) and Chiarella (1992) to a stochastic model, of which the market price equilibria can be characterised by stationary measures. We choose this very basic model of fundamentalist and speculative behaviour as it captures in a very simple way the essential aspects of the heterogeneous boundedly rational agents paradigm. Economically, in the agent-based financial market model with stochastic noise, we study how the distributional properties of the model, which can be characterised by the stationary distribution of the market price process, change as agents’ behaviour changes and how the market price distribution is influenced by the underlying deterministic dynamics. Mathematically, we seek to understand the connection between different types of attractors and bifurcations of the underlying deterministic skeleton and changes in stationary measures of the stochastic system. By comparing both the indirect and direct approaches, we examine the consistency of the results under both approaches. We show that the stationary measure of the stochastic model displays a bifurcation of very similar nature to that of the steady state of the underlying deterministic model when the time lag of chartist expectations is far away from zero. Using the stochastic method of averaging, we show through a so-called phenomenological (P)-bifurcation analysis that the stationary measure displays a significant qualitative change near a threshold value from single-peak (unimodal) to crater-like (bimodal) joint distributions (and also marginal distributions) as chartists become more active in the market. However, when the time lag in the formation of price trends used by chartists approaches zero, the stochastic model can display very different features from those of its underlying deterministic model.

The paper unfolds as follows. Section 2 reviews the heterogeneous agents financial market models developed by Beja and Goldman (1980) and Chiarella (1992). Sections 3 and 4 examine the dynamical behaviour of the stochastic model...
compared with that of the deterministic model when the time lag is positive or goes to zero, respectively. Section 5 concludes. All proofs are contained in the Appendix.

2. The model

We consider a financial market consisting of investors with different beliefs. Based on the result of Boswijk et al. (2007) (who estimate a heterogeneous agents model using S&P500 market data) that the market mainly consists of fundamentalists and chartists, we assume that there are just two types of investors, fundamentalists and chartists, in the market. For simplicity, we also assume that there are only two types of assets, a risky asset (for instance a stock market index) and a riskless asset (typically a government bond). The fundamentalists base their investment decisions on an understanding of the fundamentals of the market, perhaps obtained through extensive statistical and economic analysis of various market factors. In contrast, the chartists do not necessarily have information about the fundamentals and their investment decisions are based on recent price trends.

The excess demand of the fundamentalists is assumed to be given by

\[ D^f_t(p(t)) = a[f(t) - p(t)], \tag{2.1} \]

where \( p(t) \) is the logarithm of the risky asset price at time \( t \), \( f(t) \) denotes the logarithm of the fundamental price\(^1\) and \( a > 0 \) is a constant measuring the risk tolerance of the fundamentalists brought about by the market price deviation from the fundamental price. The log-linear excess demand function in (2.1) reflects the fundamentalists’ superior knowledge of the fundamental price and strong belief in the convergence of the market price to the fundamental price at a speed measured by their risk tolerance.

Unlike the fundamentalists, the chartists do not necessarily know the fundamental price and they consider the opportunities afforded by the existence of continuous trading out of equilibrium. Their excess demand is assumed to reflect the potential for direct speculation on price changes, reflecting price momentum which is well documented in the literature, see for example Lee and Swaminathan (2000). Let \( \psi(t) \) denote the chartists’ assessment of the current trend in \( p(t) \). Then the chartists’ excess demand is assumed to be given by

\[ D^c_t(p(t)) = h(\psi(t)), \tag{2.2} \]

where \( h \) is a nonlinear continuous and differentiable function, satisfying \( h(0) = 0 \), \( h'(x) > 0 \) for \( x \in \mathbb{R} \), \( \lim_{x \to -\infty} h(x) = 0; h''(x)x < 0 \) for \( x \neq 0 \) and \( h^{(3)}(0) < 0 \) where \( h^{(n)} \) denotes the \( n \)-th order derivative of \( h(x) \) with respect to \( x \). These properties imply that \( h \) is an S-shaped function, indicating that when the trend in the price is above (below) zero, the chartists would like to hold a long (short) position in the risky asset. A similar nonlinear feature on the demand function could be imposed for the fundamentalists. However, for the simplicity of our analysis, we assume that the excess demand of the fundamentalists is linear. In this paper, we take \( h(x) = x \tanh(bx) \) in all numerical simulations, which satisfies all the requirements of the excess demand function of the chartists and is bounded. We should stress that the boundedness of the demand function is not necessary for our results. The important factor in obtaining our results is the change of the slope of the excess demand of the chartists, that is \( h'(-) \), which measures the intensity of the chartists’ reaction to the long/short signal \( \psi \). When \( h'(-) \) is small, the chartists react weakly to the long/short signals as they believe it may have become unsustainably large in absolute value. Under the above assumptions, the intensity of the chartists’ reaction is always bounded and in particular, the chartists are most sensitive at the equilibrium, that is \( \max_{x} h'(x) = h'(0) \), denoted as \( b \). An economic intuition for the different treatment of the excess demand function can be argued as follows. With knowledge of the fundamental price, the fundamentalists are confident about the mean-reversion of the market price to the fundamental price and are assumed to have no budget constraint. In contrast, the chartists do not necessarily have knowledge about the market fundamental price. They trade based on the market price trend by extrapolating the market price when the trend is small, but become cautious when the trend becomes too strong.

Note that the chartists’ speculation on the adjustment of the price primarily depends on an assessment of the state of the market as reflected in price trends. Typically, the assessment of the price trend is based at least in part on recent price changes and is an adaptive process of trend estimation. One of the simplest assumptions is that \( \psi \) is taken as an exponentially declining weighted average of past price changes, that is,

\[ \psi(t) = c \int_{-\infty}^{t} e^{-\tau(t-s)} dp(s), \tag{2.3} \]

where \( c \in (0, \infty) \) is the decay rate, which can also be interpreted as the speed with which the chartists adjust their estimate of the trend to past price changes. Alternatively the quantity \( \tau = 1/c \) may be viewed as the average time lag in the formation of expectations, since in a loose sense \( \psi(t) \, dt \approx dp(t - \tau) \).

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\(^1\) We consider the market price and the fundamental price to both be detrended by the risk-free rate or correspondingly, the risk-free rate is assumed to be zero. Also, in this paper, we assume that the fundamental price \( F(t) \) is either a deterministic constant or given by a stochastic process that will be specified later. In addition, the fundamentalists are assumed to know the fundamental price (when it is a constant) or its distribution with infinite precision (when it is a stochastic process).
Following Beja and Goldman (1980) and Chiarella (1992), the changes of the risky asset price are brought about by aggregate excess demand \( D(t) \) of the fundamentalists (\( D_f^c \)) and of the chartists (\( D_c^c \)), defined below, at a finite speed of price adjustment in continuous time. Furthermore in the market, the transactions and price adjustments are assumed to occur simultaneously. Depending on whether the fundamental price is deterministic or stochastic, we present both a deterministic and a stochastic model in the following analysis.

A deterministic model: We first assume that the fundamental price is a constant, that is \( F = F^* \) and the market price is determined via a market maker mechanism (see Chiarella et al., 2009 for a discussion of market clearing mechanisms). Then, based on the above assumptions, the logarithm of the market price is determined by

\[
\dot{p}(t) = D(t) = D_f^c + D_c^c,
\]

where \( D_f^c, D_c^c \) are the excess demands of the fundamentalists and chartists, respectively, defined above. Note that expression (2.3) can be expressed as the first order differential equation

\[
\dot{\psi}(t) = \phi(t) = \psi(t).
\]

In summary, we obtain a deterministic model of the asset price dynamics given by

\[
\begin{align*}
\dot{p}(t) &= a[F^* - p(t)] + h(\psi(t)), \\
\dot{\psi}(t) &= \frac{1}{\tau}[-ap(t) - \psi(t) + ht(\psi(t)) + aF^*].
\end{align*}
\]

We set \( \phi = \psi \) and then (2.6) can be transformed into the system

\[
\begin{align*}
\dot{\psi} &= \psi, \\
\dot{\phi} &= K(\psi, \phi) - \frac{a\psi}{\tau},
\end{align*}
\]

where \( K(\psi, \phi) = \frac{c + h(\psi) - b\psi}{\tau}, \quad \xi = b - b^*, \quad b = h'(0) \) and \( b^* = 1 + \alpha \).

A stochastic model: We assume that the log fundamental price \( F(t) \) follows a random walk process, that is, \( F(t+h) - F(t) \) is normally distributed with mean 0 and variance \( \sigma^2 h \), independently of past values of \( F(s) (s \leq t) \). Using the notation of the stochastic differential equation, it follows that the log fundamental value \( F(t) \) can be considered to follow the Itô stochastic differential equation (SDE)

\[
dF = \sigma \, dW,
\]

where \( \sigma > 0 \) is the standard deviation (volatility) of the fundamental returns and \( W \) is a standard Wiener process on the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Correspondingly, we obtain a stochastic model of the asset price dynamics

\[
\begin{align*}
\dot{p}(t) &= a[F^* - p(t)] + h(\psi(t)) \, dt, \\
\dot{\psi}(t) &= \frac{1}{\tau}[-ap(t) - \psi(t) + ht(\psi(t)) + aF(t)] \, dt, \\
\dot{F}(t) &= \sigma \, dW.
\end{align*}
\]

Similar to the deterministic case, setting \( \phi \, dt = d\psi \), the stochastic model (2.9) reduces to a nonlinear SDE system\(^3\) in \( \psi \) and \( \phi \), namely

\[
\begin{align*}
\dot{d\psi} &= \psi \, dt, \\
\dot{d\phi} &= K(\psi, \phi) \, dt - \frac{a\psi}{\tau} \, dt + \frac{\sigma}{\tau} \, dW.
\end{align*}
\]

By the transformation, the original deterministic and stochastic systems become (2.7) and (2.10), respectively. In the following, we mainly focus on the dynamical analysis of (2.7) and (2.10). Once the dynamics of (2.7) and (2.10) have been obtained, the dynamics of the price \( p(t) \) can be obtained by integrating the first equation in (2.6) and (2.9), respectively.

Note that when \( \sigma = 0 \), the stochastic model (2.10) reduces to the deterministic model (2.7). Therefore, the stochastic system can be regarded as a stochastic analogue of the deterministic system. By the properties of the function \( h \) and the method in Schenk-Hoppe (1996a), we can show that the solutions of (2.7) and (2.10) are globally well defined for arbitrary \( (a, b, \tau) \) (see Chiarella et al., 2008 for details).

To analyse the systems (2.7) and (2.10), we need to study their long-run characteristics. For a deterministic system, the long-run characteristics are described by the stability and induced bifurcation near steady states. However, the behaviour

\(^2\) In this paper, we use a continuous-time model. In contrast, in the literature, discrete time deterministic models (such as Brock and Hommes, 1998) have mostly been used to study the price dynamics and stochastic versions of these models (such as Wenzelburger, 2004; Böhm and Chiarella, 2005; Horst and Wenzelburger, 2008) have been developed to examine the stochastic bifurcation nature of the stochastic price dynamics. However, stability analysis (in particular the instability) of the stationary measure of stochastic models is difficult for discrete-time models in general and the stochastic method of averaging (see the following section for an introduction to this method) developed in continuous-time can be used to overcome such difficulties.

\(^3\) There are many ways of introducing noise into a deterministic system, such as an additive noise to the price dynamics or stochastic volatility and the corresponding stochastic dynamics would be different for different types of noise in general.
of a random system has both stochastic and dynamic characteristics inherited from its structure of two ingredients: a model, that is (2.7) in our case, describing a dynamical system perturbed by noise and a model of the noise itself. In our stochastic model, the noise is the standard Wiener process $W$. The stochastic characteristics can be described by stationary distributions, which describe the long term behaviour of the solution of (2.10) from the distributional viewpoint. The qualitative change of stationary distributions is examined by the phenomenological (P)-bifurcation approach through the study of the solution of the corresponding Fokker–Planck equation. From a dynamical sample paths point of view, the qualitative changes of (2.10) can be described by the dynamical (D)-bifurcation approach which examines the evolution of the whole system in time and characterises the stochastic dynamics of the SDE system, such as path-wise stability. As indicated in Schenk-Hoppé (1996b) and the references cited therein, the difference between P-bifurcation and D-bifurcation lies in the fact that the P-bifurcation approach is, in general, not related to path-wise stability, whereas the D-bifurcation approach is based on invariant measures, the multiplicative ergodic theorem, Lyapunov exponents, and the occurrence of new invariant measures. The P-bifurcation has the advantage of allowing one to visualise the changes of the stationary density functions. In general, a combined analysis of D- and P-bifurcations can provide a broader picture of the behaviour of the stochastic model. In the following analysis, we focus on the P-bifurcation approach and refer the reader to Chiarella et al. (2008) for a discussion of the D-bifurcation approach to the model of this paper.

As we indicated earlier, if we treat a steady state of the deterministic model as equivalent to a stationary measure of the stochastic model, we are interested in the consistency of the long-run behaviour between the deterministic and stochastic models in terms of the bifurcation, in particular if there are phenomena in the stochastic case that do not have a counterpart in the deterministic case. For the models we have developed, it turns out that the consistency depends on whether the average time lag $\tau$ of the chartists in the formation of expectations is either far from or close to zero, the latter being treated as a limiting case of the former one as $\tau \to 0^+$ and the market price becomes more volatile due to the presence of noise. The following two sections are devoted, respectively, to the analysis of the two cases referred to.

3. Dynamical behaviour with lagged price trend

In this section, in order to highlight the comparison between the deterministic and stochastic dynamics, we consider the case when $\tau$ is bounded away from zero and examine the dynamics of the deterministic model (2.7), followed by the dynamics of the stochastic model (2.10). The method used for our analysis is the method of averaging, developed for the study of continuous-time dynamical systems. This method can be applied to both deterministic and stochastic systems. The method of averaging is an approximation method to study the bifurcation and stability of steady states (for deterministic systems) or stationary measures (for stochastic systems) with respect to certain bifurcation parameters. By changing variables and averaging in time, the mean (or long term) behaviour of the original system is retained in the form of the dynamical equations for the averaged evolution. In particular, for deterministic systems, the problem of detecting limit cycle behaviour that bifurcates from the steady state reduces to a much simpler problem of finding the equilibrium states of the averaged equation. For stochastic systems, the corresponding problem of detecting a crater-like joint stationary distribution can be simplified to that of finding a one-dimensional bimodal stationary distribution of the radius in polar coordinates. For full details of the method of averaging, we refer the reader to Andronov et al. (1966) for the deterministic case and Arnold et al. (1996) for the stochastic case.

3.1. The deterministic dynamical behaviour

For the deterministic model (2.6), when the log fundamental price is constant with $F = F^*$, it is easy to verify that there is a unique steady state $(p^*, \psi^*) = (F^*, 0)$ of (2.6) or correspondingly $(\psi^*, \phi^*) = (0, 0)$ of (2.7). Under the assumption that the chartists use a finite speed of adjustment in adapting to the price trend, that is $0 < c < \infty$, or equivalently that the chartists always estimate the price trend with a delay $\tau > 0$, Beja and Goldman (1980) perform a local linear analysis around the steady state. By considering the nonlinear nature of the function $h(x)$, Chiarella (1992) conducts a nonlinear analysis of the same model. In particular, Chiarella (1992) finds that the intensity of the chartists’s reaction at the equilibrium $b = h(0)$ plays a very important role in determining the dynamics. In this paper, we take $b$ as the key parameter through which to examine the role of the chartists in determining the market price. For convenience of comparison to the dynamics of the stochastic model, we summarise the main result of Chiarella (1992) in the following theorem.

**Theorem 3.1.** If $F = F^*$, the deterministic model (2.7) has a unique steady state $(\psi^*, \phi^*) = (0, 0)$ and the corresponding steady state of log-price is the log fundamental price. Assume that $\tau > 0$ and let $b = h(0)$ and $b^* = 1 + a\tau$, then $(\psi^*, \phi^*)$ is locally asymptotically stable for $b < b^*$ and unstable for $b > b^*$. In addition, $(\psi^*, \phi^*)$ undergoes a supercritical Hopf bifurcation at $b = b^*$ and a stable limit cycle exists for $b > b^*$.

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4 Of course, this is not entirely true as any limiting distribution is a stationary distribution, but not every stationary measure is a limiting distribution.
In order to provide some insight into the method of averaging and extend the method to the analysis of the stochastic model, we provide an alternative and simplified proof of Theorem 3.1 presented in Chiarella (1992). To study the impact of the behaviour of the fundamentalists and chartists on the price, we analyse the characteristics of the stable limit cycle generated from the supercritical Hopf bifurcation. We consider the influence of the chartists as they more actively adjust their estimate of the price trend.

We now examine the stochastic dynamics of the model (2.9). We are concerned with the change of the stationary measure of the stochastic model, instead of that of the steady state of the deterministic model. To study this, we will apply the method of averaging to the transition probability measure underlying the SDE (2.10). To conduct the local analysis,

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5 The method of averaging is a technique that was developed originally by engineers. Its basic aim is to describe the dynamics of nonlinear systems close to the steady state by taking an expansion to a term beyond the linear approximation. A good reference is Andronov et al. (1966).

6 We would like to thank one of the referees by drawing our attention to the effect.
we first apply the standard technique of “blow up” by rescaling some parameters so that the time of the SDE (2.10) is slowed down.

Assume that \( h'(\psi) - b = h'(\psi) - h(0) \) is small, that is \( \psi \) is near 0. Following Arnold et al. (1996), we restrict our analysis to be local and change parameters by rescaling (to “blow up” the neighbourhood around \( \psi = 0 \)) and introducing the (small) parameter \( \varepsilon \) so that \( h'(\psi) - b = \varepsilon^2(h'(\psi) - b) \), \( \zeta \to \varepsilon^2\zeta \) and \( \sigma \to \varepsilon\sigma \). Then the SDE system (2.10) can be rewritten as

\[
\begin{align*}
\frac{d\psi}{\tau} &= \phi dt, \\
\frac{d\phi}{\tau} &= \varepsilon^2 K(\psi, \phi) dt - \frac{a\psi}{\tau} dt + \varepsilon\sigma dW,
\end{align*}
\]

which is of the standard form to which the stochastic method of averaging can be applied. Making the same polar coordinate transformation \( \psi = (r/\eta) \sin(\theta - \eta t), \phi = r \cos(\theta - \eta t) \) as we have done for the deterministic model (2.6) and using the techniques outlined in Khas’minskii (1963) and Arnold et al. (1996), we can obtain the result that the approximation of the radius \( r(t) \) satisfies

\[
dr = \left( U(r_0) + \frac{a^2\sigma^2}{2\tau^2r^2} \right) dt + \frac{a\sigma}{\tau} dW,
\]

the stationary probability density of which is given by

\[
p(r) = C \exp\left( \frac{2\tau^2}{a^2\sigma^2} \int_0^r U(s) ds \right),
\]

where \( C \) is a normalisation constant.

Note that \( p(r) \) attains its extremum at the point \( r = r_e \) satisfying

\[
U(r_e) = -\frac{a^2\sigma^2}{2\tau^2r_e^2}.
\]

It is clear that when \( \sigma = 0 \), Eq. (3.6) reduces to the case of the steady state solution of the polar radius in the deterministic case, that is the one given by (3.3). Let \( G(r) = -a^2\sigma^2/(2\tau^2r^2) \) and then (3.6) may be written as

\[
H(r) + \frac{\zeta}{2} - \frac{b}{2} = G(r).
\]

By (3.2) and the monotonicity of \( G(\cdot) \) for \( r \in (0, \infty) \), Eq. (3.7) has only one solution \( r = r_e \) and in particular, \( p(\cdot) \) attains its maximum value at \( r = r_e \).

When \( b \) is small, in particular \( b < b^* = 1 + \alpha \tau \), then \( H_{\max} < 0 \) and (3.7) is approximated by \(-a^2\sigma^2/(2\tau^2r^2) = \zeta/2\). Then \( p(r) \) attains its maximum value near \( r_e \approx \alpha \sigma / \sqrt{-\zeta} \), which is close to zero for small \( \sigma \), see Fig. 2(a). This in turn implies a joint stationary density of \( (\psi, \phi) \) with a single peak around the origin, as illustrated in Fig. 3(a). In particular, when \( \sigma \to 0 \), we have \( r_e \to 0 \) consistently with the deterministic case. When \( b \) is large, in particular \( b > b^* = 1 + \alpha \tau \), we have \( H_{\max} > 0 \) and \( H_{\min} < 0 \). In this case, the solution of (3.7) is far away from zero, as shown in Fig. 2(b) which can be regarded as the stochastic version of Fig. 1(a). This indicates a crater-like density whose maximum is located on a circle (around \( r = 0 \) which is the steady state of the deterministic system) with a large radius, shown in Fig. 3(g). Fig. 3(d) illustrates the distribution of
that we argue that the stochastic model shares the corresponding dynamics to those of the deterministic model.

density function corresponds to a Hopf bifurcation and (2.10) undergoes a P-bifurcation of a Hopf type. It is in this sense
treat this stationary density (when

suggested by some empirical studies, such as
interest because it has a similar structure to the catastrophe theory model of

4. Dynamical behaviour in the limit

the transition from the single-peak to the crater-like density distribution. The plots in the second and third columns of

Fig. 3 illustrate the corresponding marginal distributions of \( \psi \) and \( p \).

From the above analysis, we see the qualitative change of the stationary density when the parameter \( b \) changes. If we
treat this stationary density (when \( b \) is large) as a bifurcation from the case when \( b \) is small, the maximum radius of the
density function corresponds to a Hopf bifurcation and (2.10) undergoes a P-bifurcation of a Hopf type. It is in this sense
that we argue that the stochastic model shares the corresponding dynamics to those of the deterministic model.

4. Dynamical behaviour in the limit \( \tau \to 0^+ \)

In this section, we consider the limiting case \( \tau \to 0^+ \). This corresponds to the situation in which the chartists use
the most recent price change to estimate the trend of the price. Different from the previous case \( \tau > 0 \), the limiting case is of
interest because it has a similar structure to the catastrophe theory model of Zeeman (1974), a structure that has been
suggested by some empirical studies, such as Anderson (1989) and has been used to explain stock market crashes by
Barunik and Vosvrda (2009). As in Section 3, we first consider the dynamics of the deterministic model and then move to
the stochastic model. For the deterministic model, the limiting case is characterised by a differential-algebraic system. The
system shows a singularity and the dynamics is then characterised by the so-called singularity induced bifurcation, leading
to jump fluctuation phenomenon. For the stochastic model, the dynamics of the limiting case is analysed by using
stochastic bifurcation theory in order to characterise changes of stationary measures. Unlike the previous section, we show
that the stochastic model will display different dynamics from the deterministic model and the existence of noise increases
market volatility. We provide the details in the following discussion.

4.1. The deterministic dynamical behaviour

When \( F = F^* \), the system (2.6) can be rewritten as

\[
\begin{align*}
\dot{p} &= f(p, \psi) := a(F^*-p) + h(\psi), \\
\tau \dot{\psi} &= g(p, \psi) := a(F^*-p) + h(\psi) - \psi.
\end{align*}
\] (4.1)
When $\tau \to 0^+$, the dynamical system (4.1) is known as singularly perturbed systems. As we show in the following, the dynamics have catastrophe theory characteristics, that is the dynamics are fast in one direction (here the $\psi-$direction) and slow in the other direction (here the $p-$direction), see Yurkevich (2004) for a more detailed study of such systems.

In fact, in the limiting case $\tau \to 0^+$, the system (4.1) becomes

$$\Sigma_0: \begin{cases} \dot{p} = f(p,\psi), \\ 0 = g(p,\psi), \end{cases}$$

which is a differential-algebraic equation (DAE)\(^9\) with an algebraic constraint on the variables $p$ and $\psi$. The dynamical system $\Sigma_0$ has apparently lost a dimension in the limit $\tau \to 0^+$. By differentiating the second equation of $\Sigma_0$ with respect to $t$ and using the fact that $\dot{p} = \psi$, we can see that the dynamics of $\psi$ are given by

$$(1-h(\psi))\dot{\psi} = -a\psi,$$  

which is an ordinary differential equation. Note that when $b < 1$, $h(\psi) < 1$ for all $\psi$, so that in this case $\psi = 0$ is the unique steady state of (4.3) and it is stable, as illustrated in Fig. 4(a). However, when $b > 1$, there exist two values $\psi_{10}$ and $\psi_{20}$ with $\psi_{10} < 0 < \psi_{20}$ such that $h(\psi_{10}) = h(\psi_{20}) = 1$. These values are shown in Fig. 4(b) along with the signs of $\psi$, obtained from (4.3), for various values of $\psi$. By the analysis of the vector field of $\psi$, the flow of $\psi$ with nonzero initial value moves away from the origin toward $\psi_{10}$ or $\psi_{20}$. This means that the steady state $\psi = 0$ loses its stability at $b=1$ and becomes unstable as $b$ increases from values just below one to values just above one. In addition, note that when there is a value $\psi_s$ such that $h(\psi_s) = 1$, then the solution of the second equation of $\Sigma_0$ with respect to $\psi$ is nonunique since the implicit function theorem does not hold at $\psi = \psi_s$, which is known as a singular point. At the singular point, $\psi = \psi_s = \infty$.

For system $\Sigma_0$, its singularity, the bifurcation of the fundamental equilibrium $(p,\psi) = (F,0)$ near the singular parameter $b=1$ (the so-called singularity induced bifurcation), and its dynamics are summarised by the following theorem.\(^10\)

**Theorem 4.1.** A singularity induced bifurcation occurs at $b=1$ and the steady state $(F,0)$ loses its stability at $b=1$ and becomes unstable as $b$ increases from $1^-$ to $1^+$.  

To understand the complete dynamics of the system $\Sigma_0$, we consider the limiting dynamics of the singularly perturbed system $\Sigma_\tau$ as $\tau \to 0^+$. We shall show that the singular phenomenon for the system $\Sigma_0$ corresponds to the limiting case of the Hopf bifurcation in the system $\Sigma_\tau$ for $b^* = 1 + \alpha \tau$ ($\tau \neq 0$) when $\tau \to 0^+$.

Recall that, at $b^* = b^*(\tau) = 1 + \alpha \tau$, the system $\Sigma_\tau$ undergoes a supercritical Hopf bifurcation and a stable limit cycle appears. With a fixed $b > b^*(0) > 0$, as $\tau \to 0^+$, the limit cycle persists, as suggested by the numerical simulations in Fig. 5(a). Fig. 5(b) shows a projection of Fig. 5(a) onto the $(\psi,p)$--plane. In fact, in the $(\psi,p)$--plane, we have the following observations about the system $\Sigma_\tau$. As illustrated in Fig. 5(c), for $g(p,\psi) \neq 0$, $\psi$ moves infinitely rapidly toward the curve defined by $g(p,\psi) = 0$ since $\psi \to \infty$ as $\tau \to 0^+$, such a region is denoted as the fast region. For $g(p,\psi) = 0$, as $\tau \to 0^+$ the dynamics are governed by the differential equation for $p$, namely $\dot{p} = a(F-p) + h(\psi)$. Hence, along the curve $g(p,\psi) = 0$, $p$ moves slowly and thus it is called the slow manifold. Specifically, consider how the motion evolves from the initial point $Q$ in the fast region in Fig. 5(c). The variable $\psi$ moves instantaneously horizontally to the point $N$ on the slow manifold $g(p,\psi) = 0$. Motion is then down the slow manifold under the influence of $\dot{p} = a(F-p) + h(\psi)$. When the singular point $B$ is reached, which corresponds to $\psi_{13}$ in Fig. 4(b), $\psi$ jumps instantaneously horizontally across to $C$ on the opposite branch of the slow manifold. Motion is then up to another singular point $D$ which corresponds to $\psi_{23}$ in Fig. 4(b) and the cycle then repeats itself. Therefore, the singular phenomenon in the system $\Sigma_0$ corresponds to a jump phenomenon and the limit cycle with the jump phenomenon consisting of two slow movements along the manifold of $\Sigma_0$ shown in Fig. 5(c), $A \to B$, and $C \to D$, and two jumps at the singular points $B$ and $D$ in $\Sigma_0$, namely $B \to C$ and $D \to A$. The corresponding time series in Fig. 5(d) clearly shows the periodic slow movement in price $p$ and sudden jumps in $\psi$ from time to time. This phenomenon indicates that the model is able to generate significant transitory and predictable fluctuations around the equilibrium. Using a different approach, this jump fluctuation phenomenon in the model of fundamentalists and chartists was studied by Chiarella (1992) who points out that it is merely the relaxation oscillation well known in mechanics and expounded for example by Grasman (1987). Our analysis shows that strong reaction to price changes by the chartists can make the fundamental price unstable, leading to predictable cycles for the market prices and jumps in their estimate of the price trends.

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\(^9\) For more information about DAEs, we refer the reader to the book of Brenan et al. (1989).

\(^10\) For more information about the proof of singularity induced bifurcations, we refer the reader to Chiarella et al. (2008).
Hence, if there exists \( h \), it and taking the stochastic differential of the two sides of the subsection, we make some technical assumptions that there exists \( \tau \rightarrow 0^+ \); (c) the jump fluctuation in the phase plane and (d) the corresponding time series of \( p(t) \) and \( \psi(t) \) at \( \tau \rightarrow 0^- \). Here \( a=1 \) and \( h(x) = \pm \tanh(x) \) with \( \alpha = 1 \) and \( \beta = 2.2 \).

4.2. The stochastic dynamical behaviour

We now analyse the dynamical behaviour of the stochastic model (2.9) in the limit \( \tau \rightarrow 0^+ \). As indicated earlier, the analysis is conducted by use of stochastic bifurcation theory. To analyse the changes of the stationary measure, we use the P-bifurcation approach.

As \( \tau \rightarrow 0^+ \), we see from (2.5) that \( dp / \psi dt \) whilst from the first equation of (2.9), the log price \( p \) is governed by \( dp = [a(F-p) + h(\psi)]dt \). It follows that \( \psi = a(F-p) + h(\psi) \), which implies that \( \psi \) is a continuous process. Assuming that \( \psi \) is an Itô process of the form

\[
d\psi = \lambda dt + \sigma dW
\]

and taking the stochastic differential of the two sides of \( \psi = a(F-p) + h(\psi) \), we find that

\[
d\psi = a(F-p) + h(\psi) \sigma dW + \frac{1}{2} h'(\psi) (dW)^2
\]

\[
= a(F-p) dt + h(\psi) \sigma dt + \frac{1}{2} h'(\psi) \sigma^2 dt,
\]

from which, after some algebraic manipulations and comparing with (4.4), we obtain

\[
\sigma = \sigma(\psi) = \frac{a\sigma}{h'(\psi)} \quad \text{and} \quad \lambda = \lambda(\psi) = -\frac{a\psi}{1-h'(\psi)} + \frac{1}{2} \sigma(\psi)^2.
\]

Hence, if there exists \( \psi_+ \), such that \( h'(\psi_+) = 1 \), then (4.4) is singular at \( \psi = \psi_+ \). Similarly to the deterministic case, for the different cases of \( b < 1 \), \( b = 1 \) and \( b > 1 \), the stochastic differential equation (4.4) will have a different number of singular points and therefore exhibit different behaviour. We will now discuss each case in turn. To simplify the analysis, in this subsection, we make some technical assumptions that there exists \( x_1 < 0 < x_2 \) such that \( h^{(3)}(\psi) < 0 \) for \( \psi \in (x_1,x_2) \), \( h^{(3)}(\psi) > 0 \) otherwise, and \( \psi h^{(4)}(\psi) > 0 \) for \( \psi \in (x_1,x_2) \), as illustrated in Fig. 6. These conditions are satisfied for the hyperbolic tangent function used in the numerical simulations of this paper.

When \( b < 1 \), we have \( 1-h'(\psi) > 0 \) for any \( \psi \) and there is no singularity in \( (-\infty, +\infty) \). The only singular points are \( \pm \infty \). Based on the theory of the classification of singular boundaries,\(^1\) we obtain the following result.

\(^1\) We refer to Lin and Cai (2004) for more information about the theory of the classification of various singular boundaries, including entrance, regular, and (attractively and repulsively) natural boundaries used in our discussion.
Theorem 4.2. When $b < 1$, there exists a unique stationary density $\rho$ for $c$, given by
\[
\rho(c) = N \left(1 - h(c)\right) \exp \left(\int_0^c \frac{2y(1-h(y))}{a\sigma^2} \, dy\right).
\] (4.6)

where $N$ is a normalisation constant.

Note that when $\psi$ satisfies
\[
h''(\psi) + \frac{2\psi(1-h'(\psi))}{a\sigma^2} = 0,
\] (4.7)
the stationary density $\rho(\cdot)$ attains its extremum. This, together with the assumptions on the function $h$, leads to the following result on the P-bifurcation with respect to $\psi$.

Theorem 4.3 (P-bifurcation). Let $b_p = 1 - \sqrt{-h''(0)a\sigma^2}/2$.

1. When $h''(0) > -2/(a\sigma^2)$ and $0 < b < b_p$, the stationary density $\rho(\cdot)$ has a unique extreme point $\psi = 0$, at which $\rho(\cdot)$ attains its maximum.
2. When $\max(b_p,0) < b < 1$, the stationary density $\rho(\cdot)$ has three extreme points $\psi_1$, 0 and $\psi_2$ satisfying $\psi_1 < 0 < \psi_2$. In addition, the stationary density $\rho(\cdot)$ attains its minimum value at $\psi = 0$ and its maximum values at $\psi = \psi_1$ and $\psi_2$.

Theorem 4.3 indicates that, as the chartists place more and more weight on the recent price changes (that is $\tau \to 0^+$), the number of extreme points of the stationary density $\rho(\cdot)$ in (4.6) changes from one to three as the parameter $b$ changes (but $b < 1$). This is shown in Fig. 7. This means that, a moderate increase in activity (such that $b < 1$) of the chartists who weight the recent price changes very heavily results in a large deviation of their estimate of the price trend $\psi$ from its mean value.
is always stable and there is no bifurcation. This result illustrates that, from the P-bifurcation point of view, the stochastic model, but we leave such considerations to future research.

When \( b = 1 \), that is \( h(0) = 1 \), the drift term \( M(t) \) in (4.4) becomes unbounded and furthermore \( \psi = 0 \) is a regular boundary. With the increase of \( b \) to \( b > 1 \), there exist \( \psi_{1b} < 0 < \psi_{2b} \) satisfying \( h(\psi_{1b}) = h(\psi_{2b}) = 1 \), both of which are also regular boundaries. The appearance of the regular boundary at \( b = 1 \) corresponds to the occurrence of a singularity induced bifurcation in the corresponding deterministic case and \( \psi_{1b}, \psi_{2b} \) correspond to the jump points in the deterministic case. In fact, when \( b > 1 \), the stochastic model shares some of the features of the deterministic model. For example, we know from the previous section that the deterministic dynamics exhibit fast motion in the \( \psi \)-direction and slow motion in the \( p \)-direction. For the stochastic model, we observe a similar behaviour, as suggested by the simulations in Fig. 9. However, once a regular boundary appears, it renders the stationary solution nonunique without further stipulation on the behaviour of (4.4) (see Feller, 1952). Unlike the case of \( b < 1 \), this causes some difficulties in obtaining analytical properties of the stationary densities of the stochastic system. However, from a practical point of view, this additional freedom may be a blessing since it may permit certain types of realistic behaviour to be incorporated into the context of the financial model, but we leave such considerations to future research.

5. Conclusion

In this paper, within the framework of the heterogeneous agent paradigm, we extend the basic deterministic model of Beja and Goldman (1980) and Chiarella (1992) to a stochastic model for the market price, and examine the consistency of the stochastic dynamics under the indirect and direct approaches. By using P-bifurcation analysis, we examine the qualitative changes of the stationary measures of the stochastic model.

For the simple stochastic financial market model studied here, when the time lag used by the chartists to form their expected price trends is not zero, we show that near the steady state of the underlying deterministic model, the

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12 In Figs. 8 and 9, we have taken \( h(x) = x \tanh(bx) \) with \( b = 1 \), \( a = 1 \) and \( \sigma = 0.1 \) so that \( b = \sigma \). In Fig. 8, \( \sigma = 0.7 \), \( \sigma = 0.9 \) and \( \sigma = 0.95 \) in the first, second and third columns and in Fig. 9, \( \sigma = 1.2 \) so that \( b = \sigma > 1 \).

13 This observed difference is based on Arnold's (1998) definition of the P-bifurcation. The qualitative change of the stationary distributions under change of variables is also discussed in Wagenmakers et al. (2005) for continuous time random dynamical systems (see also Diks and Wagener, 2006 for discrete time random dynamical systems). In a way different from that of Arnold, these authors use the level-crossing function to characterise the change. Correspondingly, this difference between the deterministic and stochastic cases in the paper observed under Arnold’s definition may disappear, but this issue is beyond the scope of our discussion.

14 Note that in the deterministic case, the occurrence of the singularity induced bifurcation corresponds to the appearance of a singular point where the implicit function theorem does not hold, hence the solution for the implicit function may be nonunique.
approximation of the stochastic model shares the corresponding Hopf bifurcation dynamics of the deterministic model. When the time lag used by the chartists to form their expected price changes approaches zero, so that the chartists are placing more and more weight on recent price changes, then the system can have singular points. In this case, the stochastic model displays very different dynamics from those of the underlying deterministic model. In particular, the fundamental noise can destabilise the market equilibrium and result in a change of the stationary distribution through a P-bifurcation of the stochastic model, while the corresponding deterministic model displays no bifurcation. Our results demonstrate the important connection, but also the significant difference, of the dynamics between deterministic and stochastic models. This difference comes from the existence of noise and the fact that it increases market volatility, a feature which cannot be captured by a deterministic model.

Economically, we have considered a very simple financial market model with heterogeneous agents. This stylised model and the role played by the chartists are shared by many heterogeneous agent-based discrete-time models in recent literature. However, rigorous analysis and theoretical understanding of the stochastic behaviour underlying this intuition are important to policy analysis, investment strategies and market design, but have drawn less attention, mainly due to the difficult nature of these issues. Ultimately we need to apply the theory of stochastic bifurcation directly rather than simulate the models indirectly. Applications of the tools and concepts of stochastic bifurcation theory to financial market models to obtain some analytical insight into this intuition and to improve our understanding about the interaction of the behaviour of heterogeneous agents and noise are the main contributions of this paper.

In order to bring out the basic phenomena associated with stochastic bifurcation we have focused on a highly simplified model. The simplicity of the model limits its ability to generate the stylised facts observed in financial markets. In order to do so the model would need to be embellished in a number of ways in future research, in particular: deriving the asset demands of agents from an intertemporal optimisation framework; introducing other types of randomness such as market noise to characterise excess volatility; allowing for switching of strategies according to some fitness measure (see for example Brock and Hommes, 1997, 1998); and analysing in closer detail the stochastic dynamics to replicate the stylised facts and power-law behaviour observed in real market data.

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Appendix A

A.1. Proof of Theorem 4.2

Based on Lin and Cai (2004), it is not difficult to verify the result that the boundaries $\pm \infty$ are singular boundaries of the second kind at infinity, that is $|\mathcal{M}(\pm \infty)| = \infty$ and the diffusion exponents and drift exponents of $\pm \infty$ are, respectively, $\alpha_{\pm \infty} = 0$ and $\beta_{\pm \infty} = 1$. In addition, $\mathcal{M}(\pm \infty) \leq 0$. Therefore, $\pm \infty$ are repulsively natural, implying that there exists a
nontrivial stationary solution in \((\infty, +\infty)\). In fact, from the Fokker–Planck Equation, we know that the stationary probability is given by (4.6). Note that \(p(\psi) > 0\) for all \(\psi\), so the stationary probability density is unique, see Kliemann (1987).

A.2. Proof of Theorem 4.3

It is obvious that \(\psi = 0\) is one of the solutions to (4.7) for any of the cases considered. In the following, we only consider the case \(\psi \in (\infty, 0)\); a similar reasoning holds for the case \(\psi \in (0, +\infty)\). Let \(S(\psi) = -2\psi(1 - h'(\psi))^2 / (a\sigma^2)\). By the assumptions on \(h\), we know that \(h'(\psi)\) is a concave function on \((x_1, 0)\) and \(\lim_{\psi \to +\infty} h'(\psi) = \ell\) where \(0 \leq \ell < +\infty\). In addition, \(\lim_{\psi \to -\infty} S(\psi) = +\infty\), \(S(\psi) < 0\) for \(\psi \in (-\infty, 0)\), \(S'(\psi) > 0\) for \(\psi \in (x_1, 0)\). When

\[
\begin{align*}
&\lim_{\psi \to +\infty} h'(\psi) = \ell \\
&0 < h'(0) < \infty
\end{align*}
\]

then \(S(0) < h'(0) < 0\). By the convexity and concavity of \(S(\cdot)\) and \(h'(\cdot)\) in \([x_1, 0)\) and monotonicity of \(h\) in \((\infty, x_1)\), there is no solution of (4.7) on \((\infty, 0)\). Hence, when (A.1) is satisfied, (4.7) has a unique solution \(\psi = 0\) on \((\infty, +\infty)\) and \(p^*(0) = (N / \alpha) h'(0) / (a\sigma^2) < 0\), implying that \(\psi = 0\) is the maximum point of \(p(\cdot)\).

When \(b > 1 - \sqrt{-h'(0) a\sigma^2 / 2}\), then \(S(0) > h'(0)\). Therefore, there exists a solution of (4.7) in \((\infty, 0)\), denoted by \(\psi^*_1\), to demonstrate the uniqueness of the solution of (4.7) in \((\infty, 0)\), we consider the following two cases:

(i) If the solution \(\psi^*_1\) is on \([x_1, 0)\), by the concavity of \(h(\cdot)\) and convexity of \(S(\cdot)\) in \([x_1, 0)\), there is a unique solution of (4.7) on \([x_1, 0)\). On the other hand, on \((\infty, x_1)\), \(h(\cdot)\) is monotonically increasing and \(S(\cdot)\) is monotonically decreasing. Hence, there is no other solution on \((\infty, 0)\).

(ii) When \(\psi^*_1 \in (-\infty, x_1)\), there is no solution of (4.7) on the interval \([x_1, 0)\), otherwise there is a contradiction with the case (i). With the monotonicity of \(h(\cdot)\) and \(S(\cdot)\) on \((\infty, x_1)\), there is only one solution \(\psi^*_1\) on \((\infty, 0)\).

In addition, when \(b > 1 - \sqrt{-h'(0) a\sigma^2 / 2}\), we have \(p^*(0) > 0\). So \(\psi = 0\) is the minimum point of \(p(\cdot)\). We note that at \(\psi^*_1\), it must be the case that

\[
\begin{align*}
&\left.\frac{d^2 S(\psi)}{d \psi^2}\right|_{\psi^*_1} > 0, \\
&\left.\frac{d^2 h(\psi)}{d \psi^2}\right|_{\psi^*_1} < 0
\end{align*}
\]

Otherwise, by \(\lim_{\psi \to +\infty} h'(\psi) = \ell < +\infty\) and \(\lim_{\psi \to -\infty} S(\psi) = +\infty\), there must be another solution \(\psi < \psi^*_1\) of (4.7), which is a contradiction of the uniqueness of the solution of (4.7) on \((\infty, 0)\). Then, through (4.7) and (A.2)

\[
\begin{align*}
&\left.\frac{d^2 S(\psi)}{d \psi^2}\right|_{\psi^*_1} > 0, \\
&\left.\frac{d^2 h(\psi)}{d \psi^2}\right|_{\psi^*_1} < 0
\end{align*}
\]

Therefore, \(\psi^*_1\) is a maximum point of \(p(\cdot)\).

References


