Assignment 1

1. Since \( \|f_n - 0\|_2 = \sqrt{\int_0^1 |f_n(t)|^2} = \frac{1}{\sqrt{n}} \to 0 \) as \( n \to \infty \). Thus, \( f_n \) converges in \( L^2[0, 1] \) norm to 0. Assume that given any \( \epsilon > 0 \) \( \exists N(\epsilon) \) such that \( |f_n(t) - 0| < \epsilon \forall t \in [0, 1] \) and \( \forall n > N(\epsilon) \). But, since \( |f_{N+1}(t) - 0| = 1 > \epsilon \) for \( 0 \leq t \leq 1/(N + 1) \) and \( \epsilon < 1 \), we can always establish a contradiction.

Note that the given sequence \( \{f_n(t)\} \) converges pointwise to the following \( f(t) \)

\[
 f(t) = \begin{cases} 
 0 & t \in (0, 1] \\
 1 & t = 0 
\end{cases}
\]

Since \( \lim M_n = 1 \) where \( M_n = \sup_{t \in [0, 1]} |f_n(t) - f(t)| = 1 \), \( \{f_n\} \) does not converge uniformly to \( f(t) \).

2. Since \( \|f_n - 0\|_2 = \sqrt{\int_0^1 |f_n(t)|^2} = \sqrt{\int_0^1 ndt} = 1/\sqrt{n} \to 0 \) as \( n \to \infty \), we conclude that \( f_n \to 0 \) in \( L^2[0, 1] \) norm as \( n \to \infty \). \( \lim_{n \to \infty} f_n(0) = \lim_{n \to \infty} \sqrt{n} = \infty \).

3. Given \( \epsilon > 0 \), choose \( N(\epsilon) = 1/\epsilon \) so that \( |f_n(t) - 0| < \epsilon \forall n > N \) and \( \forall t \in [0, 1] \). Thus \( \{f_n(t)\} \) converges pointwise uniformly to 0.

Since \( \|f_n(t) - 0\|_1 = \int_0^\infty f_n(t) = \int_0^n 1/ndt = 1, \|f_n - 0\|_1 \to 0 \) as \( n \to \infty \). Thus, \( \{f_n\} \) does not converge in \( L^1[0, \infty] \) norm to 0.


5. An inner product calculation. Note that both \( \phi(t) \) and \( \psi(t) \) have norm 1. Problem 4 shows that we need to multiply both \( \psi(2t) \) and \( \psi(2t - 1) \) by \( 2^{1/2} \) to normalize them. After normalization, these four functions form an ON basis of a subspace of \( L^2[0, 1] \). \( \hat{f}(t) = \frac{1}{2} \phi(t) - \frac{1}{4} \psi(t) - \frac{1}{8} \psi(2t) - \frac{1}{8} \psi(2t - 1) \); where \( \hat{f}(t) \) is the projection of \( t \) on the subspace spanned by these four functions.

6. Apply Gram-Schmidt orthogonalization process. The first four set of polynomials: \( 1, x, (3x^2 - 1)/2, (5x^3 - 3x)/2 \). Of course, you can normalize these polynomials to get an ON basis.

7. Since \( P_0, P_1, \ldots, P_{n+1} \) form an ON basis for the set of polynomials of order \( n + 1 \) we can write

\[
 tP_n(t) = \sum_{k=0}^{n+1} c_{nk} P_k(t)
\]

where

\[
 c_{nk} = \langle tP_n, P_k \rangle
\]

It follows from the fact that \( P_n \) is orthogonal to every polynomial of order less than \( n \), that

\[
 c_{nk} = \langle tP_n, P_k \rangle = \langle P_n, tP_k \rangle = 0
\]
for \(k < n - 1\). Therefore, equation 2 simplifies to

\[
tP_n(t) = c_{n,n-1}P_{n-1}(t) + c_{n,n}P_n(t) + c_{n,n+1}P_{n+1}(t)
\]  

(5)

Using the notation of the question:

\[
tP_n(t) = a_nP_n(t) + b_nP_n(t) + c_nP_{n-1}(t)
\]  

(6)

where \(a_n = \langle tP_n, P_{n+1}\rangle = \langle P_n, tP_{n+1}\rangle = \langle tP_{n+1}, P_n\rangle\); and \(c_n = \langle tP_n, P_{n-1}\rangle\). So, \(c_n = a_{n-1}\).

8. Note that for \(t^{3}\), there are discontinuities at \(\pm k\pi\) for \(k = 1,2,\ldots\), so more terms are needed to get better approximation.

9. See Sec 1.7 of the notes on Linear Operators.

We have to consider the following two cases:

(a) Assume that (1) holds. We need to prove that (2) can NOT hold. If (1) holds, then for any \(v \in V\) \(\exists u \in U\) such that \(Tu = v\). Assume that there is a nonzero \(v_0 \in V\) such that \(T^*v_0 = 0\). Then \(\langle u, T^*v_0 = 0\rangle U \forall u \in U\). Use the adjoint of \(T^*\), \(\langle Tu, v_0\rangle_V = 0\), and \(\forall u \in U\). Based on our assumption, \(R(T) = V\). So, \(v_0 \perp v \forall v \in V\). Thus, \(v_0 = 0\), contradicting our assumption that \(v_0 \neq 0\). Q.E.D.

(b) If \(v \notin R(T)\), that means \(\nexists u \in U\) such that \(Tu = v\). So let us find \(v_0\) such that \(T^*v_0 = 0\) (i.e. the least squares problem). Find the projection of \(v\) on \(R(T)\), and let \(v_0 = v - \text{proj}(v) \perp R(T)\). That means \(\langle Tu, v_0\rangle_V = 0 \forall u \in U\). Now using the adjoint operator, we have \(\langle Tu, v_0\rangle_V = \langle u, T^*v_0\rangle_V = 0, \forall u \in U\). Note that \(v_0 \neq 0\) because \(v \notin R(T)\) and \(\text{proj}(v) \in R(T)\). Thus, \(T^*v_0 = 0\).

10. \(\log y = \log a + bx\). You can construct a system of 7 linear equations in two unknowns \(\log a\) and \(b\). You can use matlab to find the solution. In general, if you want to estimate the solution of \(Ax = b\), then solve \(A^TAx = A^Tb\). If \(A^TA\) happens to be invertible (as is the case here), then \(x = (A^TA)^{-1}A^Tb\). The final solution is \(y(2.0) = 2.22\).