Homework Problem Set #1

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**Exercise 1** For \( n > 0 \), define the functions \( f_n \in L^2[0,\infty] \) by

\[
f_n(t) = \begin{cases} 
\sqrt{n}, & \text{for } n \leq t \leq n + \frac{1}{n} \\
0, & \text{otherwise}.
\end{cases}
\]

1. Compute \( ||f_n - f_m|| \). Does the sequence \( \{f_n\} \) converge in the \( L^2 \) norm?
2. Show that \( f_n(t) \) converges pointwise in \([0,\infty)\) and find the limit.
3. Does the sequence converge pointwise uniformly? Justify your answer.
4. Show that \( \{f_n\} \) is ON. Is it a basis?

**Exercise 2** Does the sequence

\[
f_n(x) = \frac{x}{1 + nx^2}
\]

for \( n = 1, 2, \cdots \) converge uniformly on the real line? Justify your answer.

**Exercise 3** For \( n > 0 \), let

\[
f_n(t) = \begin{cases} 
0, & \text{for } -\pi \leq t \leq 0 \\
nt, & \text{for } 0 \leq t \leq \frac{1}{n} \\
1, & \frac{1}{n} \leq t \leq \pi.
\end{cases}
\]

This sequence belongs to \( C[-\pi, \pi] \), i.e., the space of continuous real-valued continuous functions on the interval \([-\pi, \pi]\) with the usual inner product and norm.
1. Show that \( f_n \to \chi_{[0, \pi]} \) in the \( L^2 \) norm, where
\[
\chi_{[0, \pi]}(t) = \begin{cases} 
0, & \text{for } -\pi \leq t \leq 0 \\
1, & \text{for } 0 < t \leq \pi,
\end{cases}
\]
so that \( \{f_n\} \) is Cauchy in \( L^2 \).

2. Show that \( ||\chi - h|| > 0 \) for every \( h \in C[-\pi, \pi] \). Conclude that \( C[-\pi, \pi] \) is not a Hilbert space.

Exercise 4 Suppose the function \( \phi(t) \in L^2[-\infty, \infty] \) satisfies \( \int_{-\infty}^{\infty} \phi(t) \overline{\phi(t-k)} dt = \delta_{0,k} \), i.e., the integral equals 1 for \( k = 0 \) and vanishes for \( k = 1, 2, \ldots \). Show that for any fixed integer \( j \) the functions \( \phi_{jk}(t) = 2^{j/2} \omega(2^j t - k), \)
\( (k = 0, \pm1, \pm2, \ldots) \) form an ON set.

Exercise 5 Project the function \( f(t) = t^2 \) onto the subspace of \( L^2[0,1] \) spanned by the functions \( \phi(t), \psi(t), \psi(2t), \psi(2t-1) \), where
\[
\phi(t) = \begin{cases} 
1, & \text{for } 0 \leq t < 1 \\
0, & \text{otherwise}
\end{cases}
\psi(t) = \begin{cases} 
1, & \text{for } 0 \leq t < 1/2 \\
-1, & \text{for } 1/2 \leq t < 1 \\
0, & \text{otherwise}.
\end{cases}
\]
(This is related to the Haar wavelet expansion for \( f \).)

Exercise 6 Let \( L^2[-\infty, \infty, \omega(t)] \) be the space of square integrable functions on the real line, with respect to the weight function \( \omega(t) = e^{-t^2} \). The inner product on this space is thus
\[
(f, g) = \int_{-\infty}^{\infty} f(t) g(t) \overline{\omega(t)} dt.
\]
The Hermite polynomials \( H_n(t), \ n = 0,1, \ldots \) are the ON set of polynomials on \( L^2[-\infty, \infty, \omega(t)] \), obtained by applying the Gram-Schmidt process to the monomials \( 1, t, t^2, t^3, \ldots \) and defined uniquely by the requirement that the coefficient of \( t^n \) in \( H_n(t) \) is positive. (In fact they form an ON basis for \( L^2[-\infty, \infty, \omega(t)] \).) Compute the first 4 of these polynomials. NOTE: Later we will show that \( \int_{-\infty}^{\infty} e^{-it} e^{-t^2} dt = \sqrt{\pi} e^{-s^2/4} \). You can use this result, if you wish, to simplify the calculations.
Exercise 7 Note that in the last problem, \( H_0(t), H_2(t) \) contained only even powers of \( t \) and \( H_1(t), H_3(t) \) contained only odd powers. Can you find a simple proof, using only the uniqueness of the Gram-Schmidt process, of the fact that \( H_n(-t) = (-1)^n H_n(t) \) for all \( n \)?

Exercise 8 Find the \( L^2[-\pi, \pi] \) projection of the function \( f_1(t) = |t| \) onto the \((2n+1)\)-dimensional subspace spanned by the ON set

\[
\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos kt}{\sqrt{\pi}}, \frac{\sin kt}{\sqrt{\pi}} : k = 1, \ldots, n \right\}
\]

for \( n = 1 \). Repeat for \( n = 2, 3 \). Plot these projections along with \( f_1 \). (You can use MATLAB, a computer algebra system, a calculator, etc.) repeat the whole exercise for \( f_2(t) = t \). Do you see any marked differences between the graphs in the two cases?

Exercise 9 Use least squares to fit a straight line of the form \( y = bx + c \) to the data

\[
\begin{align*}
x &= 0 \quad 1 \quad 3 \quad 4 \\
y &= 0 \quad 8 \quad 8 \quad 20
\end{align*}
\]

in order to estimate the value of \( y \) when \( x = 2.0 \).

Exercise 10 Repeat the above problem to find the best least squares fit of the data to a parabola of the form \( y = ax^2 + bx + c \). Again, estimate the value of \( y \) when \( x = 2.0 \).