Problem Set #2
Math 8600
November 12, 2002

Exercise 1 (Oversampling) In this problem you will show that a positive effect of sampling a band limited signal \( f(t) \) at a rate faster than the Nyquist rate is that the expansion for \( f(t) \) in terms of the sampled values converges at a faster rate. (The negative effect is that you have to sample more often.)

1. Suppose \( f \) satisfies the hypotheses of the Shannon sampling theorem proven in the notes; in particular it is a band limited signal with \( \hat{f}(\lambda) = 0 \) for \( |\lambda| \geq \Omega \). Fix \( a > 1 \) and reprove the theorem to show that

\[
\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-i\frac{n\pi}{a\Omega}}, \quad c_{-n} = \frac{\pi}{a\Omega} f\left(\frac{n\pi}{a\Omega}\right).
\]

Solution: Here we use the definitions

\[
\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(t) e^{-i\lambda t} dt, \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\lambda t} d\lambda
\]

On the interval \([-a\Omega, a\Omega]\) we can expand \( \hat{f}(\lambda) \) in a uniformly convergent Fourier series

\[
\hat{f}(\lambda) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-i\frac{n\pi}{a\Omega}}.
\]

Since \( \hat{f}(\lambda) \) vanishes outside this interval,

\[
c_{-n} = \frac{1}{2a\Omega} \int_{-a\Omega}^{a\Omega} \hat{f}(\lambda) e^{i\frac{n\pi}{a\Omega}} d\lambda = \frac{1}{2a\Omega} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{i\frac{n\pi}{a\Omega}} d\lambda = \frac{\pi}{a\Omega} f\left(\frac{n\pi}{a\Omega}\right).
\]
Now
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda t} d\lambda = \frac{1}{2\pi} \int_{-a\Omega}^{a\Omega} \hat{f}(\lambda)e^{i\lambda t} d\lambda \]
\[ = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{i(-\pi n + a\Omega) t/a\Omega} = \sum_{n=-\infty}^{\infty} c_n \frac{\sin[a\Omega t - n\pi]}{\pi(a\Omega t - n\pi)}. \]

2. Let
\[ \hat{g}_a(\lambda) = \begin{cases} 
0 & \text{if } |\lambda| > a\Omega \\
\frac{\lambda + a\Omega}{(a-1)\Omega} & \text{if } -a\Omega \leq \lambda < -\Omega \\
1 & \text{if } -\Omega \leq \lambda < \Omega \\
-\frac{\lambda - a\Omega}{(a-1)\Omega} & \text{if } \Omega \leq \lambda \leq a\Omega.
\end{cases} \]

Show that
\[ g_a(t) = \frac{\cos\Omega t - \cos a\Omega t}{\pi(a - 1)\Omega t^2}. \]

Solution:
\[ g_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}_a(\lambda)e^{i\lambda t} d\lambda \]
\[ = \frac{1}{2\pi(a - 1)} \int_{-a\Omega}^{a\Omega} \left[ (\lambda + a\Omega) e^{i\lambda t} d\lambda + \int_{a\Omega}^{\infty} (-\lambda + a\Omega) e^{i\lambda t} d\lambda \right] + \frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{i\lambda t} d\lambda \]
\[ = \frac{1}{2\pi(a - 1)} \int_{-\Omega}^{\Omega} \left[ -2\lambda \cos\lambda t + 2a\Omega \cos\lambda t \right] d\lambda + \frac{\sin \Omega t}{\pi t} \]
\[ = \frac{\cos\Omega t - \cos a\Omega t}{\pi(a - 1)\Omega t^2}. \]

3. Since \( \hat{f}(\lambda) = 0 \) for \( |\lambda| \geq \Omega \), we see that \( \hat{f}(\lambda) = \hat{f}(\lambda)\hat{g}_a(\lambda) \). Prove that
\[ f(t) = \sum_{n=-\infty}^{\infty} \frac{\pi}{a\Omega} f\left(\frac{n\pi}{a\Omega}\right) g_a(t - \frac{n\pi}{a\Omega}). \]

Solution:
\[ \hat{f}(\lambda) = \hat{f}(\lambda)\hat{g}_a(\lambda) = \sum_{n=-\infty}^{\infty} c_n e^{-in\frac{\lambda}{a\Omega}} \hat{g}_a(\lambda). \]

Note that if \( h(\lambda) = e^{-in\frac{\lambda}{a\Omega}} \hat{g}_a(\lambda) \) then
\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-in\frac{\lambda}{a\Omega}} \hat{g}_a(\lambda) e^{i\lambda t} d\lambda = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}_a(\lambda) e^{i\lambda t - \frac{n\pi}{a\Omega}} d\lambda = g_a(t - \frac{n\pi}{a\Omega}). \]
Hence the result follows.

Since \( g_n(t) \) has a factor of \( t^2 \) in the denominator, this expression for \( f(t) \) converges faster than the expression in the Shannon theorem. Note that the \( n \)th term behaves like \( 1/n^2 \), rather than \( 1/n \).

**Exercise 2** *(Filtering with the FFT)* Let

\[
f(t) = e^{-\frac{t^2}{16}} (\sin 2t + 2 \cos 4t + 0.4 \sin t \sin 50t).
\]

Discretize \( f \) by setting \( y_k = f(2k\pi/256), k = 1, \ldots, 256 \). Use MATLAB’s FFT to compute \( \hat{y}_k \) for \( 0 \leq k \leq 256 \). (Note that \( y_{n-k} = y_k \). Thus the low frequency coefficients are \( \hat{y}_0, \ldots, \hat{y}_m \) and \( \hat{y}_{256-m}, \ldots, \hat{y}_{256} \) for some small integer \( m \). Filter out the high-frequency terms by setting \( \hat{y}_k = 0 \) for \( m \leq k \leq 255 - m \) with \( m = 6 \). Then apply the inverse FFT to these filtered \( \hat{y}_k \) to compute the filtered \( y_k \). Plot the results and compare with the original unfiltered signal. Experiment with several different values of \( m \).

**Exercise 3** *(Compression with the FFT)* Consider the signal \( f(t) \) as given in the previous problem. Let \( \text{tol} = 1.0 \). In the previous problem compress the transformed signal by setting \( \hat{y}_k = 0 \) whenever \( |\hat{y}_k| < \text{tol} \). Apply the inverse FFT to the compressed transformed signal to get a compressed signal \( y_k \). Plot the results and compare with the original uncompressed signal. Experiment with several different values of \( \text{tol} \). Keep track of the percentage of Fourier coefficients that have been filtered out.

**Exercise 4** Construct the unitary rep \( \mathbf{T}_N \) of \( H_R \) induced by the one-dimensional rep \( T_0^N(a_1, a_2, y_3) = e^{2\pi i N y_3} \) of the subgroup \( H^1 \), where \( N \) is an integer (not necessarily positive). Determine the action of \( H_R \) on the rep space. Under what conditions on \( N \) is \( \mathbf{T}_N \) irreducible?

**Solution** Here \( \mathbf{T}_N \) is defined on the space \( V \) of functions \( f \) on \( H_R \) such that \( f(B A) = T_0(B)f(A) \) for all \( B \in H', A \in H_R \), i.e.,

\[
f(a_1 + x_1, a_2 + x_2, y_3 + x_3 + a_1 x_2) = e^{2\pi i N y_3} f(x_1, x_2, x_3).
\] (1)

The operators \( \mathbf{T}_N(A), A \in H_R \) act on \( V \) according to

\[
[\mathbf{T}_N(A)f](A') = f(A'A).
\] (2)
We see that for any $A(x_1, x_2, x_3)$ we can always choose $B(a_1, a_2, y_3)$ such that $BA = A'(x_1', x_2', 0)$ where $0 \leq x_1' < 1, 0 \leq x_2' < 1$. Thus $f$ can be restricted to $X = H' \setminus H_R$ with coordinates $(x_1', x_2', 0)$. Moreover, setting $x_3 = 0, y_3 = -a_1 x_2$ in we have the periodicity condition

$$\varphi(a_1 + x_1, a_2 + x_2) = e^{-2\pi i a_1 x_2} \varphi(x_1, x_2)$$  \hspace{1cm} (3)

where $\varphi(x_1, x_2) = f(x_1, x_2, 0)$. Conversely, given $\varphi$ we can define a unique $f$ satisfying by

$$f(x_1, x_2, x_3) = \varphi(x_1, x_2)e^{2\pi i N x_3}.$$  

The $H_R$-invariant inner product on $X$ is $dx_1 dx_2$:

$$\langle \varphi_1, \varphi_2 \rangle = \int_0^1 \int_0^1 \varphi_1(x_1, x_2) \overline{\varphi_2(x_1, x_2)} dx_1 dx_2,$$  \hspace{1cm} (4)

and the operator $T_N[y] \equiv T_N(A(y_1, y_2, y_3))$ acts on these functions by

$$(T_N[y] \varphi)(x_1, x_2) = \exp[2\pi i N(y_3 + x_1 y_2)] \varphi(x_1 + y_1, x_2 + y_2).$$  \hspace{1cm} (5)

If $N = 0$, it is obvious that this representation is reducible. Indeed, each exponential $e^{i(m_1 x_1 + m_2 x_2)}$ for $m_1, m_2$ integers is mapped to a multiple of itself by the group action. Now assume that the integer $N$ is nonzero.

Consider the periodizing operators

$$P_j \psi(x_1, x_2) = \frac{e^{2\pi i j x_2}}{\sqrt{N}} \sum_{n=-\infty}^{\infty} (T_N[x_1, x_2, 0] \psi)(n) = \frac{1}{\sqrt{N}} \sum_{n=-\infty}^{\infty} e^{2\pi i (j + n N x_2)} \psi(n + x_1),$$  \hspace{1cm} (6)

which is well defined for any $\psi \in L^2(R)$ which belongs to the Schwartz space. It is straightforward to verify that $f_j = P_j \psi$ satisfies the required periodicity condition, hence $f$ belongs to $V$. Further $f_j = e^{2\pi i j x_2} F(x_1, X_2)$ where $F$ is periodic in $x_2$ with period $1/N$. We say that $V_j$ is the space of those functions in $V'$ with this form. Also

$$\langle P_j \psi(\cdot, \cdot, 0), P_k \psi'(\cdot, \cdot, 0) \rangle = \frac{1}{N} \int_0^1 dx_1 \int_0^1 dx_2 \sum_{m,n=-\infty}^{\infty} e^{2\pi i (j - k + N(n - m) x_2)} \psi(n + x_1) \overline{\psi'(m + x_1)}$$

$$= \int_0^1 dx_1 \sum_{n=-\infty}^{\infty} \psi(n + x_1) \overline{\psi'(n + x_1)} \frac{\delta_{jk}}{N} = \int_{-\infty}^{\infty} \psi(t_1) \overline{\psi'(t)} \ dt \frac{\delta_{jk}}{N}$$

$$= (\psi, \psi') \frac{\delta_{jk}}{N}, \quad 0 \leq j, k \leq |N| - 1,$$
so $P = \sum_{j=0}^{[N]-1} P_j$ can be extended to an inner product preserving mapping of $L_2(R)$ into $V$. That is, $P$ maps $\psi, \psi'$ to $\sum_j f_j, \sum_j f'_j$ respectively, and we have

$$\langle \sum_j f_j, \sum_k f'_k \rangle = \sum_j \langle f_j, f'_j \rangle = \frac{1}{N} \sum_{j=0}^{[N]-1} (\psi, \psi') = (\psi, \psi').$$

It is clear from that if $\varphi_j(x_1, x_2) = P_j \psi(x_1, x_2, 0)$ then we can recover $\psi(x_1)$ by integrating with respect to $x_2 : \psi(x_1) = \int_0^1 \varphi_j(x_1, y)e^{-2\pi i x_2 dy}$. Thus we define the mapping $P^*_j$ of $V'_j$ into $L_2(R)$ by

$$P^*_j \varphi_j(t) = \frac{1}{\sqrt{N}} \int_0^1 \varphi(t, y)e^{-2\pi i x_2 dy}, \quad \varphi \in V'_j.$$ (7)

Since $\varphi_j \in V'_j$ we have

$$P^* \varphi_j(t + a) = \frac{1}{\sqrt{N}} \int_0^1 \varphi(t, y)e^{-2\pi i x_2 dy} = \frac{1}{\sqrt{N}} \hat{\varphi}_{j+N\alpha}(t)$$

for $a$ an integer. (Here $\hat{\varphi}_n(t)$ is the $n$th Fourier coefficient of $\varphi(t, y)$.) The Parseval formula then yields

$$\int_0^1 |\varphi(t, y)|^2 dy = \sum_{j=0}^{[N]-1} \sum_{a=-\infty}^\infty |P^*_j \varphi(t + a)|^2$$

so

$$\langle \varphi, \varphi \rangle = \int_0^1 \int_0^1 |\varphi(t, y)|^2 dt dy = \int_0^1 \sum_{j=0}^{[N]-1} \sum_{a=-\infty}^\infty |P^*_j \varphi(t + a)|^2 dt$$

$$= \sum_{j=0}^{[N]-1} \int_0^\infty |P^*_j \varphi(t)|^2 dt = \sum_{j=0}^{[N]-1} (P^*_j \varphi, P^*_j \varphi).$$

Thus this is an inner product preserving mapping of $V' = \oplus_j V'_j$ back into $L_2(R)$. Moreover, it is easy to verify that

$$\langle P_j^* \psi, \varphi \rangle = (\psi, P^*_j \varphi)$$

for $\psi \in L_2(R), \varphi \in V'$, i.e., $P^* = \sum_j P^*_j$ is the adjoint of $P \sum_j P_j$. Since $P^*P = \mathbf{e}$ on $L_2(R)$ it follows that $P$ is a unitary operator mapping $L_2(R)$ onto $V'$ and $P^* = P^{-1}$ is a unitary operator mapping $V'$ onto $L_2(R)$.

Finally,

$$(P_j T^N[y][\psi](x) = e^{2\pi i [j+N(x_3+y_3+x_1 y_1)]} \sum_{n=-\infty}^\infty e^{2\pi i n(x_2+y_2)} \psi(n + x_1 + y_1)$$

$$= (T[y]P_j \psi)(x)$$

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so $\mathbf{P}_j \mathbf{T}_N[y] = \mathbf{T}_N[y] \mathbf{P}_j$ and the unitary reps $\mathbf{T}_N$ and $\mathbf{T}_N[V'_j]$ are equivalent for each $j = 0, 1, \ldots, |N| - 1$.

Thus we have shown that The representation space $V'$ splits into a direct sum of $|N|$ subspaces $V'_j$, and restricted to each of these subspaces the representation is irreducible and equivalent to $\mathbf{T}_N$. Note that $\mathbf{T}_N$ is irreducible only if $N = \pm 1$.

**Exercise 5** Suppose $f \in L_2(\mathbb{R})$ such that $f_\mathbf{P} \neq 0$ almost everywhere. Prove that the set $\{e^{2\pi i (m_1 x_1 + m_2 x_2)} | f_\mathbf{P}| / |f_\mathbf{P}|, \ m_1, m_2 = \pm 1, \pm 2, \ldots \}$ is an ON basis for the lattice Hilbert space $V'$. Find an explicit expression for the corresponding ON basis of $L_2(\mathbb{R})$ obtained from the mapping $\mathbf{P}^{-1}$.

**Solution:** We have that $f_\mathbf{P} \neq 0$ a.e., that it is square integrable on the unit square, and that

$$f_\mathbf{P}(a_1 + x_1, a_2 + x_2) = e^{-2\pi i a_1 x_2} f_\mathbf{P}(x_1, x_2),$$

Moreover,

$$f_\mathbf{P}(x_1, x_2) = \sum_{k=-\infty}^{\infty} e^{2\pi i k x_2} f(x_1 + k).$$

Note also that

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt = (f_\mathbf{P}, g_\mathbf{P}) = \int_0^1 \int_0^1 f_\mathbf{P} \overline{g_\mathbf{P}} d_1 d_2$$

and

$$\mathbf{T}_1[y_1, y_2] f_\mathbf{P}(x_1, x_2) = e^{2\pi i y_1 x_2} f_\mathbf{P}(x - 1 + y_1, x_2 + y_2),$$

$$\mathbf{T}_1[y_1, y_2] f(t) = e^{2\pi i t y_2} f(t + y_1).$$

Now let $g_\mathbf{P} \in V'$. We want to expand $g_\mathbf{P}$ in the form

$$g_\mathbf{P}(x_1, x_2) = \sum_{m_1, m_2} c_{m_1, m_2} e^{2\pi i (m_1 x_1 + m_2 x_2)} |f_\mathbf{P}| / |f_\mathbf{P}|(x_1, x_2).$$

Note that the set

$$E_{m_1 m_2}(x_1, x_2) = e^{2\pi i (m_1 x_1 + m_2 x_2)} |f_\mathbf{P}| / |f_\mathbf{P}|$$

is ON on $V'$. Indeed

$$(E_{m_1 m_2}, E_{n_1 n_2}) = \int_0^1 \int_0^1 e^{2\pi i [(m_1 - n_1) x_1 + (m_2 - n_2) x_2]} d_1 d_2 = \delta_{m_1 n_1} \delta_{m_2 n_2}.$$
To show that this set is a basis, note that since $g_p \frac{f_p}{|f_p|}$ is square integrable, our problem is equivalent to the expansion

$$g_p \frac{f_p}{|f_p|}(x_1, x_2) = \sum_{m_1, m_2} c_{m_1 m_2} e^{2\pi i (m_1 x_1 + m_2 x_2)}$$

which we know exists. Indeed

$$c_{m_1 m_2} = \langle g_p \frac{f_p}{|f_p|}, e^{2\pi i (m_1 x_1 + m_2 x_2)} \rangle = \langle g_p, E_{m_1 m_2} \rangle.$$ 

Next, note that $\frac{f_p}{|f_p|}(x_1, x_2)$ satisfies the twisted periodicity property, so that it belongs to $V'$. Hence there is a square integrable function $\tilde{f}(t)$ such that

$$\int_0^1 \frac{f_p}{|f_p|}(t, y) dy = \tilde{f}(t), \quad P \tilde{f} = \frac{f_p}{|f_p|}.$$ 

Moreover, for any $g \in L^2(R)$ we have

$$T_1[-m_2, m_1] g_p = e^{2\pi i (m_1 x_1 + m_2 x_2)} g_p, \quad T_1[-m_2, m_1] g(t) = e^{2\pi i m_1 t} g(t - m_2).$$

Therefore, our ON basis in $L^2(R)$ is

$$f_{m_1 m_2}(t) = e^{2\pi i m_1 t} \tilde{f}(t - m_2), \quad \tilde{f}(t) = \int_0^1 \frac{f_p}{|f_p|}(t, y) dy.$$ 

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The following MATLAB routines should be useful:

```matlab
function fc=compress( f, r)
% Input the vector f and ratio r: 0<= r <=1.
% The output is the vector fc in which the smallest
% 100r% of the terms f_k, in absolute value, are set
% equal to zero.
if (r<0) | (r>1)
  error ('r should be between 0 and 1')
end;
N=length(f); Nr=floor(N*r);
ff=sort(abs(f));
tol=abs(ff(Nr+1));
fc=(abs(f)>=toll).*f;
```
You can discretize the interval \([0, 2\pi]\) and read in the signal as a vector by using the commands

\[
t = \text{linspace}(0, 2\pi, 2^8);
\]

\[
f = \exp(-t.^2/10).*(\sin(2^8t) + 2*\cos(4^8t) + 0.4*\sin(t).*\sin(50^8t)));
\]

If \(\text{hatf}\) is the FFT of \(f\), you can filter out high frequency components from \(\text{hatf}\) with a command such as

\[
\text{filterhatf} = [\text{hatf}(1:m) \text{zeros}(1, 2^8-2*m) \text{hatf}(2^8-m+1:2^8)]
\]

```matlab
function L2error = fftcomp(t,f,r)
    \%
    \% Input: time vector t, signal vector f, compression rate r, (between
    \% 0 and 1)
    \% Output: graph of f, graph of the compression of f, and the relative L2
    \% error
    \% if (r<0) \| (r>1)
        \% error ('r should be between 0 and 1')
    end;
    \% hatf=fft(f);
    \% hatfc=compress(hatf,r);
    \% fc=ifft(hatfc);
    \% plot(t,f,t,fc)
    \% L2error=norm(f-fc,2)/norm(f)
```