Separation of variables, superintegrability and Bôcher contractions.

Willard Miller, [Joint with E.G. Kalnins (Waikato), Sarah Post (Hawaii), Eyal Subag (Technion) and Robin Heinonen (Minnesota)]

University of Minnesota
Quantum superintegrable systems are exactly solvable quantum eigenvalue problems. Their solvability is due to symmetry, but the symmetry is often “hidden”. The symmetry generators of 2nd order superintegrable systems in 2 dimensions close under commutation to define quadratic algebras, a generalization of Lie algebras. The irreducible representations of these algebras yields important information about the eigenvalues and eigenspaces of the quantum systems. Distinct superintegrable systems and their quadratic algebras are related by geometric contractions, induced by generalized Inönü-Wigner Lie algebra contractions which have important physical and geometric implications, such as the Askey scheme for obtaining all hypergeometric orthogonal polynomials as limits of Racah/Wilson polynomials. This can all be unified by ideas first introduced in the 1894 thesis of Bôcher to study R- separable solutions of the wave equation.
Superintegrable Systems: $H\psi = E\psi$

A quantum superintegrable system is an integrable Hamiltonian system on an $n$-dimensional Riemannian/pseudo-Riemannian manifold with potential:

$$H = \Delta_n + V$$

that admits $2n - 1$ algebraically independent partial differential operators commuting with $H$, the maximum possible.

$$[H, L_j] = 0, \quad L_{2n-1} = H, \quad n = 1, 2, \cdots, 2n - 1.$$  

Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\psi = E\psi$ to be solved exactly, analytically and algebraically.

A system is of order $K$ if the maximum order of the symmetry operators, other than $H$, is $K$. For $n = 2$, $K = 1, 2$ all systems are known.
Superintegrable Systems: $H\psi = E\psi$

- A quantum superintegrable system is an integrable Hamiltonian system on an $n$-dimensional Riemannian/pseudo-Riemannian manifold with potential:

$$H = \Delta_n + V$$

that admits $2n - 1$ algebraically independent partial differential operators commuting with $H$, the maximum possible.

$$[H, L_j] = 0, \quad L_{2n-1} = H, \quad n = 1, 2, \ldots, 2n - 1.$$

- Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\psi = E\psi$ to be solved exactly, analytically and algebraically.

- A system is of order $K$ if the maximum order of the symmetry operators, other than $H$, is $K$. For $n = 2$, $K = 1, 2$ all systems are known.
Superintegrable Systems: $H\psi = E\psi$

- A quantum superintegrable system is an integrable Hamiltonian system on an $n$-dimensional Riemannian/pseudo-Riemannian manifold with potential:

$$H = \Delta_n + V$$

that admits $2n - 1$ algebraically independent partial differential operators commuting with $H$, the maximum possible.

$$[H, L_j] = 0, \quad L_{2n-1} = H, \quad n = 1, 2, \cdots, 2n - 1.$$  

- Superintegrability captures the properties of quantum Hamiltonian systems that allow the Schrödinger eigenvalue problem $H\psi = E\psi$ to be solved exactly, analytically and algebraically.

- A system is of order $K$ if the maximum order of the symmetry operators, other than $H$, is $K$. For $n = 2$, $K = 1, 2$ all systems are known.
Free 2nd order superintegrable systems, (no potential, $K = 2$)

We apply these ideas to 2nd order systems in 2D ($2n - 1 = 3$). The complex spaces with Laplace-Beltrami operators admitting at least three 2nd order symmetries were classified by Koenigs (1896). They are:

- **The two constant curvature spaces**, six linearly independent 2nd order symmetries and three 1st order symmetries,
- **The four Darboux spaces**, four 2nd order symmetries and one 1st order symmetry,
- **Eleven 4-parameter Koenigs spaces**. No 1st order symmetries. Example

\[ ds^2 = 4x(dx^2 + dy^2), \quad ds^2 = \frac{x^2 + 1}{x^2} (dx^2 + dy^2), \]
\[ ds^2 = \frac{e^x + 1}{e^{2x}} (dx^2 + dy^2), \quad ds^2 = \frac{2 \cos 2x + b}{\sin^2 2x} (dx^2 + dy^2), \]

- **Eleven 4-parameter Koenigs spaces**. No 1st order symmetries. Example

\[ ds^2 = \left( \frac{c_1}{x^2 + y^2} + \frac{c_2}{x^2} + \frac{c_3}{y^2} + c_4 \right)(dx^2 + dy^2). \]
2nd order systems with potential, $K = 2$

- The symmetry operators of each system close under commutation to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of $H$ and their multiplicity.

- All the 2nd order superintegrable systems are limiting cases of a single system: the generic 3-parameter potential on the 2-sphere, $S_9$ in our listing. Analogously all quadratic symmetry algebras of these systems are contractions of $S_9$.

\[ S_9 : \quad H = \Delta_2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad s_1^2 + s_2^2 + s_3^2 = 1, \]

\[ L_1 = (s_2 \partial_{s_3} - s_3 \partial_{s_2})^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2, \quad L_3, \]

\[ H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3 \]
2nd order systems with potential, $K = 2$

- The symmetry operators of each system close under commutation to generate a quadratic algebra, and the irreducible representations of this algebra determine the eigenvalues of $H$ and their multiplicity.

- All the 2nd order superintegrable systems are limiting cases of a single system: the generic 3-parameter potential on the 2-sphere, $S^9$ in our listing. Analogously all quadratic symmetry algebras of these systems are contractions of $S^9$.

\[
S^9: \quad H = \Delta^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}, \quad s_1^2 + s_2^2 + s_3^2 = 1, \\
L_1 = (s_2 \partial_{s_3} - s_3 \partial_{s_2})^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2, \ L_3, \\
H = L_1 + L_2 + L_3 + a_1 + a_2 + a_3
\]
3 types of 2nd order superintegrable systems:

1. Nondegenerate:
   \[ V(x) = a_1 V_1(x) + a_2 V_2(x) + a_3 V_3(x) + a_4 \]

2. Degenerate:
   \[ V(x) = a_1 V_1(x) + a_2 \]

3. Free:
   \[ V = a_1, \text{ no potential} \]
Nondegenerate systems \((2n - 1 = 3\) generators\)

The symmetry algebra generated by \(H, L_1, L_2\) always closes under commutation. Define 3rd order commutator \(R\) by \(R = [L_1, L_2]\). Then

\[
[L_j, R] = A_1^{(j)} L_1^2 + A_2^{(j)} L_2^2 + A_3^{(j)} H^2 + A_4^{(j)} \{L_1, L_2\} + A_5^{(j)} HL_1 + A_6^{(j)} HL_2
\]

\[
+ A_7^{(j)} L_1 + A_8^{(j)} L_2 + A_9^{(j)} H + A_{10}^{(j)}, \quad \{L_1, L_2\} = L_1 L_2 + L_2 L_1,
\]

\[
R^2 = b_1 L_1^3 + b_2 L_2^3 + b_3 H^3 + b_4 \{L_1, L_2\} + b_5 \{L_1, L_2\} + b_6 L_1 L_2 L_1 + b_7 L_2 L_1 L_2
\]

\[
+ b_8 H\{L_1, L_2\} + b_9 HL_1^2 + b_{10} HL_2^2 + b_{11} H^2 L_1 + b_{12} H^2 L_2 + b_{13} L_1^2 + b_{14} L_2^2 + b_{15} \{L_1, L_2\}
\]

\[
+ b_{16} HL_1 + b_{17} HL_2 + b_{18} H^2 + b_{19} L_1 + b_{20} L_2 + b_{21} H + b_{22},
\]

This structure is an example of a quadratic algebra. If we know the expansion for \(R^2\) we can compute the other structure relations.
Nondegenerate systems (2n – 1 = 3 generators)

The symmetry algebra generated by $H, L_1, L_2$ always closes under commutation. Define 3rd order commutator $R$ by $R = [L_1, L_2]$. Then

$$[L_j, R] = A_1^{(j)} L_1^2 + A_2^{(j)} L_2^2 + A_3^{(j)} H^2 + A_4^{(j)} \{L_1, L_2\} + A_5^{(j)} H L_1 + A_6^{(j)} H L_2$$

$$+ A_7^{(j)} L_1 + A_8^{(j)} L_2 + A_9^{(j)} H + A_{10}^{(j)}, \quad \{L_1, L_2\} = L_1 L_2 + L_2 L_1,$$

$$R^2 = b_1 L_1^3 + b_2 L_2^3 + b_3 H^3 + b_4 \{L_1^2, L_2\} + b_5 \{L_1, L_2^2\} + b_6 L_1 L_2 L_1 + b_7 L_2 L_1 L_2$$

$$+ b_8 H \{L_1, L_2\} + b_9 H L_1^2 + b_{10} H L_2^2 + b_{11} H^2 L_1 + b_{12} H^2 L_2 + b_{13} L_1^2 + b_{14} L_2^2 + b_{15} \{L_1, L_2\}$$

$$+ b_{16} H L_1 + b_{17} H L_2 + b_{18} H^2 + b_{19} L_1 + b_{20} L_2 + b_{21} H + b_{22},$$

This structure is an example of a quadratic algebra. If we know the expansion for $R^2$ we can compute the other structure relations.
Degenerate systems \((2n - 1 = 3)\)

There are 4 generators: one 1st order \(X\) and 3 second order \(H, L_1, L_2\).

\[
[X, L_j] = C_1^{(j)} L_1 + C_2^{(j)} L_2 + C_3^{(j)} H + C_4^{(j)} X^2 + C_5^{(j)}, \quad j = 1, 2,
\]

\[
[L_1, L_2] = E_1 \{L_1, X\} + E_2 \{L_2, X\} + E_3 HX + E_4 X^3 + E_5 X,
\]

Since \(2n - 1 = 3\) there must be an identity satisfied by the 4 generators. It is of 4th order:

\[
c_1 L_1^2 + c_2 L_2^2 + c_3 H^2 + c_4 \{L_1, L_2\} + c_5 HL_1 + c_6 HL_2 + c_7 X^4 + c_8 \{X^2, L_1\} + c_9 \{X^2, L_2\}
\]

\[
+ c_{10} HX^2 + c_{11} XL_1 X + c_{12} XL_2 X + c_{13} L_1 + c_{14} L_2 + c_{15} H + c_{16} X^2 + c_{17} = 0
\]

If we know the 4th order identity, we can compute the other structure relations to within an overall scale factor.
Stäckel Equivalence Classes

All 2nd order 2d superintegrable systems with potential are known. There are 59 types, on a variety of manifolds, but under the Stäckel transform, an invertible structure preserving mapping, they divide into 12 equivalence classes with representatives on flat space and the 2-sphere, 6 with nondegenerate 3-parameter potentials

\[ \{ S_9, E_1, E_2, E_3', E_8, E_{10} \} \]

and 6 with degenerate 1-parameter potentials

\[ \{ S_3, E_3, E_4, E_5, E_6, E_{14} \} \]
Example: S9

\[ H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2} \]

where \( J_3 = s_1 \partial s_2 - s_2 \partial s_1 \) and \( J_2, J_3 \) are obtained by cyclic permutations of indices.

Basis symmetries: \( (J_3 = s_2 \partial s_1 - s_1 \partial s_2, \ldots) \)

\[
L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2}, \quad L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_2 s_1^2}{s_2^2}, \quad L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},
\]

Structure equations:

\[
[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),
\]

\[
R^2 = \frac{8}{3} \{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 + \frac{52}{3} (\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3} (16+176a_1)L_1 + \frac{1}{3} (16+176a_2)L_2 + \frac{1}{3} (16+176a_3)L_3 + \frac{32}{3} (a_1 + a_2 + a_3) + 48(a_1 a_2 + a_2 a_3 + a_3 a_1) + 64a_1 a_2 a_3,
\]

\( R = [L_1, L_2] \).

Interesting 2 variable O.P.'s: Prorial-Karlin-MacGregor, (right triangle)
Example: $S9$

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{a_1}{s_1^2} + \frac{a_2}{s_2^2} + \frac{a_3}{s_3^2}$$

where $J_3 = s_1 \partial_{s_2} - s_2 \partial_{s_1}$ and $J_2, J_3$ are obtained by cyclic permutations of indices.

Basis symmetries: $(J_3 = s_2 \partial_{s_1} - s_1 \partial_{s_2}, \cdots)$

$$L_1 = J_1^2 + \frac{a_3 s_2^2}{s_3^2} + \frac{a_2 s_3^2}{s_2^2}, \quad L_2 = J_2^2 + \frac{a_1 s_3^2}{s_1^2} + \frac{a_3 s_1^2}{s_3^2}, \quad L_3 = J_3^2 + \frac{a_2 s_1^2}{s_2^2} + \frac{a_1 s_2^2}{s_1^2},$$

Structure equations:

$$[L_i, R] = 4\{L_i, L_k\} - 4\{L_i, L_j\} - (8 + 16a_j)L_j + (8 + 16a_k)L_k + 8(a_j - a_k),$$

$$R^2 = \frac{8}{3}\{L_1, L_2, L_3\} - (16a_1 + 12)L_1^2 - (16a_2 + 12)L_2^2 - (16a_3 + 12)L_3^2 +$$

$$\frac{52}{3}(\{L_1, L_2\} + \{L_2, L_3\} + \{L_3, L_1\}) + \frac{1}{3}(16 + 176a_1)L_1 + \frac{1}{3}(16 + 176a_2)L_2 + \frac{1}{3}(16 + 176a_3)L_3$$

$$+ \frac{32}{3}(a_1 + a_2 + a_3) + 48(a_1 a_2 + a_2 a_3 + a_3 a_1) + 64a_1 a_2 a_3, \quad R = [L_1, L_2].$$

Interesting 2 variable O.P.'s: Prorial-Karlin-MacGregor, (right triangle)
Example: \( E1 \)

\[
H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2) + \frac{b_1}{x^2} + \frac{b_2}{y^2}
\]

Generators:

\[
L_1 = \partial_x^2 - \omega^2 x^2 + \frac{b_1}{x^2}, \quad L_2 = \partial_y^2 - \omega^2 y^2 + \frac{b_2}{y^2}, \quad L_3 = (x\partial_y - y\partial_x)^2 + y^2 \frac{b_1}{x^2} + x^2 \frac{b_2}{y^2}
\]

Structure relations:

\[
[R, L_1] = 8L_1^2 - 8HL_1 - 16\omega^2L_3 + 8\omega^2,
\]

\[
[R, L_3] = 8HL_3 - 8\{L_1, L_3\} + (16b_1 + 8)H - 16(b_1 + b_2 + 1)L_1,
\]

\[
R^2 + \frac{8}{3}\{L_1, L_1, L_3\} - 8H\{L_1, L_3\} + (16b_1 + 16b_2 + \frac{176}{3})L_1^2 - 16\omega^2L_3^2 - (32b_1 + \frac{176}{3})HL_1
\]

\[
+ (16b_1 + 12)H^2 + \frac{176}{3}\omega^2L_3 + 16\omega^2(3b_1 + 3b_2 + 4b_1b_2 + \frac{2}{3}) = 0
\]
Example: $E1$

$$H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2) + \frac{b_1}{x^2} + \frac{b_2}{y^2}$$

Generators:

$$L_1 = \partial_x^2 - \omega^2 x^2 + \frac{b_1}{x^2}, \quad L_2 = \partial_y^2 - \omega^2 y^2 + \frac{b_2}{y^2}, \quad L_3 = (x\partial_y - y\partial_x)^2 + y^2 \frac{b_1}{x^2} + x^2 \frac{b_2}{y^2}$$

Structure relations:

$$[R, L_1] = 8L_1^2 - 8HL_1 - 16\omega^2 L_3 + 8\omega^2,$$

$$[R, L_3] = 8HL_3 - 8\{L_1, L_3\} + (16b_1 + 8)H - 16(b_1 + b_2 + 1)L_1,$$

$$R^2 + \frac{8}{3}\{L_1, L_1, L_3\} - 8H\{L_1, L_3\} + (16b_1 + 16b_2 + \frac{176}{3})L_1^2 - 16\omega^2 L_3^2 - (32b_1 + \frac{176}{3})HL_1$$

$$+(16b_1 + 12)H^2 + \frac{176}{3}\omega^2 L_3 + 16\omega^2(3b_1 + 3b_2 + 4b_1 b_2 + \frac{2}{3}) = 0$$
Relations between nondegenerate/degenerate systems

- Every 1-parameter potential can be obtained from some 3-parameter potential by parameter restriction.
- It is not simply a restriction, however, because the structure of the symmetry algebra changes.
- A formally skew-adjoint 1st order symmetry appears and this induces a new 2nd order symmetry.
- Thus the restricted potential has a strictly larger symmetry algebra than is initially apparent.
Relations between nondegenerate/degenerate systems

- Every 1-parameter potential can be obtained from some 3-parameter potential by parameter restriction.
- It is not simply a restriction, however, because the structure of the symmetry algebra changes.
- A formally skew-adjoint 1st order symmetry appears and this induces a new 2nd order symmetry.
- Thus the restricted potential has a strictly larger symmetry algebra than is initially apparent.
Relations between nondegenerate/degenerate systems

- Every 1-parameter potential can be obtained from some 3-parameter potential by parameter restriction.
- It is not simply a restriction, however, because the structure of the symmetry algebra changes.
- A formally skew-adjoint 1st order symmetry appears and this induces a new 2nd order symmetry.
- Thus the restricted potential has a strictly larger symmetry algebra than is initially apparent.
Relations between nondegenerate/degenerate systems

- Every 1-parameter potential can be obtained from some 3-parameter potential by parameter restriction.
- It is not simply a restriction, however, because the structure of the symmetry algebra changes.
- A formally skew-adjoint 1st order symmetry appears and this induces a new 2nd order symmetry.
- Thus the restricted potential has a strictly larger symmetry algebra than is initially apparent.
Representatives of Degenerate Systems

Example: $S3$ (Higgs oscillator)

\[ H = J_1^2 + J_2^2 + J_3^2 + \frac{a}{s_3^2} \]

The system is the same as $S9$ with $a_1 = a_2 = 0$, $a_3 = a$ with the former $L_2$ replaced by

\[ L_2 = \frac{1}{2} (J_1 J_2 + J_2 J_1) - \frac{a s_1 s_2}{s_3^2} \]

and

\[ X = J_3 = s_2 \partial_{s_3} - s_3 \partial_{s_2}. \]

Structure relations:

\[ [L_1, X] = 2L_2, \ [L_2, X] = -X^2 - 2L_1 + H - a, \ [L_1, L_2] = -(L_1 X + XL_1) - \left( \frac{1}{2} + 2a \right) X, \]

\[ \frac{1}{3} \left( X^2 L_1 + XL_1 X + L_1 X^2 \right) + L_1^2 + L_2^2 - HL_1 + (a + \frac{11}{12}) X^2 - \frac{1}{6} H + (a - \frac{2}{3}) L_1 - \frac{5a}{6} = 0. \]

Interesting 2-variable O.P.'s: Koschmieder, Zerneke, (disk)
Example: $E_3$ (Harmonic oscillator)

$$H = \partial_x^2 + \partial_y^2 - \omega^2(x^2 + y^2)$$

Basis symmetries:

$$L_1 = \partial_x^2 - \omega^2 x^2, \quad L_3 = \partial_{xy} - \omega^2 xy, \quad X = x\partial_y - y\partial_x.$$ 

Also we set $L_2 = \partial_y^2 - \omega^2 y^2 = H - L_1$.

Structure equations:

$$[L_1, X] = 2L_3, \quad [L_3, X] = H - 2L_1, \quad [L_1, L_3] = 2\omega^2 X,$$

$$L_1^2 + L_3^2 - L_1 H - \omega^2 X^2 + \omega^2 = 0$$
Theorem

There is a 1-1 relationship between quadratic algebras generated by 2nd order elements in the enveloping algebra of \( \text{so}(3, \mathbb{C}) \), called free, and 2nd order superintegrable systems on the complex 2-sphere. There is a 1-1 relationship between quadratic algebras generated by 2nd order elements in the enveloping algebra of \( \text{e}(2, \mathbb{C}) \) and 2nd order superintegrable systems on the 2D complex flat space.

Remark : This theorem is constructive. Given a free quadratic algebra \( \tilde{Q} \) one can compute the potential \( V \) and the symmetries of the quadratic algebra \( Q \).
What are models of quadratic algebras?

- A representation of a quadratic algebra $Q$ is a homomorphism of $Q$ into the associative algebra of linear operators on some vector space: products go to products, commutators to commutators, etc.
- A model $M$ is a faithful representation of $Q$ in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space.
- The dimensions of the finite dimensional irreducible representations determine the multiplicities of the quantum energy eigenspaces and the representations determine the eigenvalues.

What is the significance of models?

- They are easier to work with than the original abstract algebras.
- Eigenspaces in the models can be interpreted as expansion coefficients of one separable eigenbasis of the original quantum system with potential in terms of another eigenbasis.
- Racah and Wilson polynomials are related to S9 in this way.
What are models of quadratic algebras?

- A representation of a quadratic algebra $Q$ is a homomorphism of $Q$ into the associative algebra of linear operators on some vector space: products go to products, commutators to commutators, etc.
- A model $M$ is a faithful representation of $Q$ in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space.
- The dimensions of the finite dimensional irreducible representations determine the multiplicities of the quantum energy eigenspaces and the representations determine the eigenvalues.

What is the significance of models?

- They are easier to work with than the original abstract algebras.
- Eigenspaces in the models can be interpreted as expansion coefficients of one separable eigenbasis of the original quantum system with potential in terms of another eigenbasis.
- Racah and Wilson polynomials are related to $S9$ in this way.
What are models of quadratic algebras?

- A representation of a quadratic algebra $Q$ is a homomorphism of $Q$ into the associative algebra of linear operators on some vector space: products go to products, commutators to commutators, etc.
- A model $M$ is a faithful representation of $Q$ in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space.
- The dimensions of the finite dimensional irreducible representations determine the multiplicities of the quantum energy eigenspaces and the representations determine the eigenvalues.

What is the significance of models?

- They are easier to work with than the original abstract algebras.
- Eigenspaces in the models can be interpreted as expansion coefficients of one separable eigenbasis of the original quantum system with potential in terms of another eigenbasis.
- Racah and Wilson polynomials are related to S9 in this way.
What are models of quadratic algebras?

- A representation of a quadratic algebra $Q$ is a homomorphism of $Q$ into the associative algebra of linear operators on some vector space: products go to products, commutators to commutators, etc.
- A model $M$ is a faithful representation of $Q$ in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space.
- The dimensions of the finite dimensional irreducible representations determine the multiplicities of the quantum energy eigenspaces and the representations determine the eigenvalues.

What is the significance of models?

- They are easier to work with than the original abstract algebras.
- Eigenspaces in the models can be interpreted as expansion coefficients of one separable eigenbasis of the original quantum system with potential in terms of another eigenbasis.
- Racah and Wilson polynomials are related to S9 in this way.
What are models of quadratic algebras?

- A representation of a quadratic algebra $Q$ is a homomorphism of $Q$ into the associative algebra of linear operators on some vector space: products go to products, commutators to commutators, etc.
- A model $M$ is a faithful representation of $Q$ in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space.
- The dimensions of the finite dimensional irreducible representations determine the multiplicities of the quantum energy eigenspaces and the representations determine the eigenvalues.

What is the significance of models?

- They are easier to work with than the original abstract algebras.
- Eigenspaces in the models can be interpreted as expansion coefficients of one separable eigenbasis of the original quantum system with potential in terms of another eigenbasis.
- Racah and Wilson polynomials are related to S9 in this way.
What are models of quadratic algebras?

- A representation of a quadratic algebra $Q$ is a homomorphism of $Q$ into the associative algebra of linear operators on some vector space: products go to products, commutators to commutators, etc.
- A model $M$ is a faithful representation of $Q$ in which the vector space is a space of polynomials in one complex variable and the action is via differential/difference operators acting on that space.
- The dimensions of the finite dimensional irreducible representations determine the multiplicities of the quantum energy eigenspaces and the representations determine the eigenvalues.

What is the significance of models?

- They are easier to work with than the original abstract algebras.
- Eigenspaces in the models can be interpreted as expansion coefficients of one separable eigenbasis of the original quantum system with potential in terms of another eigenbasis.
- Racah and Wilson polynomials are related to S9 in this way.
The big picture: Special functions and contractions

Connection with special functions

Special functions arise in two distinct ways:
- As separable eigenfunctions of the quantum Hamiltonian.
- As eigenfunctions in the model. Often solutions of difference equations.

Connection with contractions

- Taking coordinate limits starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Wigner-type Lie algebra contractions of $o(3,\mathbb{C})$ and $e(2,\mathbb{C})$.
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture. (Kalnins-Miller-Post, 2014)
Models of Superintegrable Systems

The big picture: Special functions and contractions

Connection with special functions

Special functions arise in two distinct ways:

- As separable eigenfunctions of the quantum Hamiltonian.
- As eigenfunctions in the model. Often solutions of difference equations.

Connection with contractions

- Taking coordinate limits starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Wigner-type Lie algebra contractions of $o(3, \mathbb{C})$ and $e(2, \mathbb{C})$.
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture. (Kalnins-Miller-Post, 2014)
The big picture: Special functions and contractions

Connection with special functions

Special functions arise in two distinct ways:
- As separable eigenfunctions of the quantum Hamiltonian.
- As eigenfunctions in the model. Often solutions of difference equations.

Connection with contractions

- Taking coordinate limits starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Wigner-type Lie algebra contractions of $o(3,\mathbb{C})$ and $e(2,\mathbb{C})$.
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture. (Kalnins-Miller-Post, 2014)
The big picture: Special functions and contractions

Connection with special functions

Special functions arise in two distinct ways:
- As separable eigenfunctions of the quantum Hamiltonian.
- As eigenfunctions in the model. Often solutions of difference equations.

Connection with contractions

- Taking coordinate limits starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Wigner-type Lie algebra contractions of $o(3, \mathbb{C})$ and $e(2, \mathbb{C})$.
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture. (Kalnins-Miller-Post, 2014)
The big picture: Special functions and contractions

Connection with special functions

Special functions arise in two distinct ways:
- As separable eigenfunctions of the quantum Hamiltonian.
- As eigenfunctions in the model. Often solutions of difference equations.

Connection with contractions

- Taking coordinate limits starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Wigner-type Lie algebra contractions of $o(3, \mathbb{C})$ and $e(2, \mathbb{C})$.
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture. (Kalnins-Miller-Post, 2014)
The big picture: Special functions and contractions

Connection with special functions

Special functions arise in two distinct ways:
- As separable eigenfunctions of the quantum Hamiltonian.
- As eigenfunctions in the model. Often solutions of difference equations.

Connection with contractions

- Taking coordinate limits starting from quantum system S9 we can obtain other superintegrable systems.
- These limits induce limit relations between the special functions associated with the superintegrable systems.
- The limits induce contractions of the associated quadratic algebras, and via the models, limit relations between the associated special functions.
- For constant curvature systems the required limits are all induced by Wigner-type Lie algebra contractions of $o(3, \mathbb{C})$ and $e(2, \mathbb{C})$.
- The Askey scheme for orthogonal functions of hypergeometric type fits nicely into this picture. (Kalnins-Miller-Post, 2014)
Model interplay

The Contraction of Models
The S9 difference operator model. 1

\[ L_2 f_{n,m} = (-4t^2 - \frac{1}{2} + B_1^2 + B_3^2)f_{n,m}, \]

\[ L_3 f_{n,m} = (-4\tau^* \tau - 2[B_1 + 1][B_2 + 1] + \frac{1}{2})f_{n,m}, \]

\[ H = L_1 + L_2 + L_3 + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2) = -4(m + 1)(B_1 + B_2 + B_3 + m + 1) - 2(B_1 B_2 + B_1 B_3 + B_2 B_3) + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2). \]

Here \( n = 0, 1, \ldots, m \) if \( m \) is a nonnegative integer and \( n = 0, 1, \ldots \) otherwise. Also

\[ a_j = \frac{1}{4} - B_j^2, \quad \alpha = -(B_1 + B_3 + 1)/2 - m, \quad \beta = (B_1 + B_3 + 1)/2, \quad \gamma = (B_1 - B_3 + 1)/2, \quad \delta = (B_1 + B_3 - 1)/2 + B_2 + m + 2, \]

\[ E^A F(t) = F(t + A), \quad \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}), \quad \tau^* = \frac{1}{2t} \left[ (\alpha + t)(\beta + t)(\gamma + t)(\delta + t)E^{1/2} - (\alpha - t)(\beta - t)(\gamma - t)(\delta - t)E^{-1/2} \right], \]

\[ w_n(t^2) = (\alpha + \beta)n(\alpha + \gamma)n(\alpha + \delta)n_4F_3\left(\begin{array}{c} -n, \alpha + \beta, \alpha + \gamma, \alpha + \delta + n - 1 \\alpha + \beta, \alpha + \gamma, \alpha + \delta \end{array} \right). \]

\[ = (\alpha + \beta)n(\alpha + \gamma)n(\alpha + \delta)n_4F_3\left(\begin{array}{c} \alpha + \beta, \alpha + \gamma, \alpha + \delta \end{array} \right)(t^2), \quad \Phi_n \equiv f_{n,m}, \]

\[ \tau^* \tau \Phi_n = n(n + \alpha + \beta + \gamma + \delta - 1)\Phi_n, \]

where \((a)_n\) is the Pochhammer symbol and \(_4F_3(1)\) is a hypergeometric function of unit argument. The polynomial \(w_n(t^2)\) is symmetric in \(\alpha, \beta, \gamma, \delta\). For the finite dimensional representations the spectrum of \(t^2\) is \(\{ (\alpha + k)^2, \ k = 0, 1, \ldots, m \}\) and the orthogonal basis eigenfunctions are Racah polynomials. In the infinite dimensional case they are Wilson polynomials.
The S9 difference operator model.

\[ L_2 f_{n,m} = (-4t^2 - \frac{1}{2} + B_1^2 + B_3^2) f_{n,m}, \]

\[ L_3 f_{n,m} = (-4\tau^* \tau - 2[B_1 + 1][B_2 + 1] + \frac{1}{2}) f_{n,m}, \]

\[ H = L_1 + L_2 + L_3 + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2) = -4(m + 1)(B_1 + B_2 + B_3 + m + 1) - 2(B_1B_2 + B_1B_3 + B_2B_3) + \frac{3}{4} - (B_1^2 + B_2^2 + B_3^2). \]

Here \( n = 0, 1, \ldots, m \) if \( m \) is a nonnegative integer and \( n = 0, 1, \ldots \) otherwise. Also

\[ a_j = \frac{1}{4} - B_j^2, \quad \alpha = -(B_1 + B_3 + 1)/2 - m, \quad \beta = (B_1 + B_3 + 1)/2, \quad \gamma = (B_1 - B_3 + 1)/2, \quad \delta = (B_1 + B_3 - 1)/2 + B_2 + m + 2, \]

\[ E^A F(t) = F(t + A), \quad \tau = \frac{1}{2t}(E^{1/2} - E^{-1/2}), \quad \tau^* = \frac{1}{2t} \left[ (\alpha + t)(\beta + t)(\gamma + t)(\delta + t)E^{1/2} - (\alpha - t)(\beta - t)(\gamma - t)(\delta - t)E^{-1/2} \right], \]

\[ w_n(t^2) = (\alpha + \beta)n(\alpha + \gamma)n(\alpha + \delta)n_4 F_3 \left( \begin{array}{c} -n, \alpha + \beta, \alpha + \gamma, \alpha + \delta + n - 1; \\ \alpha + \gamma \end{array}; \alpha + \delta, \alpha + t \right) \]

\[ = (\alpha + \beta)n(\alpha + \gamma)n(\alpha + \delta)n\Phi_n(\alpha, \beta, \gamma, \delta; t^2), \quad \Phi_n \equiv f_{n,m}, \]

\[ \tau^* \Phi_n = n(n + \alpha + \beta + \gamma + \delta - 1)\Phi_n, \]

where \( (a)_n \) is the Pochhammer symbol and \( _4F_3(1) \) is a hypergeometric function of unit argument. The polynomial \( w_n(t^2) \) is symmetric in \( \alpha, \beta, \gamma, \delta \). For the finite dimensional representations the spectrum of \( t^2 \) is \( \{(\alpha + k)^2, \ k = 0, 1, \ldots, m \} \) and the orthogonal basis eigenfunctions are Racah polynomials. In the infinite dimensional case they are Wilson polynomials.

W. Miller (University of Minnesota)
The S9 difference operator model.

The action of $L_2$ and $L_3$ on an $L_3$ eigenbasis is

\[ L_2 f_{n,m} = -4K(n+1,n) f_{n+1,m} - 4K(n,n) f_{n,m} - 4K(n-1,n) f_{n-1,m} + (B_1^2 + B_3^2 - \frac{1}{2}) f_{n,m}, \]

\[ L_3 f_{n,m} = -(4n^2 + 4n[B_1 + B_2 + 1] + 2[B_1 + 1][B_2 + 1] - \frac{1}{2}) f_{n,m}, \]

\[ K(n+1,n) = \frac{(B_1 + B_2 + n + 1)(n - m)(-B_3 - m + n)(B_2 + n + 1)}{(B_1 + B_2 + 2n + 1)(B_1 + B_2 + 2n + 2)}, \]

\[ K(n-1,n) = \frac{n(B_1 + n)(B_1 + B_2 + B_3 + m + n + 1)(B_1 + B_2 + m + n + 1)}{(B_1 + B_2 + 2n)(B_1 + B_2 + 2n + 1)}, \]

\[ K(n,n) = \left[ \frac{B_1 + B_2 + 2m + 1}{2} \right]^2 - K(n+1,n) - K(n-1,n), \]
Lie algebra contractions

Let \((A; [,]_A), (B; [,]_B)\) be two complex Lie algebras. We say \(B\) is a contraction of \(A\) if for every \(\epsilon \in (0; 1]\) there exists a linear invertible map \(t_\epsilon : B \rightarrow A\) such that for every \(X, Y \in B\),

\[
\lim_{\epsilon \to 0} t_\epsilon^{-1} [t_\epsilon X, t_\epsilon Y]_A = [X, Y]_B.
\]

Thus, as \(\epsilon \to 0\) the 1-parameter family of basis transformations can become nonsingular but the structure constants go to a finite limit.
Contractions of $e(2, \mathbb{C})$ and $o(3, \mathbb{C})$

These are the symmetry algebras of free systems on constant curvature spaces. Their contractions have long since been classified. There are 6 nontrivial contractions of $e(2, \mathbb{C})$ and 4 of $o(3, \mathbb{C})$.

Example: An Inönü-Wigner-contraction of $o(3, \mathbb{C})$. We use the classical realization for $o(3, \mathbb{C})$ acting on the 2-sphere, with basis $J_1 = s_2 p_3 - s_3 p_2$, $J_2 = s_3 p_1 - s_1 p_3$, $J_3 = s_1 p_2 - s_2 p_1$, commutation relations $[J_2, J_1] = J_3$, $[J_3, J_2] = J_1$, $[J_1, J_3] = J_2$, and Hamiltonian $H = J_1^2 + J_2^2 + J_3^2$. Here $s_1^2 + s_2^2 + s_3^2 = 1$.

**Basis change:** \{\(J'_1, J'_2, J'_3\}\} = \{\(\epsilon J_1, \epsilon J_2, J_3\}\}, \(0 < \epsilon \leq 1\)

coordinate implementation \(x = \frac{s_1}{\epsilon}, y = \frac{s_2}{\epsilon}, s_3 \approx 1, J = J_3\)


Let \(\epsilon \rightarrow 0\): \([J'_2, J'_1] = 0, [J'_3, J'_2] = J'_1, [J'_1, J'_3] = J'_2,\) get $e(2, \mathbb{C})$. 
Contractions of nondegenerate systems. 1

Suppose we have a nondegenerate superintegrable system with generators $H, L_1, L_2, R = [L_1, L_2]$ and the usual structure equations, defining a quadratic algebra $Q$. If we make a change of basis to new generators $\tilde{H}, \tilde{L}_1, \tilde{L}_2$ and parameters $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ such that

$$
\begin{pmatrix}
\tilde{L}_1 \\
\tilde{L}_2 \\
\tilde{H}
\end{pmatrix} = 
\begin{pmatrix}
A_{1,1} & A_{1,2} & A_{1,3} \\
A_{2,1} & A_{2,2} & A_{2,3} \\
0 & 0 & A_{3,3}
\end{pmatrix} 
\begin{pmatrix}
L_1 \\
L_2 \\
H
\end{pmatrix} + 
\begin{pmatrix}
B_{1,1} & B_{1,2} & B_{1,3} \\
B_{2,1} & B_{2,2} & B_{2,3} \\
B_{3,1} & B_{3,2} & B_{3,3}
\end{pmatrix} 
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix},
$$

$$
\begin{pmatrix}
\tilde{a}_1 \\
\tilde{a}_2 \\
\tilde{a}_3
\end{pmatrix} = 
\begin{pmatrix}
C_{1,1} & C_{1,2} & C_{1,3} \\
C_{2,1} & C_{2,2} & C_{2,3} \\
C_{3,1} & C_{3,2} & C_{3,3}
\end{pmatrix} 
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
$$

for some $3 \times 3$ constant matrices $A = (A_{i,j}), B, C$ such that $\det A \cdot \det C \neq 0$, we will have the same system with new structure equations of the same form for $\tilde{R} = [\tilde{L}_1, \tilde{L}_2], [\tilde{L}_j, \tilde{R}], \tilde{R}^2$, but with transformed structure constants.
Choose a continuous 1-parameter family of basis transformation matrices $A(\epsilon), B(\epsilon), C(\epsilon)$, $0 < \epsilon \leq 1$ such that $A(1) = C(1)$ is the identity matrix, $B(1) = 0$ and $\det A(\epsilon) \neq 0$, $\det C(\epsilon) \neq 0$.

Now suppose as $\epsilon \to 0$ the basis change becomes singular, (i.e., the limits of $A, B, C$ either do not exist or, if they exist do not satisfy $\det A(0) \det C(0) \neq 0$) but the structure equations involving $A(\epsilon), B(\epsilon), C(\epsilon)$, go to a limit, defining a new quadratic algebra $Q'$.

We call $Q'$ a contraction of $Q$ in analogy with Lie algebra contractions.

There is a similar definition of a contraction of a degenerate superintegrable system.
Choose a continuous 1-parameter family of basis transformation matrices $A(\epsilon), B(\epsilon), C(\epsilon), 0 < \epsilon \leq 1$ such that $A(1) = C(1)$ is the identity matrix, $B(1) = 0$ and $\det A(\epsilon) \neq 0$, $\det C(\epsilon) \neq 0$.

Now suppose as $\epsilon \to 0$ the basis change becomes singular, (i.e., the limits of $A, B, C$ either do not exist or, if they exist do not satisfy $\det A(0) \det C(0) \neq 0$) but the structure equations involving $A(\epsilon), B(\epsilon), C(\epsilon)$, go to a limit, defining a new quadratic algebra $Q'$.

We call $Q'$ a contraction of $Q$ in analogy with Lie algebra contractions.

There is a similar definition of a contraction of a degenerate superintegrable system.
Choose a continuous 1-parameter family of basis transformation matrices $A(\epsilon), B(\epsilon), C(\epsilon), 0 < \epsilon \leq 1$ such that $A(1) = C(1)$ is the identity matrix, $B(1) = 0$ and $\det A(\epsilon) \neq 0, \det C(\epsilon) \neq 0$.

Now suppose as $\epsilon \to 0$ the basis change becomes singular, (i.e., the limits of $A, B, C$ either do not exist or, if they exist do not satisfy $\det A(0) \det C(0) \neq 0$) but the structure equations involving $A(\epsilon), B(\epsilon), C(\epsilon), \epsilon$, go to a limit, defining a new quadratic algebra $Q'$.

We call $Q'$ a contraction of $Q$ in analogy with Lie algebra contractions.

There is a similar definition of a contraction of a degenerate superintegrable system.
Lie algebra contractions $\Rightarrow$ quadratic algebra contractions

Constant curvature spaces:

Theorem

(Kalnins-Miller, 2014) Every Lie algebra contraction of $A = e(2, \mathbb{C})$ or $A = o(3, \mathbb{C})$ induces uniquely a contraction of a free quadratic algebra $\tilde{Q}$ based on $A$, which in turn induces uniquely a contraction of the quadratic algebra $Q$ with potential. This is true for both classical and quantum algebras.

A complication: **Darboux spaces:** The Lie symmetry algebra is only 1-dimensional so Wigner contractions don’t apply and geometrical free quadratic algebra contractions must be determined on a case-by-case basis.
Figure: The Askey scheme and contractions of superintegrable systems

Askey Scheme of Hypergeometric Orthogonal Polynomials

Partial list of contractions of superintegrable systems
Figure: The Askey contraction scheme
Example of contraction hierarchy, S=sphere, E=flat space, D=Darboux space
Systems of Laplace type are of the form

\[ H\psi \equiv \Delta_n \psi + V\psi = 0. \]

Here \( \Delta_n \) is the Laplace-Beltrami operator on a conformally flat \( n \)D Riemannian or pseudo-Riemannian manifold.

A conformal symmetry of this equation is a partial differential operator \( S \) in the variables \( \mathbf{x} = (x_1, \cdots, x_n) \) such that \( [S, H] \equiv SH - HS = R_S H \) for some differential operator \( R_S \).

The system is **conformally superintegrable** for \( n > 2 \) if there are \( 2n - 1 \) functionally independent conformal symmetries, \( S_1, \cdots, S_{2n-1} \) with \( S_1 = H \). It is second order conformally superintegrable if each symmetry \( S_i \) can be chosen to be a differential operator of at most second order.
The Bôcher approach

In his 1894 thesis Bôcher developed a geometrical method for finding and classifying the R-separable orthogonal coordinate systems for the flat space Laplace equation $\Delta_n \psi = 0$ in $n$ dimensions. It was based on the conformal symmetry of these equations. The conformal symmetry algebra in the complex case is $so(n+2, \mathbb{C})$. We will use his ideas for $n = 2$, but applied to the Laplace equation with potential

$$H\psi \equiv (\partial_x^2 + \partial_y^2 + V)\psi = 0.$$ 

The $so(4, \mathbb{C})$ conformal symmetry algebra in the case $n = 2$ has the basis

$$P_1 = \partial_x, \quad P_2 = \partial_y, \quad J = x\partial_y - y\partial_x, \quad D = x\partial_x + y\partial_y,$$

$$K_1 = (x^2 - y^2)\partial_x + 2xy\partial_y, \quad K_2 = (y^2 - x^2)\partial_y + 2xy\partial_x.$$

Bôcher linearizes this action by introducing tetrasspherical coordinates. These are 4 projective coordinates $(x_1, x_2, x_3, x_4)$ confined to the nullcone $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$. 
Tetrasspherical coordinates

They are complex projective coordinates on the null cone

\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0. \]

Relation to Cartesian coordinates \((x, y)\):

\[
\begin{align*}
    x &= -\frac{x_1}{x_3 + ix_4}, \\
    y &= -\frac{x_2}{x_3 + ix_4},
\end{align*}
\]

\[
H = \partial_{xx} + \partial_{yy} + \tilde{V} = (x_3 + ix_4)^2 \left( \sum_{k=1}^{4} \partial_{x_k}^2 + V \right)
\]

where \(\tilde{V} = (x_3 + ix_4)^2 V\).
Relation to flat space 1st order conformal symmetries

We define

\[ L_{jk} = x_j \partial_{x_k} - x_k \partial_{x_j}, \quad 1 \leq j, k \leq 4, \quad j \neq k, \]

where \( L_{jk} = -L_{kj} \). The generators for flat space conformal symmetries are related to these via

\[
\begin{align*}
P_1 &= \partial_x = L_{13} + iL_{14}, \\
P_2 &= \partial_y = L_{23} + iL_{24}, \\
D &= iL_{34}, \\
J &= L_{12}, \\
K_1 &= L_{13} - iL_{14}, \\
K_2 &= L_{23} - iL_{24}.
\end{align*}
\]

Here

\[
\begin{align*}
D &= x \partial_x + y \partial_y, \\
J &= x \partial_y - y \partial_x, \\
K_1 &= 2x D - (x^2 + y^2) \partial_x,
\end{align*}
\]

etc.
Bôcher uses symbols of the form \([n_1, n_2, \ldots, n_p]\) where \(n_1 + \ldots + n_p = 4\), to define coordinate surfaces as follows. Consider the quadratic forms

\[
\Omega = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \quad \Phi = \frac{x_1^2}{\lambda - e_1} + \frac{x_2^2}{\lambda - e_2} + \frac{x_3^2}{\lambda - e_3} + \frac{x_4^2}{\lambda - e_4}.
\]

If the parameters \(e_j\) are pairwise distinct, the elementary divisors of these two forms are denoted by \([1, 1, 1, 1]\).

Given a point in 2D flat space with Cartesian coordinates \((x_0, y_0, z_0)\), there corresponds a set of tetrasspherical coordinates \((x_0^1, x_0^2, x_0^3, x_0^4)\), unique up to multiplication by a nonzero constant. If we substitute into \(\Phi\) we see that there are exactly 2 roots \(\lambda = \rho, \mu\) such that \(\Phi = 0\). These are elliptic coordinates. They are orthogonal with respect to the metric \(ds^2 = dx^2 + dy^2\) and are \(R\)-separable for the Laplace equations \((\partial_x^2 + \partial_y^2)\Theta = 0\) or \((\sum_{j=1}^4 \partial_{x_j}^2)\Theta = 0\).
The potential $V_{[1,1,1,1]}$

Consider the potential

$$V_{[1,1,1,1]} = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2}.$$ 

It is the only potential $V$ such that equation

$$(\sum_{j=1}^{4} \partial_{x_j}^2 + V)\Theta = 0$$

is $R$-separable in elliptic coordinates for all choices of the parameters $e_j$. The separation is characterized by 2nd order conformal symmetry operators that are linear in the parameters $e_j$. In particular the symmetries span a 3-dimensional subspace of symmetries, so the system

$$H\Theta = (\sum_{j=1}^{4} \partial_{x_j}^2 + V_{[1,1,1,1]})\Theta = 0$$

must be conformally superintegrable.
If some of the $e_i$ become equal then Bôcher shows that the process of making $e_1 \rightarrow e_2$ together with suitable transformations of the $a'_i$s produces a conformally equivalent $H$. This corresponds to the choice of coordinate curves obtained by the Bôcher limiting process

$$e_1 = e_2 + \epsilon^2, \quad x_1 \rightarrow \frac{iy_1}{\epsilon}, \quad x_2 \rightarrow \frac{y_1}{\epsilon} + \epsilon y_2, \quad x_j \rightarrow y_j, \ j = 3, 4,$$

which results in the pair of quadratic forms

$$\Omega = 2y_1 y_2 + y_3^2 + y_4^2 = 0, \quad \Phi = \frac{y_1^2}{(\lambda - e_2)^2} + \frac{2y_1 y_2}{(\lambda - e_2)} + \frac{y_3^2}{(\lambda - e_3)} + \frac{y_4^2}{(\lambda - e_4)} = 0.$$
The Bôcher Method

Contraction $[1, 1, 1, 1] \rightarrow [2, 1, 1], 2$

The coordinate curves with $e_4 = \infty$ correspond to cyclides with elementary divisors $[2, 1, \infty]$, i.e.

$$\Phi = \frac{y_1^2}{(\lambda - e_2)^2} + \frac{2y_1y_2}{(\lambda - e_2)} + \frac{y_3^2}{(\lambda - e_3)} = 0.$$

The $\lambda$ roots of this form yield planar elliptic coordinates. In order to identify "Cartesian" coordinates on the cone we can choose

$$y_1 = \frac{1}{\sqrt{2}}(x_1 + ix_2), \quad y_2 = \frac{i}{\sqrt{2}}(x_1' - ix_2'), \quad y_3 = x_3, \quad y_4 = x_4.$$
Contraction \([1, 1, 1, 1] \rightarrow [2, 1, 1], 3\)

The composite linear coordinate mapping

\[
x_1 + ix_2 = \frac{i\sqrt{2}}{\epsilon} (x'_1 + ix'_2) + \frac{i\epsilon}{\sqrt{2}} (x'_1 - ix'_2), \quad x_1 - ix_2 = -\frac{i\epsilon}{\sqrt{2}} (x'_1 - ix'_2), \quad x_3 = x'_3, \quad x_4 = x'_4,
\]
satisfies \(\lim_{\epsilon \to 0} \sum_{j=1}^{4} x_j^2 = \sum_{j=1}^{4} x'_j^2 = 0\), and induces a contraction of \(so(4, \mathbb{C})\) to itself:

\[
L'_{12} = L_{12}, \quad L'_{13} = -\frac{i}{\sqrt{2}} (L_{13} - iL_{23}) - \frac{i\epsilon}{\sqrt{2}} L_{13}, \quad L'_{23} = -\frac{i}{\sqrt{2}} (L_{13} - iL_{23}) - \frac{\epsilon}{\sqrt{2}} L_{13}
\]

\[
L'_{34} = L_{34}, \quad L'_{14} = -\frac{i}{\sqrt{2}} (L_{14} - iL_{24}) - \frac{i\epsilon}{\sqrt{2}} L_{14}, \quad L'_{24} = -\frac{i}{\sqrt{2}} (L_{14} - iL_{24}) - \frac{\epsilon}{\sqrt{2}} L_{14}.
\]

This is the Bôcher contraction \([1, 1, 1, 1] \rightarrow [2, 1, 1]\).
Bôcher contractions

This is family of contractions of $so(4, \mathbb{C})$ to itself that we call Bôcher contractions. All these contractions are implemented via coordinate transformations.
Every 2D Riemannian manifold is conformally flat, so we can always find a Cartesian-like coordinate system with coordinates \((x, y) \equiv (x_1, x_2)\) such that the Laplace equation takes the form

\[
(*) \quad \tilde{H} = \frac{1}{\lambda(x, y)} \left( \partial^2_x + \partial^2_y \right) + \tilde{V}(x) = 0.
\]

However, this equation is equivalent to the flat space equation

\[
(**) \quad H \equiv \partial^2_x + \partial^2_y + V(x) = 0, \quad V(x) = \lambda(x) \tilde{V}(x).
\]

In particular, the conformal symmetries of (*) are identical with the conformal symmetries of (**) . Thus without loss of generality we can assume the manifold is flat space with \(\lambda \equiv 1\).
The conformal Stäckel transform

Suppose we have a second order conformal superintegrable system

\[ H = \frac{1}{\lambda(x, y)} (\partial_{xx} + \partial_{yy}) + V(x, y) = 0, \quad H = H_0 + V \]

where \( V(x, y) = a_1 W(x, y) + a_2 U(x, y) \) for arbitrary parameters \( a_1, a_2 \).

**Theorem:** The transformed (Helmholtz) system

\[ \tilde{H} = E, \quad \tilde{H} = \frac{1}{\tilde{\lambda}} (\partial_{xx} + \partial_{yy}) + \tilde{V} \]

with potential \( \tilde{V}(x, y) \) is truly superintegrable, where

\[ \tilde{\lambda} = \lambda U, \quad \tilde{V} = \frac{V}{U}. \]
Thus any second order conformal Laplace superintegrable system admitting a nonconstant potential $U$ can be Stäckel transformed to a Helmholtz superintegrable system.

By choosing all possible special potentials $U$ associated with the fixed Laplace system we generate the equivalence class of all Helmholtz superintegrable systems obtainable through this process.

**Theorem:** There is a one-to-one relationship between flat space conformally superintegrable Laplace systems with nondegenerate potential and Stäckel equivalence classes of superintegrable Helmholtz systems with nondegenerate potential.
The 6 nondegenerate conformally superintegrable systems

Here, systems are \( \sum_{j=1}^{4} \partial_{x_j}^2 + V \).

\[ V_{[1,1,1,1]} = \sum_{j=1}^{4} \frac{a_j}{x_j^2}, \]

\[ V_{[2,1,1]} = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3(x_3 - ix_4)}{(x_3 + ix_4)^3} + \frac{a_4}{(x_3 + ix_4)^2}, \]

\[ V_{[2,2]} = \frac{a_1}{(x_1 + ix_2)^2} + \frac{a_2(x_1 - ix_2)}{(x_1 + ix_2)^3} + \frac{a_3}{(x_3 + ix_4)^2} + \frac{a_4(x_3 - ix_4)}{(x_3 + ix_4)^3}, \]

\[ V_{[3,1]} = \frac{a_1}{(x_3 + ix_4)^2} + \frac{a_2x_1}{(x_3 + ix_4)^3} + \frac{a_3(4x_1^2 + x_2^2)}{(x_3 + ix_4)^4} + \frac{a_4}{x_2^2}, \]

\[ V_{[4]} = \frac{a_1}{(x_3 + ix_4)^2} + a_2 \frac{x_1 + ix_2}{(x_3 + ix_4)^3} + a_3 \frac{3(x_1 + ix_2)^2 - 2(x_3 + ix_4)(x_1 - ix_2)}{(x_3 + ix_4)^4}, \]

\[ V_{[0]} = \frac{a_1}{(x_3 + ix_4)^2} + \frac{a_2x_1 + a_3x_2}{(x_3 + ix_4)^3} + a_4 \frac{x_1^2 + x_2^2}{(x_3 + ix_4)^4}. \]
Helmholtz contractions from Bôcher contractions, 1

We describe how Bôcher contractions of conformal superintegrable systems induce contractions of Helmholtz superintegrable systems.

The basic idea is that the procedure of taking a conformal Stäckel transform of a conformal superintegrable system, followed by a Helmholtz contraction yields the same result as taking a Bôcher contraction followed by an ordinary Stäckel transform: The diagrams commute.

Example: Consider the conformal Stäckel transforms of system $[1,1,1,1]$ with potential $V_{[1,1,1,1]}$. Let $H$ be the initial Hamiltonian. In terms of tetraspherical coordinates the conformal Stäckel transformed potential will take the form

$$V = \frac{a_1}{x_1^2} + \frac{a_2}{x_2^2} + \frac{a_3}{x_3^2} + \frac{a_4}{x_4^2} = \frac{V_{[1,1,1,1]}}{F(x, A)} = \frac{A_1}{x_1^2} + \frac{A_2}{x_2^2} + \frac{A_3}{x_3^2} + \frac{A_4}{x_4^2},$$

and the transformed Hamiltonian will be

$$\hat{H} = \frac{1}{F(x, A)} H,$$

where the transform is determined by the fixed vector $(A_1, A_2, A_3, A_4)$. 

W. Miller (University of Minnesota)
Helmholtz contractions from Bôcher contractions, 2

We apply the Bôcher contraction \([1, 1, 1, 1] \rightarrow [2, 1, 1]\) to this system. In the limit as \(\epsilon \rightarrow 0\) the potential \(V_{[1,1,1,1]} \rightarrow V_{[2,1,1]}\), (??), and \(H \rightarrow H'\) the \([2, 1, 1]\) system. Now consider

\[
F(x(\epsilon), A) = V'(x', A)\epsilon^\alpha + O(\epsilon^{\alpha+1}),
\]

where the the integer exponent \(\alpha\) depends upon our choice of \(A\). We will provide the theory to show that the system defined by Hamiltonian

\[
\hat{H}' = \lim_{\epsilon \rightarrow 0} \epsilon^\alpha \hat{H}(\epsilon) = \frac{1}{V'(x', A)} H'
\]

is a superintegrable system that arises from the system \([2, 1, 1]\) by a conformal Stäckel transform induced by the potential \(V'(x', A)\). Thus the Helmholtz superintegrable system with potential \(V = V_{1,1,1,1}/F\) contracts to the Helmholtz superintegrable system with potential \(V_{[2,1,1]}/V'\). The contraction is induced by a generalized Inönü-Wigner Lie algebra contraction of the conformal algebra \(so(4, \mathbb{C})\).
**Figure**: Relationship between conformal Stäckel transforms and Bôcher contractions
Free quadratic algebras uniquely determine associated superintegrable systems with potential.

A contraction of a free quadratic algebra to another uniquely determines a contraction of the associated superintegrable systems.

For a 2D superintegrable systems on a constant curvature space these contractions can be induced by Lie algebra contractions of the underlying Lie symmetry algebra.

For all 2D nondegenerate superintegrable systems these contractions are induced by Bôcher contractions of the conformal algebra to itself.

Every 2D superintegrable system is obtained either as a sequence of contractions from $S_9$ or is Stäckel equivalent to a system that is so obtained.
Taking contractions step-by-step from the S9 model we can recover the Askey Scheme. However, the contraction method is more general. It applies to all special functions that arise from the quantum systems via separation of variables, not just polynomials of hypergeometric type, and it extends to higher dimensions.

The special functions arising from the models can be described as the coefficients in the expansion of one separable eigenbasis for the original quantum system in terms of another separable eigenbasis.

The functions in the Askey Scheme are just those hypergeometric polynomials that arise as the expansion coefficients relating two separable eigenbases that are both of hypergeometric type. Thus, there are some contractions which do not fit in the Askey scheme since the physical system fails to have such a pair of separable eigenbases.
Even though 2nd order 2D nondegenerate superintegrable systems admit no group symmetry, their structure is determined completely by the underlying conformal symmetry of flat space Laplace equations $(\Delta + V)\psi = 0$. 