Exercise 1 Find the Fourier series of the function \( f \) on \([-\pi, \pi]\) given by

\[
f(t) = \begin{cases} 
1, & \text{for } -1 \leq t < 1 \\
0, & \text{otherwise.}
\end{cases}
\]

Show that none of the even terms in the series are zero at \( t = \pi/2 \), although \( f(\pi/2) = 0 \), and \( f \) vanishes in a neighborhood of this point. Does this contradict the localization theorem?

Exercise 2 Expand \( f(t) = |\sin t| \) in a Fourier series on the interval \(-\pi \leq t \leq \pi\). Plot both \( f \) and the partial sums

\[
S_k(t) = \frac{a_0}{2} + \sum_{n=0}^{k} (a_n \cos nt + b_n \sin nt)
\]

for \( k = 1, 2, 5, 7 \). Observe how the partial sums approximate \( f \).

Exercise 3 Let

\[
\Pi(t) = \begin{cases} 
1, & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\
0, & \text{otherwise}
\end{cases}
\]

be the box function on the real line. We will often have the occasion to express the Fourier transform of \( f(at + b) \) in terms of the Fourier transform \( \hat{f}(\lambda) \) of \( f(t) \), where \( a, b \) are real parameters. This exercise will give you practice in correct application of the transform.

1. Sketch the graphs of \( \Pi(t) \), \( \Pi(t + 4) \), and \( \Pi(2t + 4) = \Pi(2(t + 2)) \).
2. Sketch the graphs of \( \Pi(t) \), \( \Pi(2t) \), and \( \Pi(2(t + 4)) \). Note: In the first part a left \( \frac{1}{2} \)-translate is followed by a \( 2 \)-dilate; but in the second part a \( 2 \)-dilate is followed by a left \( \frac{1}{2} \)-translate. The results are not the same.

3. Find the Fourier transforms of \( g_1(t) = \Pi(2(t + 2)) \) and \( g_2(t) = \Pi(2(t + 4)) \) from parts 1 and 2.

4. Set \( g(t) = \Pi(2t) \) and check your answers to part 3 by applying the translation rule to

\[
g_1(t) = g(t + 2), \quad g_2(t) = g(t + 4), \quad \text{noting } g_2(t) = g_1(t + 2).
\]

**Exercise 4** Let \( f \) be the function on \( L^2(R) \) defined by

\[
f(t) = \begin{cases} \cos(5t) & \text{for } -\pi \leq t \leq \pi \\ 0 & \text{otherwise} \end{cases}
\]

a. Use the addition formulas for the cosine to show that

\[
\int_{-\pi}^{\pi} \cos(mt) \cos(\lambda t) \, dt = -2(-1)^m \frac{\lambda \sin(\pi \lambda)}{m^2 - \lambda^2}
\]

for \( m \) an integer and \( \lambda \neq m \).

b. Show that

\[
\hat{f}(\lambda) = -2\frac{\lambda \sin(\lambda \pi)}{\lambda^2 - 25}
\]

Determine the decay rate of the transform as \( \lambda \to \infty \).

c. Write down the Plancherel formula for this function.

**Exercise 5** Let \( g \) be the function on \( L^2(R) \) defined by

\[
g(t) = \begin{cases} \sin(5t) & \text{for } -\pi \leq t \leq \pi \\ 0 & \text{otherwise} \end{cases}
\]

a. Compute the transform function \( \hat{g}(\lambda) \). Determine the decay rate of the transform as \( \lambda \to \infty \). Can you account for the difference in the decay rate for this problem as opposed to the rate in the last problem?
Exercise 6 (Haar wavelets on $[0,1]$) Let $\phi(t)$ be the Haar scaling function

$$\phi(t) = \begin{cases} 
1 & \text{if } 0 \leq t < 1 \\
0 & \text{otherwise.}
\end{cases}$$

and

$$\psi(t) = \phi(2t) - \phi(2t - 1)$$

be the Haar wavelet. Show directly that the system

$$\{\phi(t), \ 2^{n/2} \psi(2^n t - m); \ n = 0, 1, 2, \cdots; \ m = 0, 1, \cdots, 2^n - 1\}$$

is an ON set in $L^2[0,1]$. (In fact, it is an ON basis for $L^2[0,1]$.)

Exercise 7 Let $\phi(t)$ be the Haar scaling function defined above. Show that the set

$$\phi_{m,n}(t) = \exp(2\pi i m t) \phi(t - n), \ m, n = 0, \pm 1, \pm 2, \cdots$$

is an ON basis for $L^2(\mathbb{R})$. This basis was proposed by D. Gabor in 1946 for use in communications theory.

Exercise 8 (Another Haar-type set) Let $n$ be a fixed positive integer and $h_k(t) = \sqrt{n} \phi(nt - k), \ k = 0, 1, \cdots, n - 1$. These are just rescaled and translated versions of the basic Haar scaling function.

1. Show that $\{h_0, \cdots, h_{n-1}\}$ is an ON set in $L^2[0,1]$.

2. Let $f(t)$ be a continuous function on $[0,1]$ and form the projection $f_n(t)$ on the subspace $S_n$ of $L^2[0,1]$ spanned by $\{h_0, \cdots, h_{n-1}\}$:

$$f_n = \sum_{k=0}^{n-1} (f, h_k) h_k.$$

Show that $f_n(t) \to f(t)$ pointwise in $t$ as $n \to \infty$.

3. For $f(t) = 1/(1 + t^2)$, compute explicitly the Haar scaling function decomposition for $n = 4$ and $n = 8$. Plot the results.