There are a total of 100 points on this 55 minute exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. A standard calculator and TWO 8.5 × 11 inch sheets of notes are allowed, but no books, other notes, cell phones or other electronic devices are allowed. Do all of your calculations on this test paper.

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Problem 1  a) (20 points) Use the method of separation of variables to solve the boundary value problem

\[ u_t = ku_{xx}, \quad 0 < x < c, \; t > 0. \]

with boundary conditions

\[ u(0,t) = 0, \; u(c,t) = 0, \quad t > 0, \]

and initial condition

\[ u(x,0) = f(x), \quad 0 < x < c, \]

where \( f \) is piecewise continuous with piecewise continuous derivative.

Solution: Look for solution \( u = X(x)T(t) \) of the differential equation and boundary conditions. Separating variables we have

\[ \frac{T'}{kT} = \frac{X''}{X} = -\lambda. \]

Thus we have the equation \( T' + \lambda k T = 0 \) and the Sturm-Liouville eigenvalue problem

\[ X'' + \lambda X = 0, \quad X(0) = 0, \quad X(c) = 0. \]

The solution of the \( T \) equation is \( T(t) = T_0 \exp(-\lambda kt) \). For the S-L problem we have

Case 1: \( \lambda = \alpha^2, \; \alpha > 0 \): Thus \( X = A \sin \alpha x + B \cos \alpha x \). The conditions are \( X(0) = 0 = 0 \), \( X(c) = 0 = A \sin \alpha c, \) so \( \alpha c = n\pi, \; \lambda = n^2 \pi^2 / c^2, \) \( n = 1, 2, \cdots \), and \( X_n(x) = \sin \frac{n\pi x}{c} \). Further, \( T_n(t) = \exp(-\frac{n^2 \pi^2 kt}{c^2}) \).

Case 2: \( \lambda = 0 \). Thus \( X(x) = Ax + B, \; X(0) = 0 = B \) and \( X(c) = 0 = Ac, \) so \( A = 0 \). We conclude that \( \lambda = 0 \) is not an eigenvalue.

Case 3: \( \lambda = -\alpha^2, \; \alpha > 0 \). Thus \( X(x) = A \exp(\alpha x) + B \exp(-\alpha x) \). \( X(0) = 0 = A + B, \) so \( B = -A \). Then \( X(c) = 0 = A(\exp(\alpha c) - \exp(-\alpha c)) = 2A \sinh(\alpha c), \) so \( A = 0 \). We conclude that there are no negative eigenvalues.
We look for a solution of our problem in the form

\[ u(x,t) = \sum_{n=1}^{\infty} B_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{c}\right) \exp\left(-\frac{n^2\pi^2 t}{c^2}\right). \]

The initial condition gives

\[ u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{c}\right), \quad 0 \leq x \leq c, \]

which is the Fourier sine series for \( f(x) \) on the interval \([0,c] \). Thus,

\[ B_n = \frac{2}{c} \int_0^c f(\tau) \sin\left(\frac{n\pi \tau}{c}\right) d\tau, \quad n = 1, 2, \ldots. \]

b) (10 points) If

\[ f(x) = \begin{cases} 1 & 0 < x < \frac{c}{2} \\ 0 & \frac{c}{2} < x < c. \end{cases} \]

Verify that the formal solution is

\[ u(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{n\pi x}{c}\right)}{n} \exp\left(-\frac{n^2\pi^2 t}{c^2}\right) \sin\left(\frac{n\pi x}{c}\right). \]

Solution:

\[ B_n = \frac{2}{c} \int_0^c f(\tau) \sin\left(\frac{n\pi \tau}{c}\right) d\tau = \frac{2}{c} \int_0^{c/2} \sin\left(\frac{n\pi \tau}{c}\right) d\tau = \frac{2}{c} \left(\frac{c}{n\pi}\right) \cos\left(\frac{n\pi \tau}{c}\right) \bigg|_{0}^{c/2}. \]
\[- \frac{2}{n\pi} \left( \cos\left(\frac{n\pi}{2}\right) - 1 \right) = \frac{4}{n\pi} \sin^2\left(\frac{n\pi}{4}\right).\]

In the last step we used the double angle formula \(\sin^2 \beta = \frac{\cos(2\beta) - 1}{2}\).

Thus,

\[ u(x, t) = \sum_{n} B_n X_n(x) T_n(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{n\pi}{4}\right)}{n} \exp\left(-\frac{n^2\pi^2 \ell^2}{c^2} t\right) \sin \frac{n\pi x}{c}. \]

c) \textbf{(5 points)} Give an argument that your formal solution of part b) is an actual solution of the heat equation.

\textbf{Solution:} Term-by-term differentiation of the series is valid for \(t > 0\) because \(n^\ell \exp\left(-\frac{n^2\pi^2 \ell^2}{c^2} t\right) \to 0\) as \(n \to \infty\) for any \(\ell = 1, 2, \cdots\). The differentiated series converges uniformly in \(x\) on \([0, c]\) for any fixed \(t > 0\).
d) (10 points) Show how the solution of part a) changes if the initial condition and right hand boundary condition remain the same, but the left hand boundary condition is replaced by

\[ u(0,t) = 1, \quad t > 0. \]

Solution: We look for the simplest solution \( u = \Phi(x,t) \) of \( u_t = ku_{xx} \) satisfying \( u(0,t) = 1, \ u(c,t) = 0 \). We try \( u = \Phi(x) \), (independent of \( t \)). Then \( \Phi'' = 0 \), so \( \Phi(x) = Ax + B \), and \( \Phi(0) = 1 = B \), and \( \Phi(c) = 0 = Ac + B \), so \( a = -1/c \). Thus

\[ \Phi(x) = 1 - \frac{x}{c}. \]

We will have a solution \( u \) of our problem provided \( u = \tilde{u}(x,t) + \Phi(x) \) where \( \tilde{u} \) satisfies part a), except that now the intitial condition must be \( \tilde{u}(x,0) = f(x) - \Phi(x), \ 0 \leq x \leq c \). Thus

\[ u(x,t) = (1 - \frac{x}{t}) + \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{c} \right) \exp \left( -\frac{n^2\pi^2 t}{c^2} \right), \]

\[ B_n = \frac{2}{c} \int_0^c \left( f(\tau) - \left( 1 - \frac{\tau}{c} \right) \right) \sin \left( \frac{n\pi \tau}{c} \right) d\tau, \]
Problem 2 Let

\[ f(x) = \begin{cases} 
  x & \text{if } -\pi \leq x \leq \frac{\pi}{2}, \\
  \pi + 2 - 2\left(\frac{1+\pi}{\pi}\right)x & \text{if } \frac{\pi}{2} < x < \pi,
\end{cases} \]

and let

\[ S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \]

be the Fourier series for \( f(x) \) in the interval \([-\pi, \pi]\).

a) (10 points) For each \( x \) in the interval \(-\pi \leq x \leq \pi\), state the exact numerical value of \( S(x) \).

**Solution:** Extend \( f \) to be \( 2\pi \)-periodic on the real line. Then

\[
\begin{align*}
\lim_{x \to -\pi^-} f(x) &= \lim_{x \to -\pi^+} f(x) = \pi + 2 - 2(1 + \pi) = -\pi, \\
\lim_{x \to -\frac{\pi}{2}^-} f(x) &= -\pi, \\
\lim_{x \to \frac{\pi}{2}^+} f(x) &= \frac{\pi}{2}, \\
\lim_{x \to \frac{\pi}{2}^-} f(x) &= \pi + 2 - 2\left(\frac{1+\pi}{\pi}\right)\frac{\pi}{2} = \pi + 2 - 1 - \pi = 1.
\end{align*}
\]

Therefore, using the pointwise convergence theorem for piecewise continuous functions with piecewise continuous derivative on the interval \([-\pi, \pi]\), we have

\[
S(x) = \begin{cases} 
  x & \text{if } -\pi < x < \frac{\pi}{2}, \\
  \pi + 2 - 2\left(\frac{1+\pi}{\pi}\right)x & \text{if } \frac{\pi}{2} < x < \pi, \\
  -\pi & \text{if } x = \pm \pi, \\
  \frac{1+\pi}{2} & \text{if } x = \frac{\pi}{2}.
\end{cases}
\]
b) (15 points) Describe the behavior of the partial sums $S_N(x)$ as $N \to \infty$ near each of the points $x = -\pi$, $x = \frac{\pi}{2}$, $x = \pi$, including possible overshoots and undershoots.

Solution:

1. Near $x = -\pi$ we have $S_N(x) \to f(x)$ uniformly in a neighborhood.
2. Near $x = +\pi$ we again have $S_N(x) \to f(x)$ uniformly in a neighborhood.
3. Near $x = \pi/2$, $S_N(x)$ overshoots by about 18% as $x \to \pi/2$ from below and undershoots by about 18% as $x \to \pi/2$ from above. The width of the overshoot/undershoot region shrinks to 0 in the limit as $n \to \infty$.

c) (5 points) Does the Parseval identity hold for this function and series? Why?

Solution: The Parseval identity holds for any piecewise continuous function with piecewise continuous derivative, hence for this function.
Problem 3 (25 points) Derive the eigenvalues and corresponding eigenfunctions for the Sturm-Liouville problem

\[ X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi, \]

\[ X(0) = 0, \quad X'(\pi) = 0. \]

Consider all possibilities for \( \lambda \).

Solution:

Case 1: \( \lambda = \alpha^2, \alpha > 0. \) The general solution is \( X(x) = A \sin \alpha x + B \cos \alpha x. \) The condition \( X(0) = 0 \) implies \( B = 0 \) and since \( X'(x) = \alpha A \cos \alpha x - \alpha B \sin x, \) the condition \( X'(\pi) = 0 \) implies \( \alpha A \cos \alpha \pi = 0. \) Thus \( \alpha \pi = (2k - 1)\pi/2, \) \( k = 1, 2, \cdots \). Therefore

\[ \lambda_k = \frac{(2k - 1)^2}{4}, \quad X_k(x) = \sin\left(\frac{(2k - 1)x}{2}\right) \quad k = 1, 2, \cdots. \]

Case 2: \( \lambda = 0. \) Then \( X(x) = Ax + B. \) The condition \( X(0) = 0 \) implies \( B = 0 \) and since \( X'(x) = A, \) the condition \( X'(\pi) = 0 \) implies \( A = 0. \) Thus 0 is not an eigenvalue.

Case 3: \( \lambda = -\alpha^2, \alpha > 0. \) Then \( X(x) = A \exp(\alpha x) + B \exp(-\alpha x). \) The condition \( X(0) = 0 \) implies \( B = -A \) and since \( X'(x) = \alpha(A \exp(\alpha x) - B \exp(-\alpha x)), \) the condition \( X'(\pi) = 0 \) implies \( \alpha A [\exp(\alpha \pi) + \exp(-\alpha \pi)] = 0, \) so \( A = 0. \) Thus there are no negative eigenvalues.