Name:  

Math 4567. Midterm Exam I (Solutions)  

February 19, 2010  

There are a total of 100 points on this 55 minute exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. A standard calculator and ONE 8.5 × 11 inch sheet of notes are allowed, but no books, other notes, cell phones or other electronic devices are allowed. Do all of your calculations on this test paper.

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**Problem 1** Let \( S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \) be the Fourier series for the function

\[
f(x) = \begin{cases} 
0 & -\pi < x \leq -\frac{\pi}{2} \\
1 & -\frac{\pi}{2} < x \leq \frac{\pi}{2} \\
0 & \frac{\pi}{2} < x \leq \pi.
\end{cases}
\]

**a) (10 points)** State the precise numerical value of \( S(x) \) for each \( x \) in the interval \(-\pi \leq x \leq \pi\).

**Solution:** \( f(x) \) is piecewise continuous on \((-\pi, \pi)\) with piecewise continuous derivative. The discontinuities are at \( x = \pm \pi/2 \) where the midpoints of the discontinuity are 1/2. Thus

\[
S(x) = \begin{cases} 
0 & -\pi \leq x < -\frac{\pi}{2} \\
\frac{1}{2} & x = -\frac{\pi}{2} \\
1 & -\frac{\pi}{2} < x < \frac{\pi}{2} \\
\frac{1}{2} & x = \frac{\pi}{2} \\
0 & \frac{\pi}{2} < x \leq \pi.
\end{cases}
\]

**b) (15 points)** Compute the Fourier coefficients \( a_j, b_n \) for \( f(x) \).

**Solution:**

\[
a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \, dt = 1,
\]

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos nt \, dt = \frac{1}{\pi} \frac{\sin nt}{n} \bigg|_{-\pi/2}^{\pi/2} = \frac{2 \sin(n\pi/2)}{n\pi} = \begin{cases} 
\frac{2(-1)^{k+1}}{(2k-1)n} & n = 2k - 1 \text{ odd} \\
0 & n \text{ even}
\end{cases}
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin nt \, dt = 0.
\]

Thus,

\[
S(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cos(2k - 1)x}{2k - 1}, \quad -\pi < x < \pi.
\]
c) (10 points) Using the fact that \( \int_0^x f(t) \, dt = x \) for \( -\frac{\pi}{2} < x < \frac{\pi}{2} \), integrate the Fourier series for \( f(x) \) term-by-term to obtain the expansion

\[
\frac{\pi x}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin(2k-1)x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.
\]

Why is this justified?

Solution: Since \( f \) and \( f' \) are piecewise continuous on \( (-\pi/2, \pi/2) \) and extended by \( \pi \)-periodicity then term-by-term integration is OK. Integrating \( S(t) \) from 0 to \( x \) we get

\[
x = \frac{x}{2} + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{(2k-1)\pi} \int_0^x \cos(2k-1)t \, dt = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi(2k-1)^2} \sin(2k-1)t|_0^x
\]

so

\[
\frac{\pi x}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin(2k-1)x.
\]

Problem 2 The Fourier cosine series for the function \( f(x) = 1 - x^2 \) on the domain \( 0 \leq x < 1 \) is

\[
f(x) \sim S(x) = \frac{2}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(n\pi x).
\]

a) (10 points) State the precise numerical value of \( S(x) \) for each \( x \) in the closed interval \( 0 \leq x \leq 1 \). Justify your statement.

Solution: \( f(x) \) is continuous on \( 0 < x \leq 1 \) and since it is even, we can extend it to \(-1 \leq x \leq 1\) with \( f(-1) = f(1) = 0 \) and make it \( 1 \)-periodic to get a function continuous everywhere, Obviously \( f' \) is piecewise continuous. Therefore \( S(x) = 1 - x^2 \) for \( 0 \leq x \leq 1 \).

b) (20 points) Is it permissible the differentiate the series for \( f(x) \) term-by-term to derive the Fourier sine series \( S'(x) \) for \( f'(x) = -2x \) on the interval \( 0 < x < 1 \)? If so, do it and state the precise numerical value of \( S'(x) \) for each \( x \) in the closed interval \( 0 \leq x \leq 1 \).
Solution: Since the extended function \( f(x) \) is continuous everywhere and has a piecewise continuous derivative on \((0,1)\) it is permissible to differentiate term-by-term. Thus we obtain

\[
-2x \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x), \quad 0 < x < 1
\]

Since this is a Fourier sine series, it converges to the odd extension of \(-2x\) to the domain \([-1,1]\) and made \(2\pi\)-periodic. Since \(f'(1-) = -2\) and \(f'(1+) = +2\), the Fourier series converges to \(-2x\) for \(0 \leq x < 1\) and to 0 for \(x = 1\).

Problem 3 The Fourier series for the triangular wave \( f(x) = |x|, \quad -\pi \leq x \leq \pi \), is

\[
f(x) \sim S(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)x.
\]

a) (10 points) Does the series \( S(x) \) converge uniformly on the interval \(-\pi \leq x \leq \pi\)? Justify your answer.

Solution: \( f(x) \) is continuous on \([-\pi,\pi]\) and \(f(-\pi) = f(\pi) = \pi\) Also \(f'(x)\) is piecewise continuous on this interval. Hence the Fourier series of \(f\) converges uniformly on the interval.

b) (10 points) Without actually performing the operation, justify that term-by-term differentiation of the series \( S(x) \) permitted. To what numerical value does \( S'(x) \) converge for each \(x\) in the closed interval \(-\pi \leq x \leq \pi\)?

Solution: Since the extended function \( f(x) \) is continuous everywhere and has a piecewise continuous derivative on \((-\pi,\pi)\) it is permissible the differentiate term-by-term. The derivative is

\[
f'(x) = \begin{cases} 
1 & \text{if } 0 < x < \pi \\
-1 & \text{if } -\pi < x < 0.
\end{cases}
\]

This function is piecewise continuous with a piecewise continuous derivative on \((-\pi,\pi)\). The discontinuities are at \(x = 0, \pm\pi\). and the midpoints of these discontinuities is 0 Thus the differentiated series converges to 1 for \(0 < x < \pi\), to -1 for \(-\pi < x < 0\) and to 0 for \(x = 0, \pm\pi\).
c) (15 points) Evaluate the sum
\[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \]
of the Fourier coefficients from part a) with a minimum of computation. Justify your answer.

Solution: Since the series for \( f(x) \) converges uniformly, Parseval’s theorem holds. Thus
\[ \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2). \]

Evaluating the integral we have
\[ \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^2}{3}. \]