Chapter 5 page 113, Problem 1 The initial temperature of a slab $0 \leq x \leq \pi$ is everywhere 0 and the face $x = 0$ is kept at that temperature. Heat is supplied through the face $x = \pi$ at a constant rate $ku_x(\pi, t) = A > 0$. Write $u(x, t) = U(x, t) + \Phi(x)$ and use the solution to the problem

\begin{align*}
(\ast) \quad & U_t = kU_{xx}, \quad 0 < x < \pi, t > 0, \\
& U(0, t) = 0, \quad U_x(\pi, t) = 0, \quad 0 < x < \pi,
\end{align*}

and $U(x, 0) = F(x)$ where

$$F(x) = \begin{cases} f(x) & \text{when } 0 < x < \pi \\ f(2\pi - x) & \text{when } \pi < x < 2\pi, \end{cases}$$

which is

$$U(x, t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n^2k}{4}t\right) \sin \frac{nx}{2},$$

$$B_n = \frac{1 - (-1)^n}{\pi} \int_0^\pi f(x) \sin \frac{nx}{2} \, dx.$$ 

to derive the final solution $u(x, t)$

\textbf{Solution:} We first find a function $u = \Phi(x)$ that satisfies the nonhomogeneous condition $ku_x(\pi, t) = A$ and the homogeneous condition $u(0, t) = 0$. The differential equation is $\Phi''(x) = 0$, so $\Phi(x) = Bx + C$. The nonhomogeneous boundary condition says $KB = A$ and the homogeneous condition says $C = 0$ thus $\Phi(x) = \frac{A}{k}x$. Then, setting
\( u(x, t) = U(x, t) + \Phi(x) \) we see that \( u(x, t) \) will be a solution of our original problem, provided \( U(x, t) \) satisfies problem (\( * \)) where \( U(x, 0) = f(x) = -\Phi(x) = -\frac{\partial}{\partial x} x \) for \( 0 < x < \pi \). Thus

\[
u(x, t) = \frac{A}{k} x + \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n^2 k}{4} t\right) \sin \frac{n x}{2},
\]

where

\[
B_n = -\frac{A}{k} \left( \frac{1 - (-1)^n}{\pi} \right) \int_0^x \sin \frac{n x}{2} \, dx.
\]

Note that \( B_n = 0 \) unless \( n = 2m - 1 \) is odd. Since

\[
\int_0^\pi x \sin \frac{n x}{2} \, dx = 2 \left\{ \int_0^{\pi/2} x \cos \frac{n x}{2} \, dx + \int_0^{\pi/2} \cos \frac{n x}{2} \, dx \right\} = 2 \left[ -\pi \cos \frac{n \pi}{2} + \frac{2}{n} \sin \frac{n \pi}{2} \right] = (\frac{2}{2m-1})^2 (-1)^{m+1},
\]

we get the solution

\[
u(x, t) = \frac{A}{k} \left\{ x + \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{(2m-1)^2} \int_0^{\pi/2} f(x) \cos \frac{n x}{2} \, dx \right\} \sin \left( \frac{2m-1}{2} x \right).
\]

**Chapter 5, page 122, Problem 1** The faces and edges \( x = 0 \) and \( x = \pi \), \((0 < y < \pi) \) of a square plate \( 0 \leq x \leq \pi \), \( 0 \leq y \leq \pi \) are insulated. The edges \( y = 0 \) and \( y = \pi \), \((0 < x < \pi) \) are kept at temperatures \( 0 \) and \( f(x) \), respectively. Let \( u(x, y) \) be the steady state temperature distribution in the plate. Show that

\[
u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh n y \cos n x,
\]

\[
A_0 = \frac{1}{\pi^2} \int_0^{\pi} f(x) \, dx, \quad A_n = \frac{2}{\pi \sinh n \pi} \int_0^{\pi} f(x) \cos n x \, dx, \quad n = 1, 2, \ldots
\]

Find \( u(x, y) \) if \( f(x) = u_0 \).

**Solution:** The problem is

\[
u_{xx} + u_{yy} = 0, \quad 0 < x < \pi, \quad 0 < y < \pi,
\]

2
\[ u_x(0, y) = 0, \ u_x(\pi, y) = 0, \ 0 < y < \pi; \]
\[ u(x, 0) = 0, \ 0 < x < \pi \]
\[ u(x, \pi) = f(x), \ 0 < x < \pi. \]

We use separation of variables \( u = X(x)Y(y) \) to find solutions satisfying the homogeneous conditions. The Sturm-Liouville eigenvalue problem is
\[ X''(x) + \lambda X(x) = 0, \ X'(0) = X'(\pi) = 0. \]

From past work we know that the eigenvalues are \( \lambda_n = n^2, \ n = 1, 2, \ldots \) with eigenfunctions \( X_n(x) = \cos nx \), and \( \lambda_0 = 0 \) with eigenfunction \( X_0(x) = 1 \). The corresponding equations for \( Y(y) \) are
\[ Y''(y) - \lambda Y(y) = 0, \ Y(0) = 0. \]

As has been shown earlier, for \( \lambda_n = n^2 \) we have \( Y_n(y) = \sinh ny \) and for \( \lambda_0 = 0 \) we have \( Y_0(y) = y \). Thus we can write
\[ u(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos nx \sinh ny, \]
where
\[ u(x, \pi) = f(x) = A_0 \pi + \sum_{n=1}^{\infty} A_n \cos nx \sinh n\pi. \]

Thus
\[ A_n \sinh n\pi = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, \ldots, \]
\[ A_0 \pi = \frac{1}{\pi} \int_0^{\pi} f(x) \, dx. \]

If \( f(x) = u_0 \) then \( A_n = 0 \), \( A_0 \pi = u_0 \), so the solution is \( u(x, y) = \frac{u_0}{\pi} y. \)

**Chapter 5, page 128, Problem 2** Let the faces of a wedge shaped plate \( 0 \leq \rho \leq a, \ 0 \leq \phi \leq \alpha \) be insulated Find the steady temperature \( u(\rho, \phi) \) in the plate when \( u = 0 \) on the rays \( \phi = 0, \ \phi = \alpha \ (0 < \rho < \alpha) \) and \( u = f(\phi) \) on the arc \( \rho = a \ (0 < \phi < \alpha) \). Assume \( f \) is piecewise smooth and \( u \) is bounded.
Solution: Our problem in polar coordinates is to find a function \(u(\rho, \phi)\) for
\[
\rho^2 u_{\rho\rho} + \rho u_{\rho} + u_{\phi\phi} = 0, \quad 0 < \rho < a, \quad 0 < \phi < \alpha,
\]
\[
u(\rho, 0) = u(\rho, \alpha) = 0, \quad 0 < \rho < a,
\]
\[
u(a, \phi) = f(\phi), \quad 0 < \phi < \alpha,
\]
where \(f\) is piecewise smooth and \(|u| < M\), i.e., \(u\) is bounded.
Separating variables, \(u = R(\rho)\Phi(\phi)\) satisfies the homogeneous conditions if \(\Phi\) satisfies the Sturm-Liouville problem
\[
\Phi'' + \lambda \Phi = 0, \quad \Phi(0) = \Phi(\alpha) = 0,
\]
and \(R\) satisfies
\[
\rho^2 R'' + \rho R' - \lambda R = 0
\]
and \(R\) is bounded. It is straightforward to show that the eigenvalues are \(\lambda_n = \frac{n^2 \pi^2}{\alpha^2}\) with eigenfunctions \(\Phi_n(\phi) = \sin \frac{n\pi \phi}{\alpha}, \ n = 1, 2, \ldots\). The change of variable \(\rho = e^s\) gives the corresponding equation for \(R\) as \(R_{ss} - \lambda_n R = 0\). The general solutions are
\[
R_n(\rho) = A\rho^{n\pi/\alpha} + B\rho^{-n\pi/\alpha},
\]
and the boundedness requirement yields \(R_n(\rho) = \rho^{n\pi/\alpha}\). Thus
\[
u(\rho, \phi) = \sum_{n=1}^{\infty} B_n \rho^{n\pi/\alpha} \sin \frac{n\pi \phi}{\alpha},
\]
and
\[
u(a, \phi) = f(\phi) = \sum_{n=1}^{\infty} B_n a^{n\pi/\alpha} \sin \frac{n\pi \phi}{\alpha},
\]
where
\[
B_n a^{n\pi/\alpha} = \frac{2}{\alpha} \int_{0}^{\alpha} f(\psi) \sin \frac{n\pi \psi}{\alpha} \ d\psi.
\]
Thus
\[
u(\rho, \phi) = \frac{2}{\alpha} \sum_{n=1}^{\infty} \left(\frac{\rho}{a}\right)^{n\pi/\alpha} \sin \frac{n\pi \phi}{\alpha} \int_{0}^{\alpha} f(\psi) \sin \frac{n\pi \psi}{\alpha} \ d\psi.
\]
Chapter 5, page 133, Problem 4 A string is stretched between points 0 and π on x-axis and, initially at rest, is released from the position \( y = f(x) \). The equation of motion is

\[
y_{tt} = y_{xx} - 2\beta y_t, \quad 0 < x < \pi, \quad t > 0,
\]

where \( 0 < \beta < 1 \) and \( \beta \) is constant. Show that

\[
y(x, t) = e^{-\beta t} \sum_{n=1}^{\infty} B_n \left( \cos \alpha_n t + \frac{\beta}{\alpha_n} \sin \alpha_n t \right) \sin nx,
\]

\[
\alpha_n = \sqrt{n^2 - \beta^2}, \quad B_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx \, dx, \quad n = 1, 2, \ldots.
\]

Solution: Set \( z(x, t) = e^{\beta t} y(x, t) \). Then \( z(x, t) \) satisfies

\[
z_{tt} = z_{xx} + \beta^2 z, \quad 0 < \pi, \quad t > 0
\]

\[
z(0, t) = z(\pi, t) = 0, \quad t > 0
\]

\[
z_t(x, 0) = \beta f(x), \quad z(x, 0) = f(x), \quad 0 < x < \pi,
\]

where \( f(0) = f(\pi) = 0 \). We look for a solution of the form

\[
z(x, t) = \sum_{n=1}^{\infty} A_n(t) \sin nx, \quad A_n(t) = \frac{2}{\pi} \int_0^\pi z(x, t) \sin nx \, dx.
\]

Then

\[
A_n''(t) = \frac{2}{\pi} \int_0^\pi z_{tt}(x, t) \sin nx \, dx = \frac{2}{\pi} \int_0^\pi \left( z_{xx}(x, t) + \beta^2 z(x, t) \right) \sin nx \, dx
\]

\[
= \frac{2}{\pi} \int_0^\pi z_{xx}(x, t) \sin nx \, dx + \frac{2\beta^2}{\pi} \int_0^\pi z(x, t) \sin nx \, dx.
\]

\[
= \frac{2}{\pi} [z_x(x, t) \sin n\pi]_0^\pi - n \int_0^\pi z_x(x, t) \cos nx \, dx] + \beta^2 A_n(t)
\]

\[
= \frac{2}{\pi} [-nz(x, t) \cos nx]_0^\pi - n^2 \int_0^\pi z(x, t) \sin nx \, dx] + \beta^2 A_n(t)
\]

\[
= (-n^2 + \beta^2) A_n(t).
\]
Thus
\[ A_n''(t) + (n^2 - \beta^2)A_n(t) = 0, \]
so
\[ A_n(t) = B_n \cos \alpha_n t + C_n \sin \alpha_n t \]
where \( \alpha_n = \sqrt{n^2 - \beta^2}, \ n = 1, 2, \cdots \). The condition
\[ z(x, 0) = f(x) = \sum_{n=1}^{\infty} B_n \sin nx \]
gives
\[ B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \cdots. \]
The condition
\[ z_t(x, 0) = \beta f(x) = \sum_{n=1}^{\infty} C_n \alpha_n \sin x \]
gives \( C_n = \beta B_n / \alpha_n \). (Here we are assuming that it is permissible to differentiate the sum term-by-term. This assumption could be avoided by taking \( z_t(x, t) = \sum_{n=1}^{\infty} E_n(t) \sin nx \) and obtaining \( E_n(t) \) by integration by parts, just as we did for \( A_n(t) \).) Thus we obtain the formal solution
\[ y(x, t) = e^{-\beta t} z(x, t) = e^{-\beta t} \sum_{n=1}^{\infty} B_n \left( \cos \alpha_n t + \frac{\beta}{\alpha_n} \sin \alpha_n t \right) \sin nx, \]

Chapter 5, page 136, Problem 1 Solve the problem
\[ y_{tt} = a^2 y_{xx} + Ax \sin \omega t, \quad 0 < x < c, \ t > 0, \]
\[ y(0, t) = y(c, t) = 0, \ y(x, 0) = y_t(x, 0) = 0. \]
Show that resonance occurs for \( \omega = \omega_n \), where
\[ \omega_n = \frac{n\pi a}{c}, \quad n = 1, 2, \cdots. \]

Solution: We look for a solution in the form
\[ y(x, t) = \sum_{n=1}^{\infty} B_n(t) \sin \frac{n\pi x}{c}, \]
\[ B_n(t) = \frac{2}{c} \int_0^c y(x,t) \sin \frac{n\pi x}{c} \, dx. \]

Then
\[ B''_n(t) = \frac{2}{c} \int_0^c y_{tt}(x,t) \sin \frac{n\pi x}{c} \, dx = \frac{2}{c} \int_0^c \left[ a^2 y_{xx}(x,t) + A x \sin \omega t \right] \sin \frac{n\pi x}{c} \, dx \]
\[ = \frac{2A(-1)^{n+1}}{n\pi} \sin \omega t - \frac{a^2 n^2 \pi^2}{c^2} B_n(t), \]

Where we have integrated by parts several times and applied the boundary conditions. Thus
\[ (*) \, B''_n(t) + \frac{a^2 n^2 \pi^2}{c^2} B_n(t) = \frac{2A(-1)^{n+1}}{n\pi} \sin \omega t. \]

This is a nonhomogeneous equation. We need only find one solution and then add to it the general solution \( H_n \cos \frac{an\pi x}{c} + K_n \sin \frac{an\pi x}{c} \) of the homogeneous equation to get the general solution. We look for a solution of the form \( B_n(t) = C_n \sin \omega t. \) Substituting this into equation (\(*)\) and setting \( \omega_n = an\pi/c \) we find a solution if \( C_n = 2A(-1)^{n+1}/(\omega_n^2 - \omega^2) \). Thus
\[ B_n(t) = \frac{2A(-1)^{n+1}}{\omega_n^2 - \omega^2} \sin \omega t, \]
and the general solution is
\[ B_n(t) = A_n \cos \frac{an\pi t}{c} + C_n \sin \frac{an\pi t}{c} + \frac{2A(-1)^{n+1}}{\omega_n^2 - \omega^2} \sin \omega t. \]

The boundary conditions are \( B_n(0) = B'_n(0) = 0 \), and these are satisfied for \( A_n = 0 \) and \( C_n = 2(-1)^n \omega/[\omega_n(\omega_n^2 - \omega^2)] \). Thus the final solution is
\[ B_n(t) = \frac{2(-1)^n \omega}{\omega_n(\omega_n^2 - \omega^2)} \sin \frac{an\pi t}{c} + \frac{2A(-1)^{n+1}}{\omega_n^2 - \omega^2} \sin \omega t, \]

unless \( \omega = \omega_n \) for some \( n \). In that case we have resonance and the solution becomes unbounded.

To see this, we look for a particular solution of \((*)\) in the case \( \omega = \omega_n \).
Take the trial solution \( B_n(t) = D_n t \cos \omega_n t \). Then we find a solution provided \( D_n = (-1)^n A/[n\pi \omega_n] \):
\[ (\dagger) \quad B_n(t) = \frac{(-1)^n A}{n\pi \omega_n} t \cos \omega_n t. \]
To this solution we can add a general solution of the homogeneous equation, but the resonant solution quickly dominates the bounded solution of the wave equation as $t$ gets large.

**Chapter 5, page 146, Problem 1** Write $\lambda = -\alpha^2$, $\alpha > 0$ and show that the Sturm-Liouville problem

$$X'' + \lambda X = 0, \ X(-\pi) = X(\pi), \ X'(-\pi) = X'(\pi),$$

has no solutions.

**Solution:** The general solution of the differential equation is

$$X(x) = Ae^{\alpha x} + Be^{-\alpha x},$$

so

$$X'(x) = \alpha(Ae^{\alpha x} - Be^{-\alpha x}).$$

The conditions $X(-\pi) = X(\pi)$, $X'(-\pi) = X'(\pi)$ can be written as

$$A \sinh \alpha \pi = B \sinh \alpha \pi, \ A \sinh \alpha \pi = -B \sinh \alpha \pi,$$

respectively. Since $\sinh \alpha \pi \neq 0$ for $\alpha \neq 0$, we have $A = -B = B$, so $A = B = 0$. Thus there are no negative eigenvalues.

**Chapter 8, page 209, Problem 1** Find the eigenvalues and eigenfunctions:

$$X'' + \lambda X = 0, \ X(0) = 0, \ X'(1) = 0.$$

**Solution:** If $\lambda = 0$ then $X(x) = Ax + B$ and $X'(x) = A$. Thus the boundary conditions are $B = 0$, $A = 0$ and $\lambda = 0$ is not an eigenvalue.

If $\lambda = -\alpha^2$, $\alpha > 0$ then $X(x) = Ae^{\alpha x} + Be^{-\alpha x}$, $X'(x) = \alpha(Ae^{\alpha x} - Be^{-\alpha x})$. Thus the boundary conditions are $A + B = 0$ and $Ae^{\alpha} - e^{-\alpha}$, or $B = -A$ where $A \cosh \alpha = 0$. Since $\cosh \alpha \neq 0$ we have $A = B = 0$ and $\lambda = -\alpha^2$ is not an eigenvalue.

If $\lambda = \alpha^2$, $\alpha > 0$ then $X(x) = A \cos \alpha x + B \sin \alpha x$, $X'(x) = \alpha(-A \sin \alpha x + B \cos \alpha x)$, and the boundary conditions can be read as

$$A = 0, \ \alpha(B \cos \alpha) = 0,$$
or \( \cos \alpha = 0 \), so \( \lambda_n = \alpha_n^2 \) where

\[
\alpha_n = (2n - 1) \frac{\pi}{2}, \quad X_n(x) = \sin \alpha_n x \quad n = 1, 2, \cdots.
\]

Since \( \int_0^1 X_n^2(x) \, dx = \frac{1}{2} \int_0^1 (1 - \cos (2n - 1)x) \, dx = \frac{1}{2} \) the normalized eigenfunctions are \( \phi_n(x) = \sqrt{2} \sin \alpha_n x \).