Chapter 8 page 201, Problem 1 (a) Consider the Sturm - Liouville problem

\[ [xX'(x)]' + \frac{\lambda}{x}X(x) = 0, \quad 1 < x < b, \]
\[ X(1) = 0, \quad X(b) = 0, \]

and use the substitution \( x = \exp s \) to convert the problem to

\[ \frac{d^2X}{ds^2} + \lambda X = 0, \quad 0 < s < \ln b, \]
\[ X|_{s=0} = 0, \quad X|_{s=\ln b} = 0. \]

Show that the eigenvalues and eigenfunctions of the original problem are

\[ \lambda_n = \alpha_n^2, \quad X_n(x) = \sin(\alpha_n \ln x), \quad n = 1, 2, \cdots, \]

where \( \alpha_n = n\pi/\ln b. \)

(b) By making the substitution

\[ s = \frac{\pi \ln x}{\ln b} \]

give a direct verification that the eigenfunctions \( X_n(x) \) of part (a) are orthogonal on the interval \( 1 < x < b \), with weight function \( p(x) = 1/x \).

Solution:
(a) We have \( \frac{d}{ds} = \frac{dx}{ds} \frac{d}{dx} = x \frac{d}{dx} \), so
\[
x[xX'(x)]' + \lambda X(x) = 0 \leftrightarrow \frac{d^2 X}{ds^2} + \lambda X = 0, \quad 0 < s < \ln b,
\]
since \( s = \ln x \). Thus, in the new coordinates the boundary conditions are
\[
X|_{s=0} = 0, \quad X|_{s=\ln b} = 0.
\]
For the original problem we solve the eigenvalue problem.
Case 1: \( \lambda = \alpha^2, \alpha > 0 \). The solution of the differential equation is
\[
X = A \cos \alpha s + B \sin \alpha s = A \cos(\alpha \ln x) + B \sin(\alpha \ln x).
\]
Then \( X(1) = 0 = A \), and \( X(b) = 0 = B \sin(\alpha \ln b) \), so we can have a nonzero solution only for \( \alpha \ln b = n\pi \), or \( \alpha = \alpha_n = n\pi/\ln b \), with \( X_n(x) = \sin \left( \frac{n\pi}{\ln b} \right) \), \( n = 1, 2, \cdots \).

(b) Since \( s = \frac{\pi \ln x}{\ln b} \), it follows that \( ds = \pi dx/x \ln b \). We have for \( m \neq n \),
\[
\int_1^b X_n(x)X_m(x) \frac{dx}{x} = \int_1^b \sin \left( \frac{n\pi \ln x}{\ln b} \right) \sin \left( \frac{m\pi \ln x}{\ln b} \right) \frac{dx}{x}
= \frac{\ln b}{\pi} \int_0^\pi \sin ns \sin ms \, ds
= 0,
\]
if \( m \neq n \).

Chapter 8, page 201, Problem 2 Let
\[
\mathcal{L}[X] = (rX')' + qX
\]
so that the Sturm-Liouville differential equation can be written as
\[
\mathcal{L}[X] + \lambda pX = 0.
\]
Derive Lagrange’s identity
\[
X\mathcal{L}[Y] - Y\mathcal{L}[X] = \frac{d}{dx} [r(XY' - YX')].
\]
Solution:

\[ X\mathcal{L}[Y] - Y\mathcal{L}[X] = X(rY')' + qXY - Y(rX')' - qYX \]

\[ = Xr'Y' + XrY'' - Yr'X' - YX'' = r'(XY' - YX') + r(XY'' - YX'') \]

Since

\[ \frac{d}{dx}[r(XY' - YX')] = r'(XY' - YX') + r(XY'' - YX'') \]

this establishes the identity.

**Chapter 8, page 201, Problem 3 (a)** Let \( \mathcal{L} \) be the operator of the previous problem, defined on a space of functions on \( a < x < b \), satisfying the conditions

\[ a_1X(a) + a_2X'(a) = 0, \quad b_1X(b) + b_2X'(b) = 0, \quad |a_1| + |a_2| > 0, \quad |b_1| + |b_2| > 0, \]

and with inner product with weight function \( p(x) = 1 \). Show that

\[ (X, \mathcal{L}[Y]) = (\mathcal{L}[X], Y) \]

(b) Let \( \lambda_m \neq \lambda_n \) be eigenvalues of the problem \( \mathcal{L}[X] + \lambda pX = 0 \) with boundary conditions

\[ a_1X(a) + a_2X'(a) = 0, \quad b_1X(b) + b_2X'(b) = 0, \quad |a_1| + |a_2| > 0, \quad |b_1| + |b_2| > 0. \]

Show that if \( X_m, X_n \) are the corresponding eigenfunctions, then

\[ (pX_m, X_n) = 0. \]

Solution:

(a)

\[ (X, \mathcal{L}[Y]) - (\mathcal{L}[X], Y) = \int_a^b \frac{d}{dx}[r(XY' - YX')] \, dx = [r(XY' - YX')]_a^b = \]

\[ r(b)(X(b)Y'(b) - Y(b)X'(b)) - r(a)(X(a)Y'(a) - Y(a)X'(a)). \]
Now suppose $a_1 \neq 0$. Then

\[
X(a) = -\frac{a_2 X'(a)}{a_1}, \quad Y(a) = -\frac{a_2 Y'(a)}{a_1}
\]

\[
\rightarrow X(a)Y''(a) - Y(a)X'(a) = -\frac{a_2 X'(a)Y'(a)}{a_1} + \frac{a_2 X'(a)Y'(a)}{a_1} = 0.
\]

If $a_2 \neq 0$ then

\[
X'(a) = -\frac{a_1 X(a)}{a_2}, \quad Y'(a) = -\frac{a_1 Y(a)}{a_2}
\]

\[
\rightarrow X(a)Y'(a) - Y(a)X'(a) = -\frac{a_1 X(a)Y(a)}{a_2} + \frac{a_1 X(a)Y(a)}{a_2} = 0.
\]

Thus always $X(a)Y'(a) - Y(a)X'(a) = 0$. A similar argument applied to the endpoint $b$ gives $X(b)Y'(b) - Y(b)X'(b) = 0$. Thus, $(X, \mathcal{L}[Y]) - (\mathcal{L}[X], Y) = 0$.

(b) We have

\[
\mathcal{L}[X_m] + \lambda_m pX_m = 0, \quad \mathcal{L}[X_n] + \lambda_n pX_n = 0.
\]

Thus

\[
(X_m, \mathcal{L}[X]) - (\mathcal{L}[X_m], X_n) = -(X_m, \lambda_n pX_n) + (\lambda_m pX_m, X_n) = [\lambda_m - \lambda_n](pX_m, X_n)
\]

However, from part (a) we have $(X_m, \mathcal{L}[X_n]) - (\mathcal{L}[X_m], X_n) = 0$, so $[\lambda_m - \lambda_n](pX_m, X_n) = 0$. Since $\lambda_m \neq \lambda_n$ it follows that $(pX_m, X_n) = 0$.

Chapter 8, page 209, Problem 4 Solve the S-L problem

\[
X'' + \lambda X = 0, \quad X(0) = 0, \quad X(1) - X'(1) = 0.
\]

Solution:

Case 1: $\lambda = \alpha^2 > 0$, $\alpha > 0$. Then

\[
X(x) = A \cos \alpha x + B \sin \alpha x, \quad X'(x) = -\alpha A \sin \alpha x + \alpha B \cos \alpha x.
\]

The conditions

\[
X(0) = 0 = A, \quad X(1) - X'(1) = 0 = B \sin \alpha - \alpha B \cos \alpha,
\]
imply $\alpha = \tan \alpha$. Similar to what is shown in the book, the solutions are $\alpha_n$, $n = 1, 2, \cdots$ such that $(n - 1)\pi < \alpha_n < (2n - 1)\frac{\pi}{2}$. The eigenvalues are $\lambda_n = \alpha_n^2$ Here $X_n(x) = \sin \alpha_n x$, so

$$||X_n||^2 = (X_n, X_n) = \int_0^1 \sin^2(\alpha_n x) dx = \frac{1}{2} \int_0^1 (1 - \cos 2\alpha_n x) dx$$

$$= \frac{1}{2} \left(1 - \frac{1}{2\alpha_n} \sin 2\alpha_n\right) = \frac{1}{2} \left(1 - \cos^2 \alpha_n\right),$$

since $\sin \alpha_n = \alpha \cos \alpha_n$. But

$$\cos^2 \alpha_n = \frac{1}{1 + \tan^2 \alpha_n} = \frac{1}{1 + \alpha_n^2},$$

so

$$||X_n||^2 = \frac{1}{2} \left(1 - \frac{1}{1 + \alpha_n^2}\right) = \frac{1}{2} \frac{\alpha_n^2}{1 + \alpha_n^2}$$

and the normalized eigenfunctions are

$$\phi_n(x) = \frac{\sqrt{2(\alpha_n^2 + 1)}}{\alpha_n} \sin \alpha_n x.$$

Case 2: $\lambda = 0$. Then $X(x) = Ax + B$. The conditions

$$X(0) = 0 = B, \quad X(1) - X'(1) = 0 = A - A$$

imply $\lambda_0 = 0$, $X_0(x) = x$. We have

$$||X_0||^2 = (X_0, X_0) = \int_0^1 x^2 dx = \frac{1}{3},$$

so the normalized eigenfunction is $\phi_0(x) = \sqrt{3}x$.

Case 3: $\lambda = -\alpha^2 < 0$, $\alpha > 0$. From the left hand boundary condition, we must have $X(x) = \sinh \alpha x$. The remaining boundary condition is then $\sinh \alpha - \alpha \cosh \alpha = 0$ or $\alpha = \tanh \alpha$. The issue is then the points of intersection of the curves $y = \alpha$ and $y = \tanh \alpha$. These curves clearly intersect at $\alpha = 0$. If they intersect again at some $\alpha_0 > 0$ then the function $g(x) = \alpha - \tanh \alpha$ is continuous on the closed interval $0 \leq \alpha \leq \alpha_0$ and differentiable on the open interval $(0, \alpha_0)$. Furthermore
\[ g(0) = g(\alpha_0) = 0. \] By the Mean Value Theorem of calculus, there must be a value \( c \in (0, \alpha_0) \) such that \( g'(c) = 0 \) but \( g'(\alpha) = \tanh^2 \alpha > 0 \) for all \( \alpha > 0 \). Thus no such \( c \) can exist, so there is no negative eigenvalue \(-\alpha_0^2\).

**Chapter 8, page 215, Problem 3** Use the normalized eigenfunctions of Problem 2, page 209, namely

\[
X'' + \lambda X = 0, \; X(0) = 0, \; hX(1) + X'(1) = 0, \; h > 0, \\
\lambda_n = \alpha_n^2, \; \tan \alpha_n = -\alpha_n / h, \; \phi_n(x) = \sqrt{2h / (h + \cos^2 \alpha_n)} \sin \alpha_n x. \; n = 1, 2, \ldots,
\]

to derive

\[
1 = 2h \sum_{n=1}^{\infty} \frac{1 - \cos \alpha_n}{\alpha_n (h + \cos^2 \alpha_n)} \sin \alpha_n, \; 0 < x < 1.
\]

**Solution**: We have

\[
1 = \sum_{n=1}^{\infty} c_n \phi_n(x), \; c_n = \int_{0}^{1} 1 \cdot \phi_n(s) ds, \; 0 < x < 1.
\]

Now

\[
c_n = \sqrt{2h / (h + \cos^2 \alpha_n)} \int_{0}^{1} \sin \alpha_n s ds = -\sqrt{2h / (h + \cos^2 \alpha_n)} \cos \alpha_n / \alpha_n.
\]

Thus

\[
1 = \sum_{n=1}^{\infty} \frac{2h}{\alpha_n (h + \cos^2 \alpha_n)} (1 - \cos \alpha_n).
\]

**Chapter 8, page 221, Problem 2** Use the normalized eigenfunctions of the S-L problem

\[
X'' + \lambda X = 0, \; X(0) = 0, \; X'(\pi) = 0
\]

to solve the boundary value problem

\[
u_t(x, t) = ku_{xx}(x, t), \; 0 < x < \pi, \; t > 0,
\]
\[ u(0, t) = 0, \ u_x(\pi, t) = 0, \ u(x, 0) = f(x). \]

**Solution:** The normalized eigenfunctions are a renormalization of those in the previous problem:

\[ \phi_n(x) = \sqrt{\frac{2}{\pi}} \sin \alpha_n x, \ \alpha_n = \frac{(2n - 1)}{2}, \ n = 1, 2, \cdots. \]

The corresponding separated functions \( T_n(t) \) satisfy

\[ T_n'(t) + \alpha_n^2 k T_n(t) = 0, \]

so

\[ T_n(t) = \exp(-\alpha_n^2 k t). \]

Thus

\[ u(x, t) = \sum_{n=1}^{\infty} B_{2n-1} \exp(-\alpha_n^2 k t) \phi_n(x), \]

with

\[ B_{2n-1} = \frac{\sqrt{2}}{\pi} \int_{0}^{\pi} u(x, t) \phi_n(x) \, dx. \]

Since \( u(x, 0) = f(x) \), we have

\[ B_{2n-1} = \frac{\sqrt{2}}{\pi} \int_{0}^{\pi} f(x) \phi_n(x) \, dx = \frac{\sqrt{2}}{\pi} \int_{0}^{\pi} f(x) \sin \left(\frac{(2n - 1)x}{2}\right) \, dx \]

for \( n = 1, 2, \cdots \).

**Chapter 8, page 228, Problem 1** Use the expansion of \( x \),

\[ x = \frac{2}{c} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\alpha_n^2} \sin \alpha_n x, \quad 0 < x < c \]

in terms of the eigenfunctions of the S-L problem

\[ X'' + \lambda X = 0, \ X(0) = 0, \ X'(c) = 0, \]

\[ \lambda_n = \alpha_n^2, \ \phi_n(x) = \sqrt{\frac{2}{c}} \sin \alpha_n x, \quad n = 1, 2, \cdots, \]

where

\[ \alpha_n = \frac{(2n - 1)\pi}{2c}, \]

to show that the temperature function

\[ u(x, t) = \frac{A}{K} \left[ x + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha_n^2} \exp(-\alpha_n^2 k t) \sin \alpha_n x \right], \quad 0 < x < 1, \ t > 0 \]
with $\alpha_n = \frac{(2n-1)\pi}{2}$, can be written as

$$u(x,t) = \frac{2A}{K} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\alpha_n^2} [1 - \exp(-\alpha_n^2 kt)] \sin \alpha_n x, \quad 0 < x < 1, \quad t > 0.$$ 

**Solution:** Set $c = 1$ in the expansion for $x$, substitute this in the expansion for $u(x,t)$ and write the sum of two infinite series as a single series to get

$$u(x,t) = \frac{A}{K} \left[ x + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha_n^2} \exp(-\alpha_n^2 kt) \sin \alpha_n x \right] =$$

$$= \frac{2A}{K} \left[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\alpha_n^2} \sin \alpha_n x + \sum_{n=1}^{\infty} \frac{(-1)^n}{\alpha_n^2} \exp(-\alpha_n^2 kt) \sin \alpha_n x \right]$$

$$= \frac{2A}{K} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\alpha_n^2} [1 - \exp(-\alpha_n^2 kt)] \sin \alpha_n x, \quad 0 < x < 1, \quad t > 0.$$ 

**Chapter 6, page 157, Problem 2** Show that the function

$$f(x) = \begin{cases} 
1 & \text{when } |x| < 1, \\
0 & \text{when } |x| > 1, \\
\frac{1}{2} & \text{when } x = \pm 1,
\end{cases}$$

satisfies the conditions of the Fourier integral pointwise convergence theorem. Establish

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin \alpha (1 + x) + \sin \alpha (1 - x)}{\alpha} \, d\alpha = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} \, d\alpha.$$ 

**Solution:** $f$ is piecewise continuous on every bounded interval and

$$\int_{-\infty}^{\infty} |f(x)| \, dx = \int_{-1}^{1} 1 \, dx = 2 < \infty,$$

so

$$\frac{f(x+) + f(x-)}{2} = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(s) \cos \alpha (s - x) \, ds \, d\alpha,$$
at each x such that $f'_R(x)$ and $f'_L(x)$ exist, and these derivatives exist at all x. Further, this function satisfies
\[
\frac{f(x+) + f(x-)}{2} = f(x)
\]
for all x. Now
\[
\int_{-\infty}^{\infty} f(s) \cos \alpha(s - x) \, ds = \int_{-1}^{1} \cos \alpha(s - x) \, ds = \left[ \frac{\sin \alpha(s - x)}{\alpha} \right]_{-1}^{1} = \frac{\sin \alpha(1-x) + \sin \alpha(1+x)}{\alpha},
\]
so
\[
f(x) = \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin \alpha(1-x) + \sin \alpha(1+x)}{\alpha} \, d\alpha.
\]
From the addition formulas for $\sin x$ we have
\[
\sin \alpha(1-x)+\sin \alpha(1+x) = \sin \alpha \cos \alpha x - \cos \alpha \sin \alpha x + \sin \alpha \cos \alpha x + \cos \alpha \sin \alpha x = 2 \sin \alpha \cos \alpha x,
\]
so
\[
f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin \alpha \cos \alpha x}{\alpha} \, d\alpha.
\]

**Chapter 6, page 162, Problem 1** Show that the function
\[
f(x) = \begin{cases} 
1 & \text{when } 0 < x < b, \\
0 & \text{when } x > b, \\
\frac{1}{2} & \text{when } x = b,
\end{cases}
\]
satisfies the conditions of the Fourier sine integral pointwise convergence theorem. Establish
\[
f(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{1 - \cos b\alpha}{\alpha} \sin \alpha x \, d\alpha, \quad x > 0.
\]

**Solution:** $f$ is piecewise smooth on every bounded interval over the positive $x$ axis and is absolutely integrable. For every $x > 0$ $f$ satisfies
\[
\frac{f(x+) + f(x-)}{2} = f(x)
\]
Thus
\[ f(x) = \frac{2}{\pi} \int_0^\infty \sin \alpha x \int_0^\infty f(s) \sin \alpha s \, ds \, d\alpha, \quad x > 0. \]

Now
\[ \int_0^\infty f(s) \sin \alpha s \, ds = \int_0^b \sin \alpha s \, ds = -\frac{\cos \alpha s}{\alpha} \bigg|_0^b = 1 - \frac{\cos \alpha b}{\alpha}, \]
so
\[ f(x) = \frac{2}{\pi} \int_0^\infty \sin \alpha x \frac{1 - \cos \alpha b}{\alpha} \, d\alpha, \quad x > 0. \]