6705 Definition of Infinite Series

A series is the indicated sum of a sequence. For a finite series this sum is a finite sum. We start with a finite list of numbers:

\[ a_1, a_2, a_3 \ldots a_n. \]

We add these numbers together and we get a finite sum

\[ a_1 + a_2 + a_3 + \ldots + a_n. \]

For example, the following list of numbers is a sequence:

\[
\frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \frac{32}{81}, \ldots, \frac{2^{n+1}}{3^n}.
\]

The sum of this list of numbers is the following series.

\[
\frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + \ldots + \frac{2^{n+1}}{3^n}.
\]

We can denote this very same finite series using summation notation as follows:

\[
\sum_{k=1}^{n} \frac{2^{k+1}}{3^k}.
\]

This notation indicates that in order to obtain the sum we substitute the numbers 1,2,3,... n into the formula \(\frac{2^{k+1}}{3^k}\) and add the resulting numbers together, that is,

\[
\sum_{k=1}^{n} \frac{2^{k+1}}{3^k} = \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \frac{32}{81} + \ldots + \frac{2^{n+1}}{3^n}.
\]

What happens to such a sum for very large values of \(n\)? How can we define such a sum if we want to take the limit as \(n \rightarrow +\infty\)? This is a finite geometric series and so we know how to compute its sum. We use the formula \(\sum_{k=1}^{n} a_1 r^{k-1} = a_1 \frac{1 - r^n}{1 - r}\). The sum is

\[
\sum_{k=1}^{n} \frac{2^{k+1}}{3^k} = 4 \left[ 1 - \left(\frac{2}{3}\right)^n \right].
\]
Does \( \lim_{n \to \infty} 4[1 - \left(\frac{2}{3}\right)^n] \) exist? Recall that \( \lim_{n \to \infty} (2/3)^n = 0 \) since \( 0 < 2/3 < 1 \). It follows that \( \lim_{n \to \infty} 4[1 - (2/3)^n] = 4 \). Using this fact, we get

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{2^{k+1}}{3^k} = \lim_{n \to \infty} 4[1 - \left(\frac{2}{3}\right)^n] = 4.
\]

This would seem to be a logical way to define the sum of all (an infinite number) of the terms of \( \sum_{k=1}^{\infty} \frac{2^{k+1}}{3^k} \). Using this idea we define the sum of any infinite series.

Definition 1. An infinite series is the sum of all the terms in an infinite sequence. The sum of an infinite series is denoted by

\[
\sum_{k=1}^{\infty} a_k.
\]

Definition 2. The value of an infinite sum is defined as follows:

\[
\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k.
\]

Look carefully at this definition. It tells us that there are two steps to finding the sum of an infinite series. The first step is to find a formula for the finite sum \( \sum_{k=1}^{n} a_k \). The second step is to take the limit as \( n \to \infty \).

The finite sums \( \sum_{k=1}^{n} a_k \) are sometimes referred to as subtotals. The symbol \( \lim_{n \to \infty} \sum_{k=1}^{n} a_k \) indicates that we keep adding more and more terms of the sum together and hope that the subtotals keep getting closer and closer to some fixed number, that is, the sequence of subtotals approaches a limit.

Example 1. Find the sum of the infinite series

\[
\sum_{k=1}^{\infty} 5(3/4)^{k-1}.
\]
Solution. First, we need to find the formula for the general finite sum

\[ \sum_{k=1}^{n} 5(\frac{3}{4})^{k-1} \]

where \( n \) can be any positive integer. Fortunately, this is a finite geometric series with \( a_1 = 5 \) and \( r = \frac{3}{4} \). Recall that the sum formula for a finite geometric series is

\[ \sum_{k=1}^{n} a_1 r^{k-1} = a_1 \frac{1 - r^n}{1 - r}. \] (3.4)

Substituting \( a_1 = 5 \) and \( r = \frac{3}{4} \) into the formula (3.4), we get the sum of the finite series to be

\[ \sum_{k=1}^{n} 5(\frac{3}{4})^{k-1} = 5 \frac{1 - (\frac{3}{4})^n}{1 - \frac{3}{4}} = 20[1 - (\frac{3}{4})^n]. \]

The subtotal for the sum of first \( n \) numbers is \( 20[1 - (\frac{3}{4})^n] \) definition 2 then say that

\[ \sum_{k=1}^{\infty} 5(\frac{3}{4})^{k-1} = \lim_{n \to \infty} \sum_{k=1}^{n} 5(\frac{3}{4})^{k-1} = \lim_{n \to \infty} 20[1 - (\frac{3}{4})^n]. \]

We need to know the limit \( \lim_{n \to \infty} (\frac{3}{4})^n \). Since \( 0 < \frac{3}{4} < 1 \), we know that \( \lim_{n \to \infty} (\frac{3}{4})^n = 0 \). Therefore,

\[ \sum_{k=1}^{\infty} 5(\frac{3}{4})^{k-1} = 20[1 - 0] = 20. \]

When an infinite series has a sum as this one does, then we say that the series is convergent. We would also say that the sum of this infinite series is 20.

Example 2. Find the sum of the infinite series

\[ \sum_{k=1}^{\infty} 8^{k+15-k}. \]
Solution. Definition 2 of the sum of an infinite series says that

\[ \sum_{k=1}^{\infty} 8^{k+1}5^{-k} = \lim_{n \to \infty} \sum_{k=1}^{n} 8^{k+1}5^{-k}. \]

First, we need to find a general formula for the finite sum. In order to do this, we write the finite geometric series in the standard form \( \sum a_1 r^{k-1} \).

\[ \sum_{k=1}^{n} 8^{k+1}5^{-k} = \sum_{k=1}^{n} 8^2(8^{k-1})5^{-1}(5^{-k+1}) = \sum_{k=1}^{n} \frac{64}{5} \left( \frac{8}{5} \right)^{k-1}. \]

This series is a geometric series with \( a_1 = 64/5 \) and \( r = 8/5 \). Substituting these numbers into the sum formula (3.4) for a finite geometric series, we get

\[ \sum_{k=1}^{n} \frac{64}{5} \left( \frac{8}{5} \right)^{k-1} = \frac{64}{5} \frac{1 - (8/5)^n}{1 - (8/5)} = \frac{64}{3} \left[ \left( \frac{8}{5} \right)^n - 1 \right]. \]

This completes the first step in finding the infinite sum. Next, we must take the limit. Applying Definition 2 for the sum of an infinite series, we get

\[ \sum_{k=1}^{\infty} \frac{64}{5} \left( \frac{8}{5} \right)^{k-1} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{64}{5} \left( \frac{8}{5} \right)^{k-1} = \lim_{n \to \infty} \frac{64}{3} \left[ \left( \frac{8}{5} \right)^n - 1 \right]. \]

We need the value of \( \lim_{n \to \infty} (8/5)^n \). Since \( 8/5 > 1 \), we know that \( \lim_{n \to \infty} (8/5)^n \) does not exist. Recall that \( \lim_{n \to \infty} r^n = 0 \) if \( |r| < 1 \) and \( \lim_{n \to \infty} r^n \) does not exist if \( |r| > 1 \). It follows that

\[ \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{64}{5} \right) \left( \frac{8}{3} \right)^{k-1} \]

does not exist. In this case, we say that the series \( \sum_{k=1}^{\infty} \left( \frac{64}{5} \right) \left( \frac{8}{3} \right)^{k-1} \) is divergent.

Example 3. Does the following infinite series converge or diverge

\[ \sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+3)}. \]
Solution. Definition 2 of the sum of the infinite series says that

$$\sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+3)} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{(4k-1)(4k+3)}.$$

As this equation clearly indicates finding \(\sum_{k=1}^{\infty}\) involves two steps. The first step is to find the finite sum \(\sum_{k=1}^{n}\) and the second step is to take \(\lim_{n \to \infty}\).

We need a formula for the finite sum

$$\sum_{k=1}^{n} \frac{1}{(4k-1)(4k+3)}.$$

We can show using mathematical induction that a formula for this finite sum is

$$\sum_{k=1}^{n} \frac{1}{(4k-1)(4k+3)} = \frac{n}{3(4n+3)}.$$

The definition of infinite series tells us that

$$\sum_{k=1}^{\infty} \frac{1}{(4k-1)(4k+3)} = \lim_{n \to \infty} \frac{n}{3(4n+3)} = \lim_{n \to \infty} \frac{1}{3\left(4 + \frac{3}{n}\right)} = \frac{1}{12}.$$

This infinite series converges and its sum is 1/12.

Example 4. Does the infinite series \(\sum_{k=1}^{\infty} (3k - 11)\) converge?

Solution. Definition 2 of the sum of an infinite series says that

$$\sum_{k=1}^{\infty} (3k - 11) = \lim_{n \to \infty} \sum_{k=1}^{n} (3k - 11).$$
We need a formula for the general finite sum \( \sum_{k=1}^{n} (3k - 11) \). Writing out the sum can sometimes help.

\[
\sum_{k=1}^{n} (3k - 11) = -8 - 5 - 2 + 1 + 4 + \ldots + (3n - 11).
\]

We see that this is an arithmetic sequence with first term \( a_1 = -8 \) and common difference \( d = 3 \). The formula for the sum of \( n \) of an arithmetic series is

\[
S_n = \frac{n}{2} [2a_1 + (n - 1)d].
\]

Substituting \( a_1 = -8 \) and \( d = 3 \) into this formula we get

\[
\sum_{k=1}^{n} (3k - 11) = \frac{n}{2} [2(-8) + (n - 1)3] = \frac{n}{2} (3n - 19).
\]

We have a formula for the sum of \( n \) terms. This completes the first step. In order to find the infinite sum, we take the limit:

\[
\lim_{n \to \infty} \sum_{k=1}^{n} (3k - 11) = \lim_{n \to \infty} \frac{n}{2} (3n - 19).
\]

As \( n \) gets larger and larger the numbers \( (n/2)(3n - 19) \) get even larger and larger. The limit does not exist. The series

\[
\sum_{k=1}^{\infty} (3k - 11)
\]

is divergent.

Example 5. Show that the series \( \sum_{k=1}^{\infty} \frac{1}{k(k + 1)} \) converges and find its value.

Solution. We can show using mathematical induction that

\[
\sum_{k=1}^{n} \frac{1}{k(k + 1)} = 1 - \frac{1}{n + 1}.
\]
for all values of \( n \). Definition 2 says that

\[
\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1.
\]

The infinite series \( \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \) converges and its value is 1.

Problem. Find the repeating decimal \( 8.\overline{45} \) as a fraction. Recall that \( 8.\overline{45} = \frac{45}{100} + \frac{45}{(100)^2} + \frac{45}{(100)^3} + \ldots \).

Solution. The repeating decimal \( 0.\overline{45} \) is defined as

\[
.\overline{45} = \frac{45}{100} + \frac{45}{10,000} + \frac{45}{1,000,000} + \ldots \\
= \sum_{k=1}^{\infty} \frac{45}{100} \left( \frac{1}{100} \right)^{k-1} = \sum_{k=1}^{\infty} \frac{45}{100} \left( \frac{1}{100} \right)^{k-1}.
\]

The finite sum \( \sum_{k=1}^{n} \frac{45}{100} \left( \frac{1}{100} \right)^{k-1} \) is a geometric series with \( a_1 = 45/100 \) and \( r = 1/100 \). The formula (\(*\)) for the sum of a finite geometric series tells us that

\[
\sum_{k=1}^{n} \frac{45}{100} \left( \frac{1}{100} \right)^{k-1} = \frac{45 \left( 1 - (1/100)^n \right)}{100 \left( 1 - (1/100) \right)} = \frac{45}{99} \left[ 1 - \left( \frac{1}{100} \right)^n \right].
\]

Definition 2 of the sum of an infinite series says

\[
\sum_{k=1}^{\infty} \frac{45}{100} \left( \frac{1}{100} \right)^{k-1} = \lim_{n \to \infty} \frac{45}{99} \left[ 1 - \left( \frac{1}{100} \right)^n \right] = \frac{45}{99} = \frac{5}{11}.
\]

Recall that \( \lim_{n \to \infty} (1/100)^n = 0 \). The limit exists and so the series is convergent to the value 5/11. It follows that

\[
8.\overline{45} = 8 + .\overline{45} = 8 + (5/11) = (93/11).
\]
Theorem. If the infinite series \( \sum_{k=1}^{\infty} a_k \) is convergent, then \( \lim_{k \to \infty} a_k = 0 \).

Proof. Let \( s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \). It follows that \( s_n - s_{n-1} = a_n \). When we say \( \sum_{k=1}^{\infty} a_k \) is convergent, then following Definition 2 we are saying that there exists a limit \( s \) such that \( \lim_{n \to \infty} s_n = s \). Also \( \lim_{n \to \infty} s_{n-1} = s \). Therefore,

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (s_n - s_{n-1}) = \lim_{n \to \infty} s_n - \lim_{n \to \infty} s_{n-1} = s - s = 0.
\]

Thus \( \lim_{k \to \infty} a_k = 0 \). This theorem does not say: If \( \lim_{k \to \infty} a_k = 0 \), then \( \sum_{k=1}^{\infty} a_k \) is convergent. In fact this last statement is false. This theorem may be restated as follows: If \( \lim_{k \to \infty} a_k \) does not exist or \( \lim_{k \to \infty} a_k \neq 0 \), then the series \( \sum_{k=1}^{\infty} a_k \) is divergent (not convergent). This is really two theorems.

Theorem 1. If \( \lim_{k \to \infty} a_k \) does not exist, then the series \( \sum_{k=1}^{\infty} a_k \) is divergent.

Theorem 2. If \( \lim_{k \to \infty} a_k = L \) and \( L \neq 0 \), then \( \sum_{k=1}^{\infty} a_k \) is divergent.

Example 6. Show that the series \( \sum_{k=1}^{\infty} \frac{2k+1}{8k + 9} \) is divergent.

Solution. Replace \( a_k \) by \( \frac{2k+1}{8k + 9} \) in Theorem 2 and we get the true statement.

(a) If \( \lim_{k \to \infty} \frac{2k+1}{8k + 9} = \frac{1}{4} \) and \( \frac{1}{4} \neq 0 \), then \( \sum_{k=1}^{\infty} \frac{2k+1}{8k + 9} \) is divergent.
Clearly \( \lim_{k \to \infty} \frac{2k + 1}{8k + 9} = \lim_{k \to \infty} \frac{2 + 1/k}{8 + 9/k} = \frac{2}{8} = \frac{1}{4} \). Also \( \frac{1}{4} \neq 0 \). Since the hypotheses of the statement (a) is true we are able to conclude that the conclusion is true. Therefore,

\[
\sum_{k=1}^{\infty} \frac{2k + 1}{8k + 9} \text{ is divergent.}
\]

Example 7. Show that the series \( \sum_{k=1}^{\infty} (-1)^k \frac{3k + 1}{7k + 4} \) is divergent.

Solution. Let us find the limit \( \lim_{k \to \infty} (-1)^k \frac{3k + 1}{7k + 4} \). First, note that

\[
\lim_{k \to \infty} \frac{3k + 1}{7k + 4} = \lim_{k \to \infty} \frac{3 + 1/k}{7 + 4/k} = \frac{3}{7}.
\]

But multiplying by \((-1)^k\) causes the signs to alternate. The sequence tries to get close to \(3/7\) and then to \(-3/7\) as \(k\) gets large. It follows that

a) \( \lim_{k \to \infty} (-1)^k \frac{3k + 1}{7k + 4} \) does not exist.

Replace \(a_k\) with \((-1)^k \frac{3k + 1}{7k + 4}\) in Theorem 1. The result is

b) If \( \lim_{k \to \infty} (-1)^k \frac{3k + 1}{7k + 4} \) does not exist, then \( \sum_{k=1}^{\infty} (-1)^k \frac{3k + 1}{7k + 4} \) diverges. From the true statements (a) and (b) we conclude:

The series \( \sum_{k=1}^{\infty} (-1)^k \frac{3k + 1}{7k + 4} \) diverges.
Exercises

1. a) Find a formula for the sum $S_n$ of $n$ terms of the geometric series

$$S_n = \sum_{k=1}^{n} 2^{3k+1} \cdot 5^{k-1}.$$ 

b) Find $\lim_{n \to \infty} S_n$.

c) Explain why the infinite series $\sum_{k=1}^{\infty} 2^{3k+1} \cdot 5^{k-1}$ is convergent. What is its sum?

2. a) Find a formula for the sum $S_n$ of $n$ terms of the geometric series

$$\sum_{k=1}^{n} 4^{k-1} \cdot 3^{k+1}.$$ 

b) Find $\lim_{n \to \infty} S_n$.

c) Explain why the infinite series $\sum_{k=1}^{\infty} 4^{k-1} \cdot 3^{k+1}$ is divergent.

3. a) Find the formula for $S_n$ the sum of the first $n$ terms of the arithmetic series

$$\sum_{k=1}^{n} (2k - 7).$$ 

b) Find $\lim_{n \to \infty} S_n$.

c) Explain why the infinite series $\sum_{k=1}^{\infty} (2k - 7)$ is divergent.

4. Given that $\sum_{k=1}^{n} \frac{1}{(5k - 3)(5k + 2)} = \frac{n}{2(5n + 2)}$.

Explain why $\sum_{k=1}^{\infty} \frac{1}{(5k - 3)(5k + 2)} = \frac{1}{10}$.
5. Show that the following infinite geometric series is convergent. Be sure to clearly include both steps.

\[ \sum_{k=1}^{\infty} 15(5/8)^{k-1}. \]

6. Given that \( \sum_{k=1}^{n} \frac{1}{(4k + 5)(4k + 9)} = \frac{n}{9(4n + 9)} \). Is the infinite series \( \sum_{k=1}^{\infty} \frac{1}{(4k + 5)(4k + 9)} \) convergent or divergent?

7. Show that the series \( \sum_{k=1}^{\infty} \frac{k + 1}{8k + 5} \) is divergent using Theorem 2.

8. Show that the series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k + 2)}{10k + 9} \) is divergent using Theorem 1.

9. Find the repeating decimal \( 6.\overline{4} = 6.444 \) as a fraction.
6707 The Comparison Test

We can often determine if a given series of positive terms is convergent or divergent by comparing it with another simpler series. The idea is to compare a series whose convergence or divergence is in question with a series that is known to be convergent or that is known to be divergent.

Theorem 1. The Comparison Test. Suppose that \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are series of positive terms, that is, \( a_n > 0 \) and \( b_n > 0 \).

1. If \( \sum_{n=1}^{\infty} b_n \) is convergent and if \( a_n \leq b_n \) for all \( n \), then \( \sum_{n=1}^{\infty} a_n \) is also convergent.

2. If \( \sum_{n=1}^{\infty} b_n \) is divergent and if \( a_n \geq b_n \) for all \( n \), then \( \sum_{n=1}^{\infty} a_n \) is also divergent.

The proof of this theorem involves the idea that a bounded increasing sequences will always have a limit. We have chosen to skip this idea in our study and as a consequence we are unable to give a proof of this theorem. Both parts of this theorem are also true if we replace “for all \( n \)” with “\( n \geq \) some positive integer” such as “for \( n \geq 5 \)”.

Theorem 2. If \( \sum a_n \) is known to be convergent (divergent) and \( c \) is a fixed constant, then \( \sum c a_n \) is convergent (divergent) and \( \sum c a_n = c \sum a_n \).

In the previous section on the definition of infinite series we found that the series \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) is convergent and its sum is 1. Starting with this sum Theorem 2, tells us that the series \( \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \) is convergent and its sum is 2.

When we say that part (1) or part (2) of the comparison test is true we are saying that we can replace \( a_n \) by a specific expression involving \( n \) and replace \( b_n \) with another specific expression involving \( n \) and the sentence
that results will be a true statement. For example, if we replace \( a_n \) by \( \frac{3n+5}{4n^3+7} \) and \( b_n \) by \( \frac{1}{n^2} \) in part (1), we get the sentence

If \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent and if \( \frac{3n + 5}{4n^3 + 7} \leq \frac{1}{n^2} \) for all \( n \), then \( \sum_{n=1}^{\infty} \frac{3n + 5}{4n^3 + 7} \) is convergent.

This is a true statement. Note that this statement does not say that
\[
\sum_{n=1}^{\infty} \frac{3n + 5}{4n^3 + 7}
\]
is convergent.

Another example. If we replace \( a_n \) with \( \frac{2n + 1}{5n^2 + 8} \) and \( b_n \) with \( \frac{1}{3n} \) in part (2) of the Comparison test, we get:

If \( \sum_{n=1}^{\infty} \frac{1}{3n} \) is divergent and if \( \frac{2n + 1}{5n^2 + 8} \geq \frac{1}{3n} \) for all \( n \), then \( \sum_{n=1}^{\infty} \frac{2n + 1}{5n^2 + 8} \) is divergent. This is a true statement. This statement does not say that
\[
\sum_{n=1}^{\infty} \frac{2n + 1}{5n^2 + 8}
\]
is divergent.

We will denote infinite series using both \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{n=1}^{\infty} a_n \). These two notations have the same meaning that is, \( \sum_{k=1}^{\infty} \frac{2k + 1}{5k^2 + 8} \) is the same as
\[
\sum_{n=1}^{\infty} \frac{2n + 1}{5n^2 + 8}.
\]

Problem. Show that \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent using the Comparison Test.

Solution. The comparison test contains two separate statements. In order to prove that a series converges we must use the statement which is part (1). We start by substituting for \( a_n \) and \( b_n \). We are trying to prove that \( \sum \frac{1}{n^2} \) converges. This means we must replace \( a_n \) with \( \frac{1}{n^2} \). We need to figure out what to use to replace \( b_n \). For this example we will just assume that this is known.
Replacing \( a_n \) by \( 1/n^2 \) and \( b_n \) by \( 2/n(n+1) \) is part (1) of the Comparison Test, we get the following true statement. If \( \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \) is convergent and if \( \frac{1}{n^2} < \frac{2}{n(n+1)} \), then \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent. As stated above we showed in the previous section that \( \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \) is convergent. Next, \( \frac{1}{n^2} \leq \frac{2}{n(n+1)} \) is true if \( n(n+1) \leq 2n^2 \) which is true if \( n \leq n^2 \) or \( 1 \leq n \). Since \( n \geq 1 \), it follows that \( \frac{1}{n^2} < \frac{2}{n(n+1)} \) for all \( n \). Since both parts of the hypotheses of the above statement are true the conclusion is true. Therefore we conclude:

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent}. \]

The statement “If \( \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \) is convergent and \( \frac{1}{n^2} < \frac{2}{n(n+1)} \), then \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent” is called an implication or conditional statement. The phrase after “if” and before “then” is called the hypothesis. The phrase after “then” is called the conclusion. For this implication the hypothesis is “\( \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \) is convergent and \( \frac{1}{n^2} < \frac{2}{n(n+1)} \)” the conclusion is “\( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent”.

The statement which is this problem conforms to our general thinking. We know that \( \sum_{n=1}^{\infty} \frac{2}{n(n+1)} = 2 \) and \( \frac{1}{n^2} < \frac{2}{n(n+1)} \). The individual terms in \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) are less than the individual terms in \( \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \) and so the subtotals for \( \sum_{n=1}^{\infty} 1/n^2 \) can only add up to something less than 2. The infinite sum \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) must be less than 2.
For a series of positive terms the more terms you add together the larger the subtotals get. If a series of positive terms is divergent (not convergent), then its sum must be plus infinity. For a divergent series of positive terms the subtotals approach infinity.

Corollary. The series \( \sum_{n=1}^{\infty} \frac{c}{n^2} \) is convergent for any fixed number \( c \).

Theorem 3. The harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent.

A proof of Theorem 3 uses the inequality \( \sum_{k=1}^{n} \frac{1}{k} > \int_{1}^{n+1} \frac{dx}{x} \geq \ln n \) and the fact that \( \lim_{n \to \infty} \ln n = +\infty \).

Example 1. Show that \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}} \) is divergent by comparing it to the harmonic series. This means use the Comparison Theorem.

Solution. In order to prove that a series diverges using the comparison test we must substitute into part 2. In the end we want to say that \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}} \) is divergent. This means we must use \( a_n = \frac{1}{\sqrt{2n+1}} \). We are comparing with the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \). This means that we must use \( b_n = \frac{1}{n} \). Replacing \( a_n \) by \( 1/\sqrt{2n+1} \) and \( b_n \) by \( 1/n \) in part (2) of the Comparison Test Theorem, we get the following true statement:

If \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent and if \( \frac{1}{\sqrt{2n+1}} \geq \frac{1}{n} \), then \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}} \) is divergent.

The fact that part (2) of the comparison test is a true theorem, means that the above statement which was obtained by substituting into the comparison test part (2) is a true statement. The hypothesis of this statement is: "\( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent and \( \frac{1}{\sqrt{2n+1}} \geq \frac{1}{n} \). The conclusion is "\( \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}} \)
is divergent”. We need to show that the hypothesis is true.

By Theorem 3 \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent.

The inequality \( \frac{1}{\sqrt{2n+1}} \geq \frac{1}{n} \) is true if \( n \geq \sqrt{2n+1} \) which is true if \( n^2 \geq 2n + 1 \). Now \( n^2 \geq 2n + 1 \) is true for all \( n \) except \( n = 1 \) and \( n = 2 \). We can still apply the Comparison Theorem even though the condition \( n^2 \geq 2n + 1 \) is not satisfied for a couple of values of \( n \). Since both parts in the hypothesis of the above statement are true, it follows that the conclusion is true. Therefore, we conclude:

\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}} \text{ is divergent.}
\]

Theorem 4. The p-series theorem. The p series \( \sum_{n=1}^{\infty} \frac{c}{n^p} \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \), where \( c \) is a constant.

This p series theorem says for example that \( \sum \frac{1}{n^3} \) and \( \sum \frac{4}{n^{5/2}} \) are convergent and that \( \sum \frac{1}{\sqrt{n}} \) and \( \sum \frac{5n^{-2/3}}{n} \) are divergent.

Example 2. Show that the following series is convergent by comparing it with the series \( \sum_{n=1}^{\infty} \frac{6}{n^3} \):

\[
\sum_{n=1}^{\infty} \frac{5n + 3}{n^2(n^2 + 2)}.
\]

Solution. We are trying to conclude that a series is convergent using the comparison test. In order to do this we must substitute into part (1). The statement of the problem makes it clear what expressions must be substituted for \( a_n \) and \( b_n \). Replacing \( a_n \) by \( \frac{5n + 3}{n^2(n^2 + 2)} \) and \( b_n \) by \( \frac{6}{n^3} \) in part (1) of the comparison test we obtain the true statement.

(a) If \( \sum_{n=1}^{\infty} \frac{6}{n^3} \) is convergent and if \( \frac{5n + 3}{n^2(n^2 + 4)} < \frac{6}{n^3} \), then \( \sum_{n=1}^{\infty} \frac{5n + 3}{n^2(n^3 + 4)} \) is convergent.
(b) The series \( \sum_{n=1}^{\infty} \frac{6}{n^3} \) is convergent because it is a \( p \) series with \( p = 3 \) and \( c = 6 \).

The inequality \( \frac{5n + 3}{n^2(n^2 + 4)} < \frac{6}{n^3} \) is true if \( 5n^2 + 3n < 6n^2 + 24 \) which is true if \( 3n < n^2 + 12 \) which is clearly true for all \( n \). Therefore,

\[
\frac{5n + 3}{n^2(n^2 + 2)} < \frac{6}{n^3} \quad \text{for all } n.
\]

The statement (a), (b), and (c) taken together say that

\[
\sum_{n=1}^{\infty} \frac{5n + 3}{n^2(n^2 + 2)} \quad \text{is convergent.}
\]

Example 3. Is the following series convergent or divergent?

\[
\sum_{n=1}^{\infty} \frac{3n + 1}{n(n^2 + 4)}.
\]

Solution. The first step in the solution of this problem is to decide the question: do we think that the series is convergent or do we think that it is divergent? There are two parts or statements to the Comparison Theorem. Based on the answer to this question we decide which of these two parts we want to substitute into. In order to decide this we first get an estimate of the size of the terms in the series. We find this estimate as follows:

\[
\frac{3n + 1}{n(n^2 + 4)} = \frac{3 + 1/n}{n^2 + 4} \approx \frac{3}{n^2}.
\]

Since \( \sum 3/n^2 \) is convergent, our guess is that \( \sum_{n=1}^{\infty} \frac{3n + 1}{n(n^2 + 4)} \) is convergent.

We now try to prove that this guess is correct using the Comparison Theorem. We replace \( a_n \) by \( \frac{3n+1}{n(n^2+4)} \) and \( b_n \) by \( \frac{4}{n^2} \) (a little larger than \( \frac{3}{n^2} \)) in the Comparison Theorem and we get the following true statement.
(a) If $\sum_{n=1}^{\infty} \frac{4}{n^2}$ is convergent and if $\frac{3n+1}{n(n^2+4)} \leq \frac{4}{n^2}$, then $\sum_{n=1}^{\infty} \frac{3n+1}{n(n^2+1)}$ is convergent.

(b) The series $\sum_{n=1}^{\infty} \frac{4}{n^2}$ is convergent because it is a $p$-series with $p = 2$ and $c = 4$.

Cross multiplying we see that $\frac{3n+1}{n(n^2+4)} < \frac{4}{n^2}$ is true if $(3n+1)n < 4(n^2+4)$ which is true if $3n^2 + n < 4n^2 + 16$ which is true if $n \leq n^2 + 16$. Since $n \leq n^2 + 16$ is clearly true, it follows that

$$\frac{3n+1}{n(n^2+4)} < \frac{4}{n}$$

is also true for all $n$.

The statements (b) and (c) say that the hypothesis of the implication (a) are true. The statements (a), (b), and (c) taken together say that

$$\sum_{n=1}^{\infty} \frac{3n+1}{n(n^2+4)}$$

is convergent.

Problem. Does the series $\sum_{n=1}^{\infty} \frac{2n+1}{5n^2 + 4}$ converge or diverge? Justify your answer.

Solution. First, we must decide, do we think the series converges or do we think the series diverges? We want to use the Comparison Theorem. In order to decide this question we estimate the size of the terms of the series as follows:

$$\frac{2n+1}{5n^2 + 4} = \frac{2 + 1/n}{5n + 4/n} \approx \frac{2}{5n}.$$

Since $\sum 2/5n$ diverges, we guess that $\sum_{n=1}^{\infty} \frac{2n+1}{5n^2 + 4}$ also diverges.

We now set out to prove that this guess is really correct. We replace $a_n$ by $\frac{2n+1}{5n^2 + 4}$ and $b_n$ by $\frac{1}{3n}$ in Part (2) of the Comparison Theorem. Note that $\frac{1}{3n} < (2/5n)$. This gives the following true statement:
(a) If \( \sum_{n=1}^{\infty} \frac{1/3}{n} \) is divergent and \( \frac{2n + 1}{5n^2 + 4} \geq \frac{1}{3n} \), then \( \sum_{n=1}^{\infty} \frac{2n + 1}{5n^2 + 4} \) is divergent.

(b) The series \( \sum \frac{1/3}{n} \) is divergent because it is a \( p \) series with \( p = 1 \) and \( c = 1/3 \).

Now \( \frac{1}{3n} < \frac{2n + 1}{5n^2 + 4} \) is true if \( 5n^2 + 4 < 6n^2 + 3n \) which is true if \( 4 \leq n^2 + 3n \). It is clearly true that \( 4 \leq n^2 + 3n \). It follows that

\[
\frac{1}{3n} < \frac{2n + 1}{5n^2 + 4} \quad \text{for all } n.
\]

Since both parts of the hypothesis of the above statement are true, we are able to conclude that

\[
\sum_{n=1}^{\infty} \frac{2n + 1}{5n^2 + 4} \text{ is divergent.}
\]

Problems

1. Replace \( a_n \) by \( \frac{2n+3}{(n+1)^2(n+2)^2} \) and \( b_n \) by \( \frac{5}{n^5} \) in Part 1 of the Comparison Theorem.

   (a) What is the hypothesis of the statement that results? What is the conclusion?

   (b) Is the resulting statement true? Why?

2. Replace \( a_n \) by \( \frac{n+5}{(n+1)(3n+2)} \) and \( b_n \) by \( \frac{1}{3n} \) in part 2 of the Comparison Theorem.

   (a) What is the hypothesis of the statement that results?

   (b) Does this statement say that \( \sum_{n=1}^{\infty} \frac{(n + 5)}{(n + 1)(3n + 2)} \) is divergent?”?

3. Given that the following three statements are true. The three statements (a), (b), and (c) taken together enable us to conclude that what other statement is true.
(a) If $\sum_{n=1}^{\infty} \frac{8}{n^2}$ converges and $\frac{7n^2 + 5}{n^4 + 8n^2 + 3} < \frac{8}{n^2}$ then $\sum_{n=1}^{\infty} \frac{7n^2 + 5}{n^4 + 8n^2 + 3}$ converges.

(b) The series $\sum_{n=1}^{\infty} \frac{8}{n^2}$ converges.

(c) The inequality $\frac{7n^2 + 5}{n^4 + 8n^2 + 3} < \frac{8}{n^2}$ is true for all $n$.

4. Given that each of the following three statements is true, what other statement are you able to conclude is also true?

(a) If $\sum_{n=1}^{\infty} \frac{1}{5n^{1/2}}$ diverges and $\frac{\sqrt{4n + 5}}{9n + 4} > \frac{1}{5n^{1/2}}$, then $\sum_{n=1}^{\infty} \frac{\sqrt{4n + 5}}{9n + 4}$ diverges.

(b) The series $\sum_{n=1}^{\infty} \frac{1}{5n^{1/2}}$ diverges.

(c) The inequality $\frac{\sqrt{4n + 5}}{9n + 4} > \frac{1}{5n^{1/2}}$ is true for all $n$.

5. Estimate the size of $\frac{3n^2 + 1}{(n + 4)(n^2 + 5)}$ for large $n$.

6. Show that $\frac{4n^2 + 3}{(n + 5)(n^2 + 4)} < \frac{4}{n}$.

7. For what values of $n$ is $\frac{1}{\sqrt{2n + 5}} \geq \frac{1}{n}$.

8. Show that the series $\sum_{n=1}^{\infty} \frac{7n + 4}{n(n^2 + 5)}$ is convergent by comparing it with $\sum_{n=1}^{\infty} \frac{8}{n^2}$. 
9. Does the series \( \sum_{n=1}^{\infty} \frac{5n + 8}{n(2n^2 + 3)} \) converge or diverge? Justify your answer using the Comparison Theorem.

10. Does the series \( \sum_{n=1}^{\infty} \frac{\sqrt{4n + 3}}{5n + 8} \) converge or diverge. Justify your answer using the Comparison Theorem.