An Analysis of HDG Methods for the Vorticity-Velocity-Pressure Formulation of the Stokes Problem

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Abstract

In this work we provide the a priori error estimates of a hybridizable discontinuous Galerkin (HDG) method for solving the vorticity-velocity-pressure formulation of the three-dimensional Stokes equations of incompressible fluid flow. By using a projection-based approach, we prove that, when all the unknowns use polynomials of degree \( k \geq 1 \), the \( L^2 \) norm of the errors in the approximate vorticity and pressure converge to zero with order \( k + 1/2 \) whereas the error in the approximate velocity converges with order \( k + 1 \).

1. Introduction

1.1 Statement of Problem

We consider the following vorticity-velocity-pressure formulation of the Stokes equations of incompressible fluid flow:

\[
\begin{align*}
\nu - \nabla \times \mathbf{u} &= 0 &\text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 &\text{in } \Omega, \\
\mathbf{u} &= g &\text{on } \partial \Omega, \\
\end{align*}
\]

(1a) (1b) (1c)

2. HDG Method

Let \( \mathcal{T}_h \) be a shape-regular triangulation of \( \Omega \) which consists of tetrahedra \( T \). We denote by \( E_T \) the set of faces \( F \) of all tetrahedra \( T \) of the triangulation \( \mathcal{T}_h \) and by \( |F| \), the set of boundaries of \( F \) of the elements \( T \) of \( \mathcal{T}_h \).

The HDG method \cite{Cockburn2017} seeks an approximation \((\mathbf{w}_h, \mathbf{v}_h, p_h, q_h) \in W_h \times V_h \times P_h \times \mathcal{Q}_h\) for the exact solution \((\mathbf{w}, \mathbf{v}, p, q) \in \mathcal{W} \times \mathcal{V} \times \mathcal{P} \times \mathcal{Q}\) of the problem (1), where

\[
\begin{align*}
W_h &= \{ w \in L^2(\Omega) \cap H^1(\Omega) : w|_F \in P_1(F), \quad \forall F \in \mathcal{T}_h \}, \\
V_h &= \{ v \in L^2(\Omega) \cap H^1(\Omega) : v|_F \in P_1(F), \quad \forall F \in \mathcal{T}_h \}, \\
P_h &= \{ p \in L^2(\Omega) : p|_F \in P_1(F), \quad \forall F \in \mathcal{T}_h \}, \\
\mathcal{Q}_h &= \{ q \in L^2(\Omega) : \int_{\partial T} q n \, ds = 0, \quad \forall T \in \mathcal{T}_h \}.
\end{align*}
\]

(2a) (2b) (2c) (2d)

3. Error Analysis

In this section, we present estimates of the projection of the approximation errors, namely,

\[
\begin{align*}
\mathbf{v}_h &= \mathbf{w} - \mathbf{w}_h, & \mathbf{v} &= \mathbf{w} - \mathbf{w}_h, & \mu &= \mathbf{v} - \mathbf{v}_h, & \mathbf{p} &= \mu - \mathbf{p}_h, & \mathbf{q} &= \mu. \\
\end{align*}
\]

(3a) (3b) (3c) (3d) (3e)

3.1 Error Equations

The projection of the errors satisfies the following equations:

\[
\begin{align*}
\nu \mathbf{v}_h - \nabla \times \mathbf{w}_h &= 0 &\text{in } \Omega, \\
\nabla \cdot \mathbf{v}_h &= 0 &\text{in } \Omega, \\
\mathbf{v}_h &= g &\text{on } \partial \Omega, \\
\end{align*}
\]

(4a) (4b) (4c)

Here the stabilization function \( s \) and \( \eta_{\mathbf{n}} \) are taken to be constants on each face of \( \partial \Omega \).

Features of HDG Methods \cite{Cockburn2017}

- \( \mathbf{w}_h \) is the simple \( L^2 \)-projection of \( \mathbf{w} \) into \( P_1(\mathcal{T}_h) \).
- \( \mathbf{v}_h \) is nothing but the projection used in the analysis of HDG methods for diffusion problems in \cite{Cockburn2017} with the stabilization parameter \( \epsilon \) therein replaced by \( 1/\tau \).

3.2 Error Estimates

\[\begin{align*}
\| \mathbf{w} - \mathbf{w}_h \|_{L^2(\Omega)} + \| \mathbf{v} - \mathbf{v}_h \|_{L^2(\Omega)} &\leq C \eta_{\mathbf{n}}^{1/2} h^{k+1/2} \| \mathbf{w} - \mathbf{w}_h \|_{H^{k+1}(\Omega)} + C h^{k+1/2} \| \mathbf{v} - \mathbf{v}_h \|_{H^{k+1}(\Omega)}, \\
\| \mathbf{p} - \mathbf{p}_h \|_{L^2(\Omega)} &\leq C \eta_{\mathbf{n}}^{1/2} h^{k+1/2} \| \mathbf{p} - \mathbf{p}_h \|_{H^{k+1}(\Omega)} + C h^{k+1/2} \| \mathbf{p} - \mathbf{p}_h \|_{H^{k+1}(\Omega)}, \\
\| \mathbf{q} \|_{L^2(\Omega)} &\leq C \eta_{\mathbf{n}}^{1/2} h^{k+1/2} \| \mathbf{q} \|_{H^{k+1}(\Omega)} + C h^{k+1/2} \| \mathbf{q} \|_{H^{k+1}(\Omega)}.
\end{align*}\]

References