Tropical Varieties

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The tropical semi-ring

In the tropical semi-ring \((\mathbb{R} \cup \{-\infty\}, \oplus, \odot)\) the operations are
\[ \oplus \text{ maximum} \]
\[ \odot \text{ addition} \]

Two examples:
\[ 5 \odot (3 \oplus 2) = 8 \]
\[ 5 \odot 3 \oplus 5 \odot 2 = 8 \]

The neutral element for \(\oplus\) is \(-\infty\).
The neutral element for \(\odot\) is 0.
A tropical monomial in \( n \) variables:

\[
\begin{align*}
    c \odot x^v &= c \odot x_1 \odot \cdots \odot x_1 \odot \cdots \odot x_n \odot \cdots \odot x_n \\
    &\quad \begin{array}{c}
        \text{\( v_1 \) times} \\
        \text{\( v_n \) times}
    \end{array}
\end{align*}
\]

where \( c \in \mathbb{R} \) and \( v \in \mathbb{N}^n \).

A tropical polynomial is a finite tropical sum of tropical monomials with different exponent vectors.

Evaluating a tropical monomial with zero-coefficient in a point \( \omega \in \mathbb{R}^n \):

\[
    x^v(\omega) = \omega_1 \odot \cdots \odot \omega_1 \odot \cdots \odot \omega_n \odot \cdots \odot \omega_n = v \cdot \omega \\
    \begin{array}{c}
        \text{\( v_1 \) times} \\
        \text{\( v_n \) times}
    \end{array}
\]

To evaluate a tropical polynomial in \( \omega \in \mathbb{R}^n \) we evaluate its terms and take the maximum.
Tropical polynomials

- Evaluating a polynomial $f$ with zero coefficients in $\omega \in \mathbb{R}^n$ is equivalent to solving the following optimization problem:

$$\text{maximize } \omega \cdot \nu$$

subject to $\nu \in NP(f)$

where $NP(f)$ denotes the Newton Polytope of $f$.

- Tropical polynomial functions are piecewise linear.

- The regions of linearity are the outer normal cones of $NP(f)$.

Example

$$f = 0 \circ x_1^2 x_2 \oplus x_1^6 x_2 \oplus x_1^6 x_2^2$$
$$\oplus x_1^5 x_2^4 \oplus x_1^4 x_2^6 \oplus x_1 x_2^5 \oplus x_1^2 x_2^4$$
Tropical hypersurfaces

Define the zero-set of a tropical polynomial $f$ in $n$-variables as:

$$T(f) = \{ \omega \in \mathbb{R}^n : f(\omega) \text{ is a attained by at least two terms in } f \}$$

For zero-coefficient polynomials $T(f)$ is the union of all non-maximal cones in $\text{NF}(\text{NP}(f))$, where NF denotes the normal fan. The zero-set is also called a tropical hypersurface.

Example

$T(x_1 \oplus x_2 \oplus x_3) \subseteq \mathbb{R}^3$ is the union of three 2-dimensional cones:
“Zero-coefficients” is no restriction

Hypersurfaces of zero-coefficient polynomials are star-shaped. But zero-coefficient are no real restriction:

\[ p = (3) \oplus (2) \odot y \oplus (2) \odot x \oplus (0) \odot x \odot x \]

\[ p(x, y) = \max(3, 2 + y, 2 + x, 2x) \]

Question: When is the maximum attained twice?

\[ q = t^3 \oplus t^2 \odot y \oplus t^2 \odot x \oplus x \odot x \]

\[ q(x, y, t) = \max(3t, 2t + y, 2t + x, 2x) \]

Answer: When \( q(x, y, t) \) is attained twice and \( t = 1 \).

\[ T(p) \times \{1\} = T(q) \cap (\mathbb{R}^2 \times \{1\}) \]
Tropicalization

The tropicalization of a polynomial \( f \in k[x_1, \ldots, x_n] \) is the tropical polynomial \( \text{trop}(f) \) where

- \( + \) and \( \cdot \) have been changed to \( \oplus \) and \( \ominus \).
- The coefficients have been changed to 0.

**Example**

\[
\text{trop}(1x_1^2x_2 + 2x_1^6x_2 + 3x_1^6x_2^2 + 4x_1^5x_2^4 + 5x_1^4x_2^6 + 6x_1x_2^5 + 7x_1^2x_2^4) = 0 \ominus x_1^2x_2 \oplus x_1^6x_2 \oplus x_1^6x_2^2 \oplus x_1^5x_2^4 \oplus x_1^4x_2^6 \oplus x_1x_2^5 \oplus x_1^2x_2^4
\]

- For a field with a valuation we may consider an alternative definition where we take the valuation of the coefficients to get the tropical coefficients.
Tropical varieties

- For \( f \in k[x_1, \ldots, x_n] \) we may write \( T(f) \) for \( T(\text{trop}(f)) \).
- For \( I \subseteq k[x_1, \ldots, x_n] \) the tropical variety of \( I \) is defined as

\[
T(I) := \bigcap_{f \in I} T(f)
\]

Lemma
For a principal ideal \( \langle f \rangle \subseteq k[x_1, \ldots, x_n] \) we have

\[
T(\langle f \rangle) = T(f).
\]

Is every tropical variety a finite intersection of hypersurfaces?
Example: Grassmann 2,5

Consider the 10 2x2 minors of a 2x5 matrix

\[
\begin{pmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25}
\end{pmatrix}
\]

\[
a = x_{11}x_{22} - x_{12}x_{21}
\]

\[
b = \ldots
\]

The five relations

\[
bf - ah - ce = 0
\]

\[
bg - ai - de = 0
\]

\[
\ldots
\]

generate the Grassmann-Plücker ideal \( G_{2,5} \).

The tropical variety of \( G_{2,5} \) is a subset of \( \mathbb{R}^{10} \).

It is the intersection of the hypersurfaces of the 5 relations.

To be continued...
Consider the polynomial ring $k[x_1, \ldots, x_n]$. Let $\omega \in \mathbb{R}^n$.

- The weight of a monomial $x_1^{a_1} \cdots x_n^{a_n}$ with $a \in \mathbb{N}^n$ is $\langle \omega, a \rangle$.
- The initial form $\text{in}_\omega(f)$ of a polynomial $f \in k[x_1, \ldots, x_n]$ is the sum of terms with maximal weights. Example:

$$\text{in}_{(1,2)}(x_1^4 + 2x_2^2 + x_1x_2 + 1) = x_1^4 + 2x_2^2$$

- The initial ideal of an ideal $I \subseteq k[x_1, \ldots, x_n]$ is defined as

$$\text{in}_\omega(I) = \langle \text{in}_\omega(f) \rangle_{f \in I}$$
Equivalent definition of tropical varieties

Observe: $T(f) = \{ \omega \in \mathbb{R}^n : \text{in}_\omega(f) \text{ is not a monomial} \}$.

**Theorem**

*If $I \subseteq k[x_1, \ldots, x_n]$ is an ideal then

$T(I) = \{ \omega \in \mathbb{R}^n : \text{in}_\omega(I) \text{ is monomial-free} \}$

**Proof.**

We must prove

$$\bigcap_{f \in I} T(f) = \{ \omega \in \mathbb{R}^n : \text{in}_\omega(I) \text{ is monomial-free} \}$$

- $\supseteq$: Easy.
- $\subseteq$: A tiny bit more difficult...
The Gröbner fan of an ideal

Definition (Mora, Robbiano)

- Let \( I \subseteq k[x_1, \ldots, x_n] \) be a homogeneous ideal.
- Define an equivalence relation \( \sim \) on \( \mathbb{R}^n \).

\[
 u \sim v \iff \text{in}_u(I) = \text{in}_v(I)
\]

- The closure of each equivalence class is called a Gröbner cone.
- The set of all these cones is a polyhedral complex.
- We call this the Gröbner fan of \( I \).
The Gröbner fan of an ideal

The following things are in bijection

- The full-dimensional Gröbner cones
- Monomial initial ideals of $I$
- The marked reduced Gröbner bases $I$

Example

$I = \langle a^5 + b^3 + c^2 - 1, a^2 + b^2 + c - 1, a^6 + b^5 + c^3 - 1 \rangle \subseteq \mathbb{Q}[a, b, c]$ has 360 reduced Gröbner bases and 360 full-dimensional cones in its fan. (Not homogeneous!) Intersection of fan and 2-simplex:
A polyhedral structure on the tropical variety

- From the theorem/definition:

  \[ T(I) = \{ \omega \in \mathbb{R}^n | \text{in}_{\omega}(I) \text{ is monomial-free} \} \]

  it is clear that \( T(I) \) is a union of Gröbner cones.

- We may think of the tropical variety as a polyhedral complex inheriting its structure from the Gröbner fan.

- The tropical variety is a subcomplex of the Gröbner fan.
The homogeneity space

Definition
Let \( I \subseteq k[x_1, \ldots, x_n] \) be an ideal.
The Gröbner cone equal to the equivalence class \( \{\omega \in \mathbb{R}^n : \text{in}_\omega(I) = I\} \) is called the homogeneity space of \( I \).

Lemma

- The intersection of all Gröbner cones is the homogeneity space.
- The Gröbner cones are invariant under translation by vectors in the homogeneity space.
- The gradings for which \( I \) is homogeneous are exactly given by the vectors in the homogeneity space.
Example: Grassmann 2,5 - continued

- The f-vector of the Gröbner fan of $G_{2,5}$ is $(1, 20, 120, 300, 330, 132)$.
- There is 1 five-dimensional cone (the homogeneity space).
- There are 132 ten-dimensional cones.
- Tropical variety is a 7-dimensional pure subcomplex.
- Modulo the homogeneity space it is 2-dimensional.
- Projectively we may draw it as a graph.

The f-vector is $(1, 10, 15)$.
- The initial ideals for the the 15 seven-dimensional cones are all generated by binomials.
Tropical bases

Definition
Let $C$ be Gröbner cone not contained in $T(I)$. A polynomial $f \in I$ is a witness for $C$ if $T(f) \cap \text{relint}(C) = \emptyset$.

Definition
A finite generating set $F$ of $I$ is a tropical basis if $T(I) = \bigcap_{f \in F} T(f)$

Theorem
Every ideal $I \subseteq k[x_1, \ldots, x_n]$ has a tropical basis.

Proof.
Sketch of constructive proof.
- Start with a generating set of $I$.
- For every Gröbner cone $C$ of $I$ not contained in the tropical variety choose monomial in the initial ideal.
- “Lift” the monomial to a witness $f \in I$ for $C$.
- Add $f$ to the generating set.
What if...

- the ideal $I$ is not homogeneous?

**Lemma**

The initial ideal $\text{in}_\omega(I)$ contains a monomial if and only if $\text{in}_{(0,\omega)}(hI)$ contains a monomial.

- the ideal $I \subseteq \mathbb{C}\{\{t\}\}[x_1, \ldots, x_n]$ and we use the valuation to tropicalize the polynomial?

**Lemma**

If the ideal is generated by polynomials with coefficients in $\mathbb{C}[t]$ just consider the ideal $I'$ they generate in $\mathbb{C}[x_1, \ldots, x_n, t]$. The intersection of $T(I')$ with the $t = 1$ plane will be $T(I) \times \{1\}$. 

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The initial ideal $\text{in}_\omega(I)$ contains a monomial if and only if $\text{in}_{(0,\omega)}(hI)$ contains a monomial.
Don’t forget the important theorem from Hannah’s talk:

Theorem

- Let \( \text{val} : (\mathbb{C}\{\{t\}\}^*)^n \to \mathbb{Q}^n \) be coordinate-wise valuation.
- Let \( I \subseteq \mathbb{C}[x_1, \ldots, x_n] \) be a homogeneous ideal and \( I' \subseteq \mathbb{C}\{\{t\}\}[x_1, \ldots, x_n] \) the ideal it generates.
- The following identities hold:

\[
\text{val}(V(I')) = T(I') \cap \mathbb{Q}^n = T(I) \cap \mathbb{Q}^n
\]
Computing tropical varieties in Gfan

- The Gröbner fan approach gives us algorithms for computing tropical varieties.
- This requires further development of the techniques from the Gröbner walk [Collart, Kalkbrener, Mall].
- The algorithms are described in [Bogart, Jensen, Speyer, Sturmfels, Thomas]: “Computing tropical varieties”.
- The algorithms have been implemented in Gfan.
- Gfan will be presented in the workshop next week.