CORRRELATED EQUILIBRIUM POINTS: COMMENTS AND RELATED TOPICS

By

N. Iris Auriol and Ezio Marchi

IMA Preprint Series #2465

(February 2016)

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
400 Lind Hall
207 Church Street S.E.
Minneapolis, Minnesota 55455-0436
Phone: 612-624-6066 Fax: 612-626-7370
URL: http://www.ima.umn.edu

CORRRELATED EQUILIBRIUM POINTS: COMMENTS AND RELATED TOPICS

by

N. Iris Auriol *) and Ezio Marchi)**

Abstract: Correlated equilibria, introduced by Aumann in 1974 [1], and related to various topics studied by Marchi in 1969[4] and 1972[5] have very interesting aspects. In a recent study by Marchi and Matons [6]they have used the fact that the fiber bundle where the game is defined , is very rich from an analytic and geometric viewpoint. In that article have been considered the same aspects of the theory of n-person games considering two players with only two pure strategies each (called "case 2 by 2"). In this new paper, we present a generalization of these results to the case that one of the players has two strategies and the remaining have n each (case 2 by n). We would point out that if we attack and solve it in the way that the problem was posed in [4] and [5] maybe we can not get any useful results. This is due to the fact that the underlying fiber bundle in the game is cut by the simplex and this can be displayed only in the case 2 by 2. However, Marchi and Matons [6] we have found a new way to obtain all the analytic and geometrical tools via a set of general inequalities.

We also study the relationship between Aumann correlated equilibria and Nash equilibrium points.

Key Words: correlated equilibrium, Nash equiibrium, fiber bundle, inequalities joint probabilities.

Journal of Economic Literature Classification Numbers:

AMS Classification: 90A05, 90A07

*) UNSL,San Luis, Argentina

email: niajua@gmail.com

**) Founder and Emeritus Pofessor, IMASL,UNSL, San Luis, Argentina

email: emarchi1940@gmail.com (corresponding author)

1 Introduction

In the study of correlated strategies there are some recent studies, for example, Marchi and Matons [6] consider a two-person game in normal form

$\Gamma = \{\sum_1, \sum_2, A_1, A_2\}$ where $\sum_i$ is the pure strategy set of player $i \in N$ $=\{1,2\}$, A_i is the payoff function, $|\sum_i| = 2$, briefly designated as "*case 2×2*".

The mixed extension is given standardly by

$$\widetilde{\Gamma} = \left\{ \widetilde{\sum}_1, \widetilde{\sum}_2, \ E_1 \ , E_2 \right\}$$

where $\widetilde{\sum}_i$ is the set of probability distribution over $\sum_i$, and E_i is the expectation of player i. This game involve the statistic behavior and the competition among the players. This means from a statistical viewpoint that they are not correlated. Thus in this way the correlation among their action is null or avoided. Therefore next it is introduced correlated strategies that allow more flexibility betwen both players. There are pioneering work studying such sets, for example Luce [3] , and later Aumann [1] and Marchi [4], [5] .

Indeed we are going to study the game

$$\widetilde{\widetilde{\Gamma}} = \left\{ \widetilde{\sum_1 \times \sum_2}, E_1, E_2 \right\}$$

where $\widetilde{\sum_1 \times \sum_2}$is the set of correlated strategies, that is

$$\widetilde{\sum_1 \times \sum_2} = \left\{ z = (z_{ij}) \in \mathbb{R}^{2\times 2} : z_{ij} \geq 0, \sum_{i=1}^{2}\sum_{j=1}^{2} z_{ij} = 1 \right\}$$

Hence, if $z \in \widetilde{\sum_1 \times \sum_2}$ then has the form

$$z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

This set has a strong relation with the set $\widetilde{\sum}_1 \times \widetilde{\sum}_2$. This relationship can be seen in the work of Marchi [4] which is studied exhaustively.

In all the cases we have that a point $z \in \widetilde{\sum_1 \times \sum_2}$ can be written as

$$z = \mathbf{x} \otimes \mathbf{y} + u$$

where $\mathbf{x} = (x, 1-x)$ and $\mathbf{y} = (y, 1-y)$ are the natural projections in the corresponding spaces, and $u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$. Formally:

$$
\begin{aligned}
x &= \sum_{j=1}^{2} z_{1j} \ , \ 1-x = \sum_{j=1}^{2} z_{2j} \quad , \quad \sum_{j} u_{ij} = 0, \quad \forall i \\
y &= \sum_{i=1}^{2} z_{i1} \ , \ \ 1-y = \sum_{i=1}^{2} z_{i2} \ \ , \ \sum_{i} u_{ij} = 0, \quad \forall j
\end{aligned}
$$

Here, the matrix u describes the separation of the projection $\mathbf{x} \otimes \mathbf{y}$.

The important point to be studied in this note is the fact that the Aumann correlated equilibrium z assume the same payoff as their projection $\mathbf{x} \otimes \mathbf{y}$. That is, the correlation between the strategies is irrelevant since any cooperation did not obtain more wealth.

Recall that in this case was obtained

$$
z = \begin{pmatrix} xy & x(1-y) \\ (1-x)y & (1-x)(1-y) \end{pmatrix} + \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}
$$

and then, clearly

$$
\begin{aligned}
p_{\mathbf{x}}(u) &= (u_{11}+u_{12}, u_{21}+u_{22}) = (0,0) \\
p_{\mathbf{y}}(u) &= (u_{11}+u_{21}, u_{12}+u_{22}) = (0,0)\,.
\end{aligned}
$$

Therefore

$$
u = u_{11} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}
$$

We remain that the most important condition which allow us to have a correlated strategy in the corresponding simplex is the following (to see [6]):

$$
x + y + 1 \leq z_{11} \leq \min\{x, y\}
$$

From here,

$$
\begin{aligned}
E_i(z) &= E_i(\mathbf{x} \otimes \mathbf{y}) + u_{11} E_i \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \\
&= E_i(\mathbf{x} \otimes \mathbf{y}) + u_{11}(a^i_{11} - a^i_{12} - a^i_{21} + a^i_{22})
\end{aligned}
$$

with

$$
x + y - 1 - xy \leq u_{11} \leq \min\{x, y\} - xy
$$

This last inequality fully describes the fact that the point z is indeed a mixed strategy in $\widetilde{\sum_1 \times \sum_2}$,analytically and geometrically. This means that actions of both players are regarding into a correlated context and thus have an intuitive interpretation, and more realistic from the point of view applied.

It follows that, if

$$a^i_{11} + a^i_{22} = a^i_{12} + a^i_{21},$$

then each Aumann correlated equilibrium, determines a Nash equilibrium with the same payoff.

2 Generalization

In this paragraph we show the cases in which one of the players has two strategies and other one has n strategies, with $n = 2, 3, 4$, generalizing it for any n. So we cover a very wide class of games that are important in practice.

Start with the case 2×3,i.e. with a game where $|\sum_1| = 2$ and $|\sum_2| = 3$, and so each element $z \in \widetilde{\sum_1 \times \sum_2}$ has the following form,

$$z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{pmatrix} \text{ with } z_{ij} \geq 0 \text{ and } \sum_{i=1}^{2}\sum_{j=1}^{3} z_{ij} = 1$$

Let $\mathbf{x} \in \widetilde{\sum_1}$, $\mathbf{y} \in \widetilde{\sum_2}$ be the respective projections, $\mathbf{x} = (x, 1-x)$, $\mathbf{y} =(y_1, y_2, 1 - y_1 - y_2)$.

If $(\mathbf{x}, \mathbf{y})$ is a Nash equilibrium then

$$\begin{aligned} x &= z_{11} + z_{12} + z_{13} \\ 1 - x &= z_{21} + z_{22} + z_{23} \\ y_1 &= z_{11} + z_{21} \\ y_2 &= z_{12} + z_{22} \\ 1 - y_1 - y_2 &= z_{13} + z_{23} \end{aligned}$$

Solving in a suitable way, we have

$$\begin{aligned} z_{22} &= y_2 - z_{12} \\ z_{21} &= y_1 - z_{11} \\ z_{23} &= 1 - x - y_1 + z_{11} - z_{22} \\ z_{13} &= z_{22} - y_2 + x - z_{11} \end{aligned}$$

Now, considering that $0 \leq z_{ij}, y_j, x \leq 1$, they turn out to be the following inequalities

$$\begin{aligned} y_2 - x &\leq z_{22} - z_{11} \leq 1 - x - y_1 \\ 0 &\leq z_{11} \leq y_1 \\ 0 &\leq z_{22} \leq y_2 \end{aligned} \tag{1}$$

It is important to observe that the number of free variables is just the number obtained from the product between $(2 \times 3 - 1)$ and $(2 + 3 - 2)$, which coincides with the dimension of the fiber.

Now, we have

$$z = \mathbf{x} \otimes \mathbf{y} + \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix}$$

where

$$\begin{aligned} p_{\mathbf{x}}(u) &= (u_{11} + u_{12} + u_{13}, u_{21} + u_{22} + u_{23}) = (0,0) \\ p_{\mathbf{y}}(u) &= (u_{11} + u_{21}, u_{12} + u_{22}, u_{13} + u_{23}) = (0,0,0) \end{aligned}$$

and from here

$$\begin{aligned} u_{13} &= u_{22} - u_{11} \\ u_{23} &= u_{11} - u_{22} \\ u_{21} &= -u_{11} \\ u_{12} &= -u_{22} \end{aligned}$$

Replacing then

$$\begin{aligned} z &= \begin{pmatrix} xy_1 & xy_2 & x(1 - y_1 - y_2) \\ (1-x)\,y_1 & (1-x)\,y_2 & (1-x)(1 - y_1 - y_2) \end{pmatrix} + \\ &\quad + \begin{pmatrix} u_{11} & -u_{22} & u_{22} - u_{11} \\ -u_{11} & u_{22} & u_{11} - u_{22} \end{pmatrix} \end{aligned}$$

or

$$z=\mathbf{x}\otimes\mathbf{y}+u_{11}\begin{pmatrix}1 & 0 & -1\\ -1 & 0 & 1\end{pmatrix}+u_{22}\begin{pmatrix}0 & -1 & 1\\ 0 & 1 & -1\end{pmatrix}$$

Considering the inequality (1) with

$$\begin{aligned} z_{22} &= (1-x)\,y_2+u_{22}\\ z_{11} &= xy_1+u_{11} \end{aligned}$$

it follows that

$$y_2-x\leq y_2-x\,(y_1+y_2)+u_{22}-u_{11}\leq 1-x-y_1$$

and from here

$$x\,(y_1+y_2-1)\leq u_{22}-u_{11}\leq (1-x)\,(1-y_1-y_2)$$

Therefore all the Aumann correlated equilibrium are Nash equilibrium, having all the same payoff if

$$u_{11}\left(a_{11}^i+a_{23}^i-a_{21}^i-a_{13}^i\right)=-u_{22}\left(-a_{12}^i+a_{13}^i+a_{22}^i-a_{23}^i\right)$$

Therefore this determines a linear manifold where they are the same.

This fact it is important since in this game it is irrelevant to correlate the strategies of one palyer with the remaining one. This is similar to the case studied in cooperative game theory by von Newmann and Morgestern [7] for the case where there a player dummy. Here both players get the same payoff correlating their actions or not.

Now, for the case 2×4 we have

$$z=\begin{pmatrix}z_{11} & z_{12} & z_{13} & z_{14}\\ z_{21} & z_{22} & z_{23} & z_{24}\end{pmatrix}\in\widetilde{\sum_1\times\sum_2},$$

with the projections $\mathbf{x}=(x,1-x)$, $\mathbf{y}=(y_1,y_2,y_3 1-y_1-y_2-y_3)$.That is to say:

$$\begin{aligned} x &= z_{11}+z_{12}+z_{13}+z_{14}\\ 1-x &= z_{21}+z_{22}+z_{23}+z_{24}\\ y_1 &= z_{11}+z_{21}\\ y_2 &= z_{12}+z_{22}\\ y_3 &= z_{13}+z_{23}\\ 1-y_1-y_2-y_3 &= z_{14}+z_{24} \end{aligned}$$

Then, working as the previous case we get the following important inequalities

$$\begin{aligned} x - y_2 - 1 &\leq z_{11} - z_{22} + z_{13} \leq x - y_2 \\ x + y_1 + y_3 - 1 &\leq z_{11} - z_{22} + z_{13} \leq x + y_1 + y_3 \end{aligned}$$

Hence

$$x + y_1 + y_3 - 1 \leq z_{11} - z_{22} + z_{13} \leq x - y_2 \tag{2}$$

or, equivalently

$$-x + y_2 \leq z_{22} - z_{11} - z_{13} \leq 1 - x - y_1 - y_3$$

We remind that

$$z = \mathbf{x} \otimes \mathbf{y} + u$$

and

$$u = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \end{pmatrix}$$

Given that $p_{\mathbf{x}}(u) = 0$ and $p_{\mathbf{y}}(u) = 0$, it follows that

$$\begin{aligned} u &= \begin{pmatrix} u_{11} & -u_{22} & u_{13} & -u_{11} + u_{22} - u_{13} \\ -u_{11} & u_{22} & -u_{13} & u_{11} - u_{22} + u_{13} \end{pmatrix} = \\ &= u_{11} \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix} + u_{22} \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} + \\ &\quad + u_{13} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{aligned}$$

Now, taking into consideration (2), it turns out

$$x + y_1 + y_3 - 1 \leq x y_1 - (1 - x)\, y_2 + x y_3 + u_{11} - u_{22} + u_{13} \leq x - y_2$$

Working these inequalities it turns out that

$$\begin{aligned} x &< (y_1 + y_2 + y_3 - 1) \leq u_{22} - u_{11} - u_{13} \leq \\ &\leq (1 - x)(1 - y_1 - y_2 - y_3) \end{aligned}$$

Therefore, under these conditions, every Aumann correlated equilibria is a Nash equilibrium point with the same payoff if $E(u) = 0$, ie if

$$u_{11}\left(a^i_{11} - a^i_{21} - a^i_{14} + a^i_{24}\right) + u_{22}\left(-a^i_{12} - a^i_{24} + a^i_{14} + a^i_{22}\right) + \\ +u_{13}\left(a^i_{13} - a^i_{23} - a^i_{14} + a^i_{24}\right) = 0$$

Finally, we consider the general case $2\times n$, one can easily see that u is of the following type:

$$\begin{aligned} u &= u_{11}\begin{pmatrix} 1 & 0 & 0 & ... & -1 \\ -1 & 0 & 0 & ... & 1 \end{pmatrix} + u_{22}\begin{pmatrix} 0 & -1 & 0 & ... & 1 \\ 0 & 1 & 0 & ... & -1 \end{pmatrix} + \\ & +\sum_{i=3}^{n} u_{1i}\begin{pmatrix} 0 & ... & 0 & ... & 1 \\ 0 & ... & 0 & ... & -1 \end{pmatrix} \end{aligned}$$

and therefore, payment equal for both types of equilibrium occurs when

$$\begin{aligned} 0 &= u_{11}\left(a^i_{11} - a^i_{21} - a^i_{1n} + a^i_{2n}\right) + u_{22}\left(-a^i_{12} - a^i_{2n} + a^i_{1n} + a^i_{22}\right) + \\ & +\sum_{i=3}^{n} u_{1i}\left((-1)^{i+1} a^i_{1i} + (-1)^i a^i_{2i} + (-1)^i a^i_{1n} + (-1)^{i+1} a^i_{2n}\right) \end{aligned}$$

Thus we have obtained the general case in which there are two players and one has an arbitrary number of pure strategies, while the rest have only two.

Conclusion 1 *We have obtained, for low dimensions, in a noncooperative two person game, one situations where it is irrelevant to coordinate the actions of the players. This could be useful to decide directly the cooperation between the players. In general, the situation is that the correlation is better. In a manner analogous to that done here can be studied when one or both receive a proportion generally better than they would get without coordinating their behavior.*

References

[1] Aumann, R.J. [1974], *Subjectiviti and Correlation in Randomized Strategies*, Journal of Mathematics Economics- 1, 67-95 ,

[2] Hart, S. and Schmeidler, D. [1989], *Existence of Correlated Equilibria*, Mathematics of Operations Research (U.S.A), vol. 14, N^{o}1,18-25.

[3] Luce, R. D. [1954], *A definition of Stability for n-person Games.* Annals of Mathematics 59, 357-366.

[4] Marchi, E. [1969], *On the Posibility of an Inusual Extension of the Minimax Theorem*, Zeit. Wahrscheinlichkeitstheorie verw. Gebiete, 12, 223-230.

[5] Marchi, E. [1972] , *The Natural Vector Bundle of the Set Product Probability*, Zeit. Wahrscheinlichkeitstheorie verw. Gebiete, 23, 7-17.

[6] Marchi, E. and Matons C., M. [2011], *An Important Observation about Aumann Correlated Equilibria Points,* Unpublished.

[7] von Newmann J. and Morgenstern O., [1944], *Theory of Game and Economic Behavior*, Princeton University Press.