ON A CHIPMAN'S PROBLEM

By

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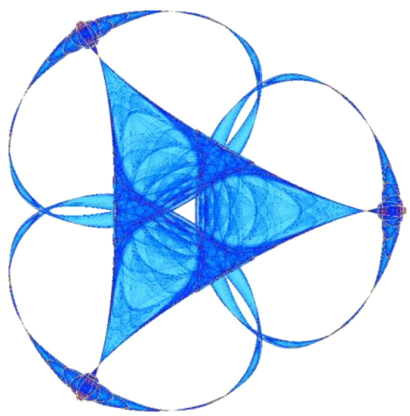

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***Abstract:*** *in this paper we study the transportation problem when the minimization is given with more that one attributions, say for example, volume and weight. We obtain new results that are interesting from the theoretical and practical points of view.*

**Key words:** transportation, optimum, linear programming, several attributes.

1. **Introduction**

Some years ago when the author of this paper visited the Institute of Mathematics and its Applications at the University of Minnesota, he had an interview with Professor Chipman, who was concerned that in classical transportation problem, as posed in the literature by Charnes and Cooper [3] or in a more modern way in Brualdi [1], the good is divisible and has only an attribute, say for example, volume or weight.

Professor Chipman noted that if we consider merchandise or good with more than one attribute, both and volume, then the problem of transportation becomes more interesting and real. To our knowledge there not exist any good and/or mathematical approaches, to this important problem, which we refer as the “Chipman’s Problem”.

Consider an arbitrary amount of good, with two attributes, volume and weight, the one can have the following picture.

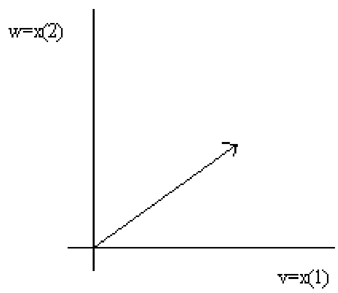


Figure 1

By $x = (vol, wei) = (x(1), x(2))$ where $x(1)$ is the volume and $x(2)$ is the weight. If the transportation company take into consideration only are attributes for computing cost, then we obtain the unitary cost $c_w$ of the weight considered alone the total cost is

$$c_1^t = (0, c_w)$$

thus, the total cost is given by

$$c_1^t x = (0, c_w)\begin{pmatrix} x(1) \\ x(2) \end{pmatrix} = c_w x(2)$$

Similarly, if we consider only volume the total cost is given by

$$c_2^t = (c_v, 0)$$

thus, the total cost is given by

$$c_2^t x = (c_v, 0)\begin{pmatrix} x(1) \\ x(2) \end{pmatrix} = c_v x(1)$$

In the first case, we say that the cost is volume free and in the second case, we say that it is weight free. We then use $\alpha \geq 0$ and $\beta \geq 0$ to represent the corresponding proportions to transport of $x = (x(1), x(2))$, that is

$$\alpha c_1^t x(1) + \beta c_2^t x(2) = (\alpha c_1^t + \beta c_2^t)x = (\alpha(0, c_w) + \beta(c_v, 0))x$$

Now if we have more than one source $i \in M = \{1,...,m\}$ and more than one destination $j \in N = \{1,...,n\}$ where the good has to arrive, we obtain the vector

$$x_{ij} = (x_{ij}(1), x_{ij}(2))$$

Here the first coordinate $x_{ij}(1)$ measures the volume to be sent and the second amount $x_{ij}(2)$ is the amount for the weight, then volume from the port *i* is going to be bound by

$$\sum_j x_{ij}(1) \leq r_i(1)$$

where $r_i(1)$ is the maximum capacity of weight from port *i* and

$$\sum_j x_{ij}(2) \leq r_i(2)$$

where $r_i(2)$ is the maximum capacity of volume from port *i*. In similar way the maximum volume to be sent to destination *j*, for $j = 1,...,n$, is

$$\sum_i x_{ij}(1) \leq s_j(1) \text{ and } \sum_i x_{ij}(2) \leq s_j(2)$$

The previous equations determine two sets of inequalities which by linear algebra can be represented by two matrices where the *x*´s form a matrix of dimension $m \times n$. Indeed, we have

$$A_1(x_{ij}(1)) \text{ and } A_1(x_{ij}(1))$$

for volume and weight, respectively. They can be represented by

$$\begin{bmatrix} 11...1 & & & \\ & 11...1 & & \\ & & 11...1 & \\ & & & 11...1 \\ 1 & 1 & 1 & 1 \\ . & . & . & . \\ . & . & . & . \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

as in transportation problem. See Charnes and Cooper [3].

The "Chipman's Problem" now takes the form

$$\min_{x(1),x(2)} \sum_{ij} \alpha c_{ij}(1) x_{ij}(1) + \sum_{ij} \beta c_{ij}(2) x_{ij}(2) = c_{\alpha\beta}^t x \qquad (1a)$$

where $\alpha$ and $\beta$ are the parameters previously described. The constrains for the transportation problem are now represented by

$$A_1(x(1)) = \begin{pmatrix} r_1 \\ s_1 \end{pmatrix} \text{ and } A_2(x(2)) = \begin{pmatrix} r_2 \\ s_2 \end{pmatrix} \qquad (1b)$$

and both are of size $(m+n)(m \times n)$ with $x(1) = (x_{ij}(1)) \geq 0$ and $x(2) = (x_{ij}(2)) \geq 0$. The problem to be dealt with is not in a standard form. This can be written as a linear program problem as follows

$$\min c_{\alpha\beta}^t x$$

sub. to

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} = \begin{pmatrix} r_1 \\ s_2 \\ r_2 \\ s_2 \end{pmatrix}$$

To study the properties, we have now described, we note that *(1a)* and *(1b)* take the form of inequalities. Since the inequality holds true, we have

$$\sum_i \min_x f_i(x) \leq \min_x \sum_i f_i(x)$$

However in this case since $x(1)$ and $x(2)$ have disjoint domains then the equality holds and the problem now becomes

$$\min_{x(1)} c^t(1)x(1) + \min_{x(2)} c^t(2)x(2)$$

$$A_1 x(1) = \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}$$

$$(2)$$

$$A_2 x(2) = \begin{pmatrix} r_2 \\ s_2 \end{pmatrix}$$

Thus, the multiplication by $\alpha$ and $\beta$ determines the “Chipman’s Problem”.

## 2. **Feasibility**

Now we want to see when the set of transportation plans satisfying *(2)* is not empty. For this we follow Brualdi [2]. The first problem to be examinated is to show that Chipman transportation problem has a solution. For this we demonstrate the following

***Theorem 2.1:*** *Under the following conditions*

*1)* $\sum_i r_i(1) = \sum_j s_j(1)$

2) $\sum_i r_i(2) = \sum_j s_j(2)$

*the "Chipman*'s *Problem" has a solution. As assumed in Brualdi [2], the numbers*

$r_i(1)$, $r_i(2)$, $s_i(1)$ *and* $s_i(2)$ *are non negative.*

**Proof:** Then for $i \in M$ and $j \in N$ and let $\bar{x}_{ij}(1)$ and $\bar{x}_{ij}(2)$ denote the number of units transported in the model. Let

$$a_i(1) = \sum_j \bar{x}_{ij}(1) \text{ and } a_i(2) = \sum_j \bar{x}_{ij}(2)$$

refer to shipments from the origin and the corresponding amounts for the destinations

$$b_j(1) = \sum_i \bar{x}_{ij}(1) \text{ and } b_j(2) = \sum_i \bar{x}_{ij}(2).$$

Now, for the entire plan consider

$$d(1) = \sum_i a_i(1) \text{ and } d(2) = \sum_i a_i(2)$$

and

$$b(1) = \sum_j b_j(1) \text{ and } b(2) = \sum_j b_j(2).$$

Then, the balance of shipments turns out to be $a(1) = b(1)$ and $a(2) = b(2)$.

Now define

$$\bar{x}_{ij}(1) = \frac{a_i(1)b_i(1)}{d(1)} \text{ and } \bar{x}_{ij}(2) = \frac{a_i(2)b_i(2)}{d(2)}.$$

Since $d(1)$ and $d(2)$ are strictly positive $a_i(1)$, $a_i(2)$, $b_i(1)$ and $b_i(2)$ are non negative numbers. We have $\bar{x}_{ij}(1) \geq 0$ and $\bar{x}_{ij}(2) \geq 0$. Moreover

$$\bar{x}_{i1}(1) + \bar{x}_{i2}(1) + \ldots + \bar{x}_{im}(1) = \frac{a_i(1)b_1(1)}{d(1)} + \frac{a_i(1)b_2(1)}{d(1)} + \ldots + \frac{a_i(1)b_m(1)}{d(1)} =$$
$$\frac{a_i(1)}{d(1)} \sum_i b_i(1) = a_i(1)\frac{d(1)}{d(1)} = a_i(1)$$

In the same way we can prove

$$\sum_i \bar{x}_{ij}(2) = a_j(2)$$

Now we can associate the "Chipman's Problem" with $x = (x(1), x(2))$ and it can be represented by two bipartite graphs. We denote then by $BG_1(x)$ and $BG_2(x)$, and call the bipartite graphs of Chipman transportation plan. This is done as follows.

We take nodes $t_1^k, ..., t_m^k$ one corresponding to each source nodes $u_1^k, ..., u_n^k$ correspond.

To each destination we then join $u_i^k$ to $t_j^k$, $k = 1,2$, for $i \leq m$ and $j \leq n$. We then have $x_{ij}(1) \neq 0$, for $i \leq m$ and $j \leq n$. Similarly in the second graph we associate the solution with an "edge" if $x_{ij}(2) \neq 0$. In other words the nodes corresponding to the source of the $j^{th}$ destination are joined by an edge. This provides in the Chipman problem a non zero amount of commodity is sent from the $i^{th}$ source to the $j^{th}$ destination. The two graphs of the transportation plan $x = (x(1), x(2))$ can be obtained from two complete bipartite graphs $K_{m,n}^1$ and $K_{m,n}^2$ by deleting the edges corresponding to $x_{ij}$ which are zero. We use the bipartite graph $BG_k(x)$ of a transportation problem mainly as a theoretical device.

Given a transportation plan $x = (x_{ij})$ an elementary cycle of $BG_k(x)$ corresponds to a set *S* of non zero transportation numbers *X* which are rook wise connected. In order for the cycle to be elementary *S* contains 0 or 2 shipments from each row and column. An example is pictured in the following figure. The edges of an elementary cycle correspond to the circled transportation numbers.

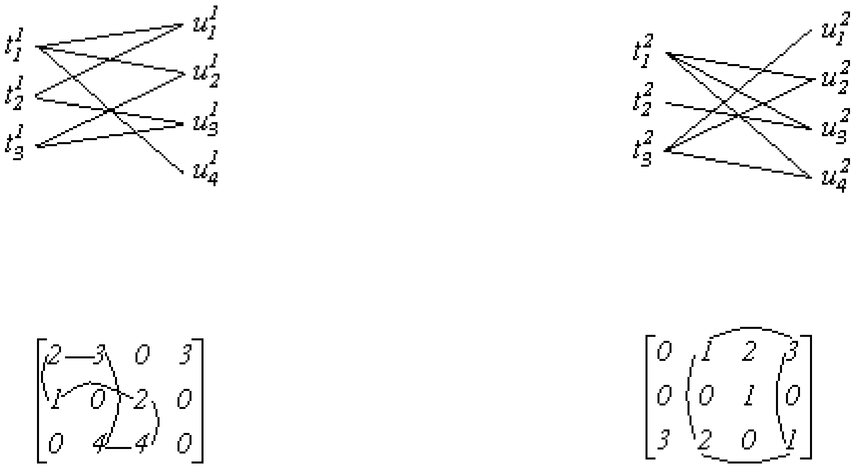


Figure 2

Now we prove the structure of the elementary cycles in $BG_k(x)$ correspond to the natural form given in *(3)*.

***Theorem 2:*** *An elementary cycle of the Chipman's problem is an elementary cycle of $BG(x)$ and vice versa.*

**Proof:** Consider an elementary cycle in $BG_1(x) \cup BG_2(x)$. It may happen that there is an elementary cycle in $BG_1(x(1))$ and not in $BG_2(x(2))$. Then an elementary cycle in $BG_1(x(1))$ is an elementary cycle in $BG(x)$. The same with the $BG_1(x(1))$ and $BG_2(x(2))$. Reciprocally, we have an elementary cycle in $BG(x)$ since the graphs are disjoint. Then is elementary in either $BG_1(x(1))$ or $BG_2(x(2))$ or both. □

We point out that an elementary cycle of $BG(x)$ of *(3)* –see below- has its support in places $A_1$ and $A_2$. It will be either 0 or 2 transportation numbers from each row and column. An example is picture in figure 3, where the edges of an elementary cycle correspond to the circled transportation numbers.

$$\begin{array}{ccc|ccc} 0 & & 0 & & & \\ 2 & 0 & 6 & & & \\ 3 & 4 & 7 & & & \\ 1 & 0 & 0 & 1 & 0 & \\ 2 & 0 & 0 & 2 & 0 & \\ 0 & 4 & 1 & 5 & 1 & 0 \\ 0 & 0 & 6 & 6 & 0 & \end{array} \qquad \begin{array}{ccc|ccc} & & 0 & & & \\ & 0 & 4 & & & \\ 0 & 3 & 5 & & & \\ 3 & 4 & 10 & & & \\ 3 & 1 & 0 & 4 & 1 & 0 \\ 0 & 3 & 1 & 4 & 1 & 0 \\ 0 & 0 & 4 & 4 & 0 & \\ 0 & 0 & 5 & 5 & 0 & \end{array}$$

Figure 3

The structure of elementary cycles in $BG(x)$ corresponds to the natural form given in *(3)*. Therefore as a consequence of theorem 12.3.2 in the book of Brualdi and our previous development we have

***Theorem 3:*** *Let $x=(x(1),x(2))$ be a transportation plan such that $BG_1(x(1))$ and $BG_2(x(2))$ have an elementary cycle. Then there is a transportation shipment plan with cost no greater than that of the transportation plan $y=(y(1),y(2))$, $cy \le cx$.*

From this theorem it follows that to determine an optimal transportation plan, which is a transportation plan of smallest cost; we need only to consider those transportation plans whose bipartite graphs have no cycles. Thus, we can confine our attention to transportation plans whose bipartite graphs are either trees or forests (graphs whose connected components are trees). We now show that for given demand and supply requirements this reduces the transportation plans that need to be considered to a finite set.

***Theorem 4:*** *Let supply and demand requirements* $r_i(1)$, $r_i(2)$, $s_j(1)$ *and* $s_j(2)$, $i=1,...,m$, $j=1,...,n$, *be given. Let* $x=(x(1),x(2))$ *be transportation plan such that* $BG_1(x(1))$ *and* $BG_2(x(2))$ *both graphs that have not cycles. Then any other transportation plan will both have different bipartite graphs.*

***Corollary 5:*** *Let supply and demand requirements* $r_i(1)$, $r_i(2)$, $s_j(1)$ *and* $s_j(2)$, $i=1,...,m$, $j=1,...,n$, *be given. Then there are only finitely many transportation plans whose both have no cycles.*

Thus, we have reduced the search of optimal transportation plans from an infinite to a finite set.

The essential part of the proof of theorem 4 is due to Jurkar and Ryser [4] and suggests a method for constructing a transportation plan $x=(x(1),x(2))$, such that both $BG_1(x(1))$ and $BG_2(x(2))$ have no cycles. Take $i$ and $j$ and determine $x_{ij}(1)$ as follows: if $r_i(1)<s_i(1)$, we let $x_{ij}(1)=r_i(1)$, $x_{ik}(1)=0$ for $k\neq j$, if $r_i(1)>s_i(1)$, we let $x_{ij}(1)=s_i(1)$, $x_{ik}(1)=0$ for $k\neq j$ and finally if $r_i(1)=s_i(1)$ we put $x_{ij}(1)=r_i(1)=s_i(1)$, $x_{ik}(1)=0$, for $k\neq i$ and $k\neq j$, $x_{ik}(1)=0$.

Thus, the transportation numbers in one column or row of reduced $x(1)$, are now determined. In the first case the supply requirements of the $i^{th}$ source is zero, while the demand requirement of the $j^{th}$ destination is reduced to $s_j(1)-r_i(1)>0$. Similar considerations apply for the remaining cases. The same procedure can be done with respect to the second component. Repeating this procedure in the transportation plan $x=(x(1),x(2))$ for which $BG_1(x(1))$ and $BG_2(x(2))$ do not have cycles. Performing this procedure in both matrices we can adopt the northwest-corner rule and consistently choose to determine the transportation plan. The transportation problem with supply requirements 3, 4, 7 and demand 1, 2, 5, 6 in the first block and 5, 4, 4, 4 with 3, 4, 10 in the second block, gives the transportation plan showed in the figure 3.

Thus, as a consequence of the previous theorem the procedure obtained above will yield all transportation plans whose bipartite graphs have no cycles. Moreover the transportation numbers so determinated, if non zero, are of the form

$$(r_{i_1}(1)+r_{i_2}(1)+\ldots+r_{i_p}(1))=(s_{j_1}(1)+s_{j_2}(1)+\ldots s_{j_p}(1))$$

or

$$(s_{j_1}(1)+s_{j_2}(1)+\ldots s_{j_p}(1))=(r_{i_1}(1)+r_{i_2}(1)+\ldots+r_{i_p}(1))$$

for some $i_1,i_2,\ldots,i_p$ and $j_1,j_2,\ldots,j_p$ with $1\le i_1<i_2<\ldots<i_p\le m$ and $1\le j_1<\ldots<j_p\le n$.

Moreover, if the supplies and demands are integers, then the solutions are also integers. In addition if the two forms just expressed are never zero except if $p=m$ and $q=n$ then the bipartite graph constructed is connected and is a tree.

**Example:** Take as an example of a Chipman's transportation problem the follwing

$$\begin{array}{ccccc|ccccc}
2 & 1 & 0 & 0 & 0 & \cancel{3} & \cancel{1} & 0 & & \\
0 & 2 & 3 & 1 & 1 & \cancel{7} & \cancel{5} & \cancel{2} & \cancel{1} & 0 \\
0 & 0 & 0 & 0 & 3 & \cancel{3} & 0 & & & \\
\hline
\cancel{2} & \cancel{3} & \cancel{3} & \cancel{1} & \cancel{4} & & & & & \\
0 & \cancel{2} & 0 & 0 & \cancel{3} & & & & & \\
 & 0 & & & 0 & & & & &
\end{array}
\qquad
\begin{array}{ccccc|ccccc}
1 & 0 & 0 & 0 & 0 & \cancel{1} & 0 & & & \\
3 & 2 & 1 & 3 & 0 & \cancel{9} & \cancel{6} & \cancel{4} & \cancel{3} & 0 \\
0 & 0 & 0 & 1 & 1 & \cancel{2} & \cancel{1} & 0 & & \\
\hline
\cancel{4} & \cancel{2} & \cancel{1} & \cancel{4} & \cancel{1} & & & & & \\
\cancel{3} & 0 & 0 & \cancel{1} & 0 & & & & & \\
0 & & & 0 & & & & & &
\end{array}$$

Figure 4

can be represented in the following tree

Figure 5

We now describe the duals to the transportation problem that enables us to determine when a transportation plan can be obtained in an optimal manner. Let $u_1^k, u_2^k, \ldots, u_m^k$, for $k = 1,2$, be real numbers, one corresponding to each source and let $v_1^k, v_2^k, \ldots, v_n^k$, for $k = 1,2$, be real numbers, one corresponding to each destination, for the first attribute such that

$$u_i^1 + v_j^1 \le c_{ij}(1) \qquad i = 1,\ldots,m; \;\; j = 1,\ldots,n$$

$$(4)$$

$$u_i^2 + v_j^2 \le c_{ij}(2) \qquad i = 1,\ldots,m; \;\; j = 1,\ldots,n$$

Now consider the sum

$$R(u,v) = \sum_{i=1}^{m} r_i(1)u_i^1 + \sum_{j=1}^{n} s_j(1)v_j^1 + \sum_{i=1}^{m} r_i(2)u_i^2 + \sum_{j=1}^{n} s_j(2)v_j^2$$

We then have

***Theorem 6:*** *Let* $x = (x(1), x(2))$ *be transportation plan of the Chipman's problem. Then for each real numbers* $u_1^k, u_2^k, ..., u_m^k$*,* $v_1^k, v_2^k, ..., v_n^k$*,* $k = 1,2$*, satisfying (4), the cost of X satisfies*

$$cx \geq R(u,v) \tag{5}$$

*Moreover, if, in addition*

$$u_i^k + v_j^k = c_{ij}(k) \text{ whenever } x_{ij}(k) = 0,\ k = 1,2, \tag{6}$$

*then X is an optimal transportation plan with cost*

$$cx = R(u,v) \tag{7}$$

**Proof:** Let $u_1^k, u_2^k, ..., u_m^k$, $v_1^k, v_2^k, ..., v_n^k$, $k = 1,2$, be real numbers satisfying *(4)*. Then

$$\sum_i \sum_j c_{ij}(1) x_{ij}(1) + \sum_i \sum_j c_{ij}(2) x_{ij}(2) \geq$$

$$\sum_i \sum_j (u_i^1 + v_j^1) x_{ij}(1) + \sum_i \sum_j (u_i^2 + v_j^2) x_{ij}(2) =$$

$$\sum_i u_i^1 \sum_j x_{ij}(1) + \sum_i u_i^2 \sum_j x_{ij}(2) + \sum_j v_j^1 \sum_i x_{ij}(1) + \sum_j v_j^2 \sum_i x_{ij}(2) = R(u,v)$$

Thus *(5)* holds true. If in addition *(6)* holds then the last inequality becomes equality. The reason is because $x_{ij}(k) = 0$ whenever $c_{ij}(k) > u_i(k) + v_j(k)$. Hence

$$\sum_i \sum_j c_{ij}(1) x_{ij}(1) + \sum_i \sum_j c_{ij}(2) x_{ij}(2) =$$

$$\sum_i u_i^1 r_i(1) + \sum_i u_i^2 r_i(2) + \sum_j v_j^1 s_j(1) + \sum_j v_j^2 s_j(2) = R(u,v)$$

which is *(7)*. □

The significance of this theorem is the following: suppose $x = (x(1), x(2))$ is a transportation plan that we wish to check for optimality. According to the theorem if we can find numbers $u_1^k, u_2^k, ..., u_m^k$, $v_1^k, v_2^k, ..., v_n^k$ satisfying *(4)* and *(6)*, then we conclude that *x* is an optimal transportation plan.

As a continuation of this paper we will study an algorithm to be described in order to reach an optimum.

The problem with several attributes follows accordingly. A further problem is when we have a joint cost depending on the volume and the weight: $C(v,w)$.

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