SENSITIVITY ANALYSIS FOR MARKET EQUILIBRIUM PROBLEMS

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SENSITIVITY ANALYSIS FOR MARKET EQUILIBRIUM PROBLEMS*

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Abstract

We utilize an abstract network framework recently introduced for the study of competitive equilibrium problems, to study the stability and sensitivity of the equilibrium in market equilibrium problems. In particular, we consider a steady-state market equilibrium problem in which many commodities are produced, a number of alternative production processes are available and one has to determine the optimal levels of production and consumption satisfying an equilibrium condition. Assuming that the factor production cost and inverse demand functions are monotone, we first show that the market equilibrium commodity demand pattern depends continuously upon the inverse demand functions and the commodity flow pattern depends continuously upon the factor production cost functions. We then focus on the delicate question of predicting the direction of the change in the equilibrium commodity flow pattern, the commodity demands, and the incurred factor production costs and commodity prices, resulting from changes in the factor production cost and inverse demand functions in order to elucidate certain market phenomena. Our analysis here depends crucially on the fact that the equilibrium conditions can be formulated as a variational inequality.

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1. Introduction

An abstract network framework for the study of a broad class of competitive equilibrium problems which, in general, do not possess an inherent network structure was recently introduced in Dafermos and Nagurney (1984a). This model was subsequently expanded in Dafermos and Nagurney (1984b) where it was also shown how two well-known problems in economics, namely, the oligopolistic market equilibrium problem which is not in any way related to a geographic network and the spatial economic equilibrium which is related to the network defined by the location of markets can be cast into this form.

In describing this abstract framework we borrow terminology from economics where we expect that most of the applications will be found. However, we note that many other equilibrium problems in engineering and operations research can also be modelled in terms of this abstract framework. Hence, in economic terms, the problem can be described as follows: A steady-state market equilibrium problem is considered in which many commodities are produced, a number of alternative production processes are available for the production of each commodity and one has to determine the optimal levels of production (supply) and consumption (demand) satisfying some equilibrium condition. Here we consider the equilibrium property that the price that the consumers are willing to pay for each commodity cannot exceed the cost of its production (Dafermos and Nagurney, 1984a,b). Many equilibrium problems in economics are of this nature; see, for example, Varian (1978).

We consider here the general case where each factor used in the production of a particular commodity has its own production cost and at the same time it contributes to its own and other commodities' costs of factors of production in an individual way. The production cost of a commodity produced via a production process is obtained as the sum of the production costs of the production factors utilized by the process in question. Similarly, we consider the general case where the consumer demand for each commodity does not depend only on the price of this commodity but can also depend upon the prices of the other commodities.
In this paper we utilize the recently derived variational inequality formulation of the equilibrium conditions for the above-described market equilibrium problem (Dafermos and Nagurney, 1984a) in order to study the stability and sensitivity of the market equilibrium problem with factor production cost functions and inverse commodity demand functions which satisfy a monotonicity condition. Here we accomplish this without actually computing the incurred changes as was done in Nagurney (1984) and the results we obtain are in agreement with those therein and slightly more general. Other sensitivity analysis results have already been obtained, albeit for some rather special models (Henderson and Quandt, 1980; Samuelson, 1983).

It is important to know whether the market equilibrium commodity demand pattern depends continuously upon the inverse commodity demand functions and whether the commodity flow pattern depends continuously upon the factor production cost functions; in other words, do small changes in the commodity prices induce small changes in the commodity demands, and do small changes in the factor production cost functions induce small changes in the commodity flows. In Section 3 we study these questions for the general market equilibrium model with factor production cost functions and inverse commodity demand functions which satisfy a monotonicity condition and show that the equilibrium pattern of commodity flows and demands depends continuously on the factor production cost and inverse demand functions (Theorem 3.1).

A more delicate question is whether one is able to predict the direction of the change in the equilibrium commodity flow pattern, the commodity demands, and the incurred factor production costs and commodity prices, resulting from change in the factor production cost and inverse demand functions. One would expect intuitively that whenever the production cost situation for a commodity is "improved," the flow of the commodity via the production process would increase, while the incurred production cost would decrease. Also, whenever the inverse demand for a commodity increases (decreases), one would expect an increase (decrease) in the demand for this commodity and an increase (decrease) in the incurred inverse
demand. However, Dafermos and Nagurney (1982) and Nagurney (1983) gave examples of the spatial economic equilibrium problem which, as mentioned earlier, can be cast into the abstract network framework, in which certain phenomena which can be viewed as analogues of Braess' paradox (1968) for transportation networks. More recently, Nagurney (1984) gave an example in which the addition of a new market results in an increase in the inverse demand, equivalently, price for all other markets (see also Hart (1975)).

As regards changes in the incurred production costs brought about by "improving" the production cost situation, by either enhancing existing production processes, adding new ones with or without the addition of a new commodity, we establish in Section 3 (Theorem 3.2) inequalities which show that, in general, the following phenomena may prevail: an improvement in the production cost situation for a commodity may result in an increase in some of the incurred production costs, and, hence, the equilibrium prices, and a decrease in some of the commodity flows. However, we show in Corollary 3.1 that if changes in the production cost structure are such that only a single production process for a single commodity is improved (or a single production process for an existing commodity is added) then the cost for this process for this commodity will necessarily decrease while the commodity flow will increase.

As regards changes in the incurred equilibrium prices brought about by changes in the inverse demand mechanism (e.g., via changes in taste or the addition of a new commodity), these same inequalities show that, in general, the following phenomena may prevail: an increase in the inverse demand function for a commodity may result in a decrease in some of the demands and a decrease in some of the prices. However, we show in Corollary 3.2 that if changes in the inverse demand mechanism are such that the inverse demand function for a single commodity is shifted upward (respectively downward), whereas the rest remain fixed, then both the incurred price and demand for this commodity will increase (respectively decrease).
A similar analysis to the one used here has also been applied in Dafermos and Nagurney (1982, 1984c, d) to study the stability and sensitivity of the traffic and spatial economic equilibrium problems.

2. The Network Model

In this section we briefly review the abstract network framework introduced in Dafermos and Nagurney (1984a) to study competitive equilibrium problems which are not, in general, related to a geographic network.

We first review how the market equilibrium problem outlined in the Introduction can be visualized as an equilibrium network flow problem. We identify the elements of our model with the elements (links, paths, nodes) of a network as follows.

We visualize the production factors utilized in the market as the links \( a, b, \) etc. of the network. Then any sequence of distinct production factors, inducing a production process for a particular commodity can be visualized as a path \( p \) of the network. Hence, the allowable paths \( p, q, \) etc. in the network correspond to all possible production processes utilized in the market producing the same or different commodities. For convenience, we assume that all the paths originate from a common node 0. We visualize each commodity produced in the market as the destination node of each of the paths which correspond to the different production processes via which the particular commodity can be produced.

We denote the commodities and their corresponding destination nodes of the network by \( w, \omega, \) etc. With each destination node \( w, \) we associate a demand \( d_w \) and the price \( \rho_w \) for the corresponding commodity \( w. \) We arrange the commodity demands and prices into column vectors \( \hat{\mathbf{d}} \) and \( \hat{\mathbf{\rho}} \) in \( \mathbb{R}^J \) where \( J \) is the total number of commodities. The demand for any commodity may depend upon the prices associated with all commodities in the market, that is,

\[
\hat{\mathbf{d}} = \hat{\mathbf{d}}(\hat{\mathbf{\rho}}) \tag{2.1}
\]

where \( \hat{\mathbf{d}} \) is a prescribed smooth function. We also assume here that the demands
uniquely determine the demand prices, that is,

\[ p = \hat{\mathbf{d}}(\mathbf{q}) \]  \hspace{0.5cm} (2.2)

where \( \hat{\mathbf{d}} \) is the inverse function of \( \hat{\mathbf{J}} \) in (2.1).

The amount of a commodity produced via a particular production process \( p \) can be visualized as the path commodity flow \( F_p \) along the path representing the production process in question. The path flows are nonnegative and must satisfy the conservation of flow equations

\[ d_w = \sum_{p \text{ terminates in } w} F_p. \]  \hspace{0.5cm} (2.3)

We arrange the path flows into a column vector \( \mathbf{F} \) in \( \mathbb{R}^Q \) where \( Q \) is the total number of paths. A nonnegative path commodity flow \( \mathbf{F} \) which satisfies (2.3) is feasible.

The path commodity flow \( \mathbf{F} \) induces on every link \( a \) and for every commodity \( w \) link commodity flows \( f_{wa}^w \) via the equation

\[ f_{wa}^w = \sum_p \delta_{ap}^w F_p \]  \hspace{0.5cm} (2.4)

where \( \delta_{ap}^w = 1 \), if path \( p \) terminates in destination \( w \) and contains link \( a \), and \( \delta_{ap}^w = 0 \) otherwise. Note that the link commodity flow \( f_{wa}^w \) is the total amount of commodity \( w \) utilizing the production factor represented by link \( a \). We arrange the link commodity flows into a column vector \( \mathbf{f} \) in \( \mathbb{R}^{NJ} \) where \( J \) is the total number of destinations and \( N \) is the total number of links in the network. A feasible distribution is a pair \((\mathbf{f}, \mathbf{q})\) induced by a feasible \( \mathbf{F} \).

The production cost of a factor, for each commodity \( w \), will be denoted by \( c_{wa}^w \). We arrange the \( c_{wa}^w \) into a column vector \( \mathbf{c} \) in \( \mathbb{R}^{NJ} \). The production cost \( c_{wa}^w \) of every factor \( a \) used in the production of a certain commodity \( w \) may, in general, depend upon the total amounts of every commodity, i.e., \( c_{wa}^w = \hat{c}_{wa}^w(\mathbf{f}) \), or, in vector form,

\[ \mathbf{c} = \hat{\mathbf{c}}(\mathbf{f}). \]  \hspace{0.5cm} (2.5)
The production cost of a commodity \( w \) produced via a process corresponding to a path \( p \) is

\[
c_p = \sum_a \delta_{ap} c_a^w
\]  

(2.6)

where \( \delta_{ap} = 1 \) if \( a \) is contained in \( p \) and 0 otherwise and \( w \) is the commodity where \( p \) terminates. In view of (2.6),

\[
c_p = c_p^w \overset{\text{def}}{=} \sum_a \delta_{ap} c_a^w
\]  

(2.7)

A market equilibrium consisting of commodity demand prices and commodity flows will be reached if the price of each commodity does not exceed the cost of its production. Hence, we adopt here the well-known market equilibrium conditions that the price charged by the producers of the commodity, which we assume here to be equal to the cost of its production, must be equal to the price that the consumers are willing to pay for the commodity. The demand for each commodity must be satisfied by the sum of the respective path commodity flows.

This equilibrium state is then characterized by the following equilibrium conditions, which must hold for every commodity \( w \) and every production process \( p \) terminating in \( w \):

\[
\begin{align*}
\hat{c}_p^w(f) & = \hat{\rho}_w(d), \text{ if } F_p > 0 \\
\geq & \hat{\rho}_w(d), \text{ if } F_p = 0
\end{align*}
\]  

(2.8)

and \( (f, d) \) also satisfies (2.3) and (2.4). Observe that the market equilibrium conditions (2.8) state that only the most efficient production processes are being used.

These equilibrium conditions have been recently formulated as a variational inequality (Dafermos and Nagurney, 1984a):

\[
\hat{c}(f)^T(f' - f) - \hat{\rho}(d)^T(d' - d) \geq 0 \text{ for all feasible } (f', d')
\]  

(2.9)
Interestingly, the variational inequality (2.9) that governs the market equilibrium discussed here has the same form as the variational inequality which characterizes the multimodal traffic equilibrium problem derived in Dafermos (1982b). The theory of variational inequalities provides us with powerful tools for qualitative results, as well as a plethora of algorithms for computations which have been employed successfully in the framework of traffic equilibrium problems (see Dafermos (1980, 1982a,b, 1983), Dafermos and Nagurney (1984c,d) and Nagurney (1982)).

We mention that, under a strong monotonicity assumption on $\hat{\mathcal{G}}$ and $-\hat{\mathcal{G}}$, there exists a unique distribution pattern which can be constructed by means of a general iterative scheme devised by Dafermos (1983) for the numerical solution of finite-dimensional variational inequalities. Details on existence, uniqueness, and the computation of market equilibria will be presented in a forthcoming publication.

3. Sensitivity Analysis

In this section we study how changes in the factor production cost functions and the inverse commodity demand functions affect the equilibrium distribution pattern in the network. On a fixed market network we change the factor production cost functions from $\hat{\mathcal{G}}(\cdot)$ to $\hat{\mathcal{G}}^*(\cdot)$, and the inverse demand functions from $\hat{\mathcal{G}}(\cdot)$ to $\hat{\mathcal{G}}^*(\cdot)$ and we are to compare the corresponding distribution patterns $(\mathfrak{f}^1, \mathfrak{d}^1)$ and $(\mathfrak{f}^*, \mathfrak{d}^*)$.

We impose upon $\hat{\mathcal{G}}(\cdot)$ and $-\hat{\mathcal{G}}(\cdot)$ the following strong monotonicity assumption

$$[\hat{\mathcal{G}}(\mathfrak{f}^1) - \hat{\mathcal{G}}(\mathfrak{f}^2)]^T[\mathfrak{f}^1 - \mathfrak{f}^2] - [\hat{\mathcal{G}}(\mathfrak{d}^1) - \hat{\mathcal{G}}(\mathfrak{d}^2)]^T[\mathfrak{d}^1 - \mathfrak{d}^2] \geq \alpha(\|\mathfrak{f}^1 - \mathfrak{f}^2\|^2 + \|\mathfrak{d}^1 - \mathfrak{d}^2\|^2)$$

(3.1)

for all $(\mathfrak{f}^1, \mathfrak{d}^1), (\mathfrak{f}^2, \mathfrak{d}^2) \in \mathcal{K}$

where $\mathcal{K}$ denotes the set of feasible distributions and $\alpha > 0$.

A sufficient condition for (3.1) to hold is that for all $(\mathfrak{f}^1, \mathfrak{d}^1), (\mathfrak{f}^2, \mathfrak{d}^2) \in \mathcal{K}$
\[
\begin{align*}
[\hat{\hat{\hat{\mathcal{L}}}^*(f^*) - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)}]^T [\hat{\hat{\hat{\mathcal{L}}}^*(f^*) - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)}] & \geq \beta \|\hat{\hat{\hat{\mathcal{L}}}^*(f^*) - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)}\|^2 \\
-\begin{bmatrix} \hat{\hat{\hat{\mathcal{L}}}^*(d^*) - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)} \end{bmatrix}^T [\hat{\hat{\hat{\mathcal{L}}}^*(d^*) - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)}] & \geq \xi \|\hat{\hat{\hat{\mathcal{L}}}^*(d^*) - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)}\|^2
\end{align*}
\tag{3.2}
\]

for some \( \beta, \xi > 0 \).

Condition (3.2) implies that the factor production cost for a commodity depends primarily on the amount of the commodity using the particular factor. Similarly, condition (3.2) implies that the price of a commodity depends primarily on the demand for the particular commodity. These assumptions are reasonable and we would expect them to hold in many applications of interest.

We now show that small changes in the factor production cost functions and the inverse commodity demand functions induce small changes in the equilibrium pattern.

**Theorem 3.1**

\[
\|\hat{f}^* - \hat{f}, \hat{d}^* - \hat{d}\| \leq \frac{1}{\alpha} \|\hat{\hat{\hat{\mathcal{L}}}^*(f^*) - \hat{\hat{\hat{\mathcal{L}}}^*(f^*)}, -\hat{\hat{\hat{\mathcal{L}}}^*(d^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)}\| 
\tag{3.3}
\]

**Proof.** \((\hat{f}, \hat{d}), (\hat{f}^*, \hat{d}^*) \) must satisfy the variational inequalities

\[
\hat{\hat{\hat{\mathcal{L}}}^*(f^*)}^T (\hat{f}^* - \hat{f}) - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)}^T (\hat{d}^* - \hat{d}) \geq 0 \quad , \quad (\hat{f}^*, \hat{d}^*) \in \mathcal{K} \tag{3.4}
\]

\[
\hat{\hat{\hat{\mathcal{L}}}^*(d^*)}^T (\hat{f}^* - \hat{f}) - \hat{\hat{\hat{\mathcal{L}}}^*(d^*)}^T (\hat{d}^* - \hat{d}) \geq 0 \quad , \quad (\hat{f}^*, \hat{d}^*) \in \mathcal{K} \tag{3.5}
\]

Rewriting (3.4) for \( \hat{f}' = f^* \), \( \hat{d}' = d^* \), and (3.5) for \( \hat{f}' = f \) and \( \hat{d}' = d \) and adding the resultant inequalities, we obtain

\[
[\hat{\hat{\hat{\mathcal{L}}}^*(f^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(f)}]^T [\hat{\hat{\hat{\mathcal{L}}}^*(f^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(f)}] - [\hat{\hat{\hat{\mathcal{L}}}^*(d^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(d)}]^T [\hat{\hat{\hat{\mathcal{L}}}^*(d^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(d)}] \geq 0 
\tag{3.6}
\]

or

\[
[\hat{\hat{\hat{\mathcal{L}}}^*(f^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(f)} + \hat{\hat{\hat{\mathcal{L}}}^*(f)} - \hat{\hat{\hat{\mathcal{L}}}^*(f)}]^T [\hat{\hat{\hat{\mathcal{L}}}^*(f^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(f)}] - [\hat{\hat{\hat{\mathcal{L}}}^*(d^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(d)} + \hat{\hat{\hat{\mathcal{L}}}^*(d)} - \hat{\hat{\hat{\mathcal{L}}}^*(d)}]^T [\hat{\hat{\hat{\mathcal{L}}}^*(d^*)} - \hat{\hat{\hat{\mathcal{L}}}^*(d)}] \geq 0 . 
\tag{3.7}
\]

Using now monotonicity condition (3.1), (3.7) yields
\[
\begin{align*}
&\left[\tilde{\xi}(\xi^*) - \hat{\xi}(\xi^*)\right]^T [\xi^* - \xi^*] - \left[\tilde{\xi}(\xi^*) - \hat{\xi}(\xi^*)\right]^T [\xi^* - \xi^*] \\
\geq& \left[\hat{\xi}(\xi^*) - \hat{\xi}(\xi^*)\right]^T [\xi^* - \xi^*] - \left[\hat{\xi}(\xi^*) - \hat{\xi}(\xi^*)\right]^T [\xi^* - \xi^*] \\
\geq& \alpha |\xi^* - \xi|^2 + |\xi^* - \xi|^2 = \alpha \|\xi^* - \xi\| \cdot (\xi^* - \xi) \cdot (\xi^* - \xi) \cdot (\xi^* - \xi) \|^2
\end{align*}
\]  
(3.8)

Applying now Schwarz’s inequality to the left hand side of (3.8), we obtain

\[
\left[\left(\tilde{\xi}(\xi^*) - \hat{\xi}(\xi^*)\right) - \left(\tilde{\xi}(\xi^*) - \hat{\xi}(\xi^*)\right)\right] \cdot \left(\tilde{\xi}(\xi^*) - \hat{\xi}(\xi^*)\right) \geq \alpha \|\tilde{\xi}(\xi^*) - \hat{\xi}(\xi^*)\| \cdot (\tilde{\xi}(\xi^*) - \hat{\xi}(\xi^*)) \|^2
\]
(3.9)

from which (3.3) follows and the proof is complete.

We now proceed to study how changes in the factor production cost functions affect the direction of the change in the equilibrium commodity flows and the incurred production costs and how changes in the inverse commodity demand functions affect the direction of the change in the equilibrium commodity demands and the incurred equilibrium commodity prices. Whenever a new production process (link or path) is added to an existing market network, which may reflect a new production process, but existing commodities or an entirely new commodity, we may as well assume that this production process was there all the time but its cost was so high that it was not used in the production of the commodity. Similarly, whenever a new commodity is added, we may as well assume that the destination node was there all the time but that the price structure was such that there was no demand for the commodity. Hence, it suffices to consider the following problem: on a fixed network we change the factor production cost functions from \(\tilde{\xi}(\cdot)\) to \(\tilde{\xi}^*(\cdot)\) which induce corresponding production process cost functions \(\hat{\xi}(\cdot)\) and \(\hat{\xi}^*(\cdot)\), and we change the inverse commodity demand functions from \(\tilde{\xi}(\cdot)\) to \(\tilde{\xi}^*(\cdot)\), and we are to compare the corresponding equilibrium distribution and flow patterns \((\xi, d, F)\) and \((\xi^*, d^*, F^*)\).

We impose upon \(\hat{\xi}(\cdot)\) and \(\tilde{\xi}(\cdot)\) the strong monotonicity condition (3.2).

**Theorem 3.2.** Consider the market network with two factor production cost functions \(\tilde{\xi}(\cdot), \tilde{\xi}^*(\cdot)\), which induce corresponding production process cost functions \(\hat{\xi}(\cdot), \hat{\xi}^*(\cdot)\), and two inverse commodity demand functions \(\tilde{\xi}(\cdot)\) and \(\tilde{\xi}^*(\cdot)\). Let \((\xi, d, F)\) and \((\xi^*, d^*, F^*)\) be the equilibrium distribution and commodity flow patterns associated with \((\tilde{\xi}(\cdot), \hat{\xi}(\cdot))\) and \((\tilde{\xi}^*(\cdot), \hat{\xi}^*(\cdot))\) respectively. Then
\[
\sum_{a, w} \left[ \hat{c}^{* w}(f^*) - \hat{c}^w(f)_a \right] [f^* a - f^a] - \sum_w \left[ \hat{\beta}^{* w}(d^*) - \hat{\beta}^w(d)_a \right] [d^* - d^a] \leq 0 \quad (3.10)
\]

\[
\sum_p \left[ \hat{c}^{* p}(f^*)_p - \hat{c}^p(f)_p \right] [F^* - F^p] - \sum_w \left[ \hat{\beta}^{* w}(d^*)_p - \hat{\beta}^w(d)_p \right] [d^* - d^p] \leq 0 \quad (3.11)
\]

\[
\sum_{a, w} \left[ \hat{c}^{* w}(f^*) - \hat{c}^w(f)_a \right] [f^* a - f^a] - \sum_w \left[ \hat{\beta}^{* w}(d^*) - \hat{\beta}^w(d)_a \right] [d^* - d^a] \leq 0 \quad (3.12)
\]

\[
\sum_p \left[ \hat{c}^{* p}(f^*)_p - \hat{c}^p(f)_p \right] [F^* - F^p] - \sum_w \left[ \hat{\beta}^{* w}(d^*)_p - \hat{\beta}^w(d)_p \right] [d^* - d^p] \leq 0 \quad (3.13)
\]

**Proof.** Inequalities (3.10) and (3.12) have been established in the course of proving Theorem 3.1. Inequalities (3.11) and (3.13) follow from (3.10) and (3.12) upon using (2.4) and (2.9).

Observe that inequalities (3.10), (3.11), (3.12) and (3.13) show that under monotonicity condition (3.1), as regards changes in the incurred production costs brought about by "improving" the production cost situation, the following phenomena may prevail: an improvement in the production cost situation for a commodity may result in an increase in some of the incurred production costs and a decrease in some of the commodity flows. Nevertheless, it follows from Theorem 3.1 that this situation can never occur under the following circumstances.

**Corollary 3.1.** Assume that the production cost for factor a and commodity w is improved, while all other factor production costs remain fixed, that is, \( \hat{c}^{* w}(f^*) < \hat{c}^w(f) \) for some a, w, and \( f^* \in \mathcal{K} \) and \( \hat{c}^{* \omega}(f^a) = \hat{c}^\omega(f^a) \) for all \( b \neq a, \omega \neq w, \) and \( f^* \in \mathcal{K} \). Assume also that \( \partial \hat{c}^\omega(f^a) / \partial f^w = 0 \) for all \( b \neq a \) and \( \omega \neq w \) and \( f^* \in \mathcal{K} \).

If we fix the inverse demands for all commodities, that is, \( \hat{\beta}^{* w}(d^*) = \hat{\beta}^w(d^*) \) for all \( w \) and \( d^* \in \mathcal{K} \), then the commodity flow of commodity w on link a cannot decrease and the incurred factor production cost cannot increase, i.e., \( f^* a \geq f^w a \) and \( \hat{c}^{* w}(f^*_a) \leq \hat{c}^w(f^a) \).

**Proof.** Since \( \hat{c}^{* \omega}(f^*) = \hat{c}^\omega(f^a) \) for all \( b \neq a, \omega \neq w, f^* \in \mathcal{K} \) and \( \hat{\beta}^{* w}(d^*) = \hat{\beta}^w(d^*) \) for all \( w \) and \( d^* \in \mathcal{K} \), inequality (3.12) yields
\[
\sum_{b,\omega} \left[ \hat{c}_{b}^{\omega}(f^*) - \hat{c}_{b}^{\omega}(f) \right] [f_{b}^{\omega} - f_{b}^{w}] = \left[ \hat{c}_{a}^{*w}(f^*) - \hat{c}_{a}^{w}(f) \right] [f_{a}^{*w} - f_{a}^{w}] \leq 0 \tag{3.14}
\]

which implies, since \(\hat{c}_{a}^{*w}(f^*) < \hat{c}_{a}^{w}(f)\), that \(f_{a}^{*w} > f_{a}^{w}\). From (3.10) it follows that

\[
\left[ \hat{c}_{a}^{*w}(f^*) - \hat{c}_{a}^{w}(f) \right] [f_{a}^{w} - f_{a}^{*w}] \geq 0
\tag{3.15}
\]

\[
\sum_{b,\omega} \left[ \hat{c}_{b}^{\omega}(f^*) - \hat{c}_{b}^{\omega}(f) \right] [f_{b}^{\omega} - f_{b}^{w}] - \sum_{w} \left[ \hat{\rho}_{w}(d^*) - \hat{\rho}_{w}(d) \right] [d_{w}^{*} - d_{w}] = 0,
\]

\(b \neq a\)
\(\omega \neq w\)

The last term on the right-hand side of (3.15) is nonnegative by the monotonicity condition (3.2). Applying the mean value theorem to the first term on the right-hand side of (3.15) and using the assumption that \(\partial \hat{c}_{b}^{\omega}(f')/\partial f_{a}^{w} = 0\) for all \(b \neq a\) and \(\omega \neq w\), and \(f' \in \mathcal{X}\), we obtain

\[
\sum_{b,\omega} \left[ \hat{c}_{b}^{\omega}(f^*) - \hat{c}_{b}^{\omega}(f) \right] [f_{b}^{\omega} - f_{b}^{w}] =
\]

\(b \neq a\)
\(\omega \neq w\)

\[
\sum_{b,\omega} \sum_{c,v} \int_{0}^{1} \left. \frac{\partial \hat{c}_{b}^{\omega}(f')}{\partial f_{c}^{v}} \right|_{f' = (1-t)f^* + tf} \ dt (f_{c}^{v} - f_{c}^{*v}) (f_{b}^{\omega} - f_{b}^{w})
\tag{3.16}
\]

On account of condition (3.2) the matrix \(\left[ \frac{\partial \hat{c}_{c}}{\partial f} \right] \) is positive definite and, hence, the right hand side of (3.16) is nonnegative. Therefore, (3.15) implies

\[
\left[ \hat{c}_{a}^{*w}(f^*) - \hat{c}_{a}^{w}(f) \right] [f_{a}^{w} - f_{a}^{*w}] \geq 0
\tag{3.17}
\]

Since we showed that \(f_{a}^{*w} > f_{a}^{w}\), we conclude that

\[
\hat{c}_{a}^{*w}(f^*) \leq \hat{c}_{a}^{w}(f).
\tag{3.18}
\]
Finally, inequalities (3.10), (3.11), (3.12) and (3.13) suggest that, as regards changes in the incurred equilibrium prices brought about by changes in the inverse demand mechanism, the following phenomena may prevail: an increase in the inverse demand for a commodity may result in a decrease in some of the incurred equilibrium prices and a decrease in some of the demands. However, it follows from Theorem 3.2 that the above-mentioned phenomena can never occur under the following circumstances.

**Corollary 3.2.** Here we assume that the inverse demand for commodity \( w \) is increased, while all other inverse commodity demand functions remain fixed, that is, \( \hat{\rho}_w^*(d_w) > \hat{\rho}_w^*(d_w') \) for some \( w \), and \( d' \in \mathcal{K} \) and \( \hat{\rho}_{w'}^*(d_{w'}) = \hat{\rho}_{w'}^*(d_{w'}) \) for all \( \omega \neq w \), \( d' \in \mathcal{K} \). Assume also that \( \partial \hat{\rho}_w^*(d_w')/\partial d_w = 0 \), for all \( \omega \neq w \), \( d' \in \mathcal{K} \). If we fix the factor production cost functions for all commodities, that is, \( \hat{c}_a^w(f') = \hat{c}_a^w(f') \) for all \( a, w \) and \( f' \in \mathcal{K} \), then the equilibrium inverse demand, equivalently, price, for commodity \( w \) cannot decrease and the commodity demand cannot decrease, i.e., \( \hat{\rho}_w^*(d_w) > \hat{\rho}_w^*(d_w) \) and \( d_w^* > d_w^* \).

**Proof.** Since \( \hat{c}_a^w(f') = \hat{c}_a^w(f') \) for all \( a, w \) and \( f' \in \mathcal{K} \) and \( \hat{\rho}_w^*(d_w) = \hat{\rho}_w^*(d_w) \) for all \( \omega \neq w \) and \( d' \in \mathcal{K} \), inequality (3.12) yields

\[
- \sum_{\omega} \left[ \hat{\rho}_w^*(d_w^*) - \hat{\rho}_w^*(d_w^*) \right] [d_w^* - d_w] = - \sum_{\omega} \left[ \hat{\rho}_w^*(d_w^*) - \hat{\rho}_w^*(d_w^*) \right] [d_w^* - d_w] \leq 0 \tag{3.19}
\]

which implies, since \( \hat{\rho}_w^*(d_w^*) > \hat{\rho}_w^*(d_w^*) \) that \( d_w^* > d_w^* \). From inequality (3.10) we get

\[
- \left[ \hat{\rho}_w^*(d) - \hat{\rho}_w^*(d_w^*) \right] [d_w^* - d_w] \geq 0 \tag{3.20}
\]

\[
\sum_{\omega} \left[ \hat{\rho}_w^*(d_w^*) - \hat{\rho}_w^*(d_w^*) \right] [d_w^* - d_w] + \sum_{a,w} \left[ \hat{c}_a^w(f_w^*) - \hat{c}_a^w(f_w^*) \right] [f_w^* - f_w^*].
\]

The last term on the right hand side in (3.20) is nonnegative under the assumption of
monotonicity. Applying the mean value theorem to the first term on the right hand side and using the assumption that \( \frac{\partial \hat{\rho}_w^*(d')}{\partial d_w} = 0 \) for all \( \omega \neq w, d' \in \mathcal{K} \), we obtain

\[
- \sum_{\omega \neq w} \left[ \hat{\rho}_w^*(d^*) - \hat{\rho}_w^*(d) \right] \left[ d^*_w - d_w \right] = \\
- \sum_{\omega \neq w} \sum_{v \neq w} \int_0^1 \frac{\partial \hat{\rho}_w^*(d')}{\partial d_v} \left. \right|_{d' = \left[(1-t)d^*_v + td_v\right]} dt (d^*_v - d_v) (d^*_w - d_w) \quad (3.21)
\]

Due to monotonicity condition (3.2), the matrix \(- \left[ \frac{\partial \hat{\rho}_w^*}{\partial d} \right] \) is positive definite and, hence, the right hand side of (3.21) is nonnegative. Therefore, (3.20) implies

\[
- \left[ \hat{\rho}_w^*(d) - \hat{\rho}_w^*(d^*) \right] \left[ d^*_w - d_w \right] \geq 0 \quad (3.22)
\]

and since, as shown above, \( d^*_w \geq d_w \), (3.22) implies that

\[
\hat{\rho}_w^*(d^*) \geq \hat{\rho}_w^*(d) . \quad (3.23)
\]

Corollary 3.3. Here we assume that the inverse demand for commodity \( w \) is decreased while all other inverse commodity demand functions remain fixed, that is, \( \hat{\rho}_w^*(d') < \hat{\rho}_w^*(d) \) for some \( w \), and \( d' \in \mathcal{K} \), and \( \hat{\rho}_w^*(d') = \hat{\rho}_w^*(d) \) for all \( \omega \neq w, d' \in \mathcal{K} \). Assume also that \( \frac{\partial \hat{\rho}_w^*(d')}{\partial d_w} = 0 \), for all \( \omega \neq w, d' \in \mathcal{K} \). If we fix the factor production cost functions for all commodities, that is, \( \hat{c}_a^w(f') = \hat{c}_a^w(f') \) for all \( a, w \) and \( f' \in \mathcal{K} \), then the equilibrium price for commodity \( w \) cannot increase and the commodity demand cannot increase, i.e., \( \hat{\rho}_w^*(d^*_w) \leq \hat{\rho}_w^*(d) \) and \( d^*_w \leq d_w \).

The proof is the same as for Corollary 3.2.

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