CRITICAL SETS AND NEGATIVE BUNDLES

BY

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CRITICAL SETS AND NEGATIVE BUNDLES

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Let $f : \mathbb{R}^{m+1} \times \mathbb{R}^k \to \mathbb{R}$ be smooth and let $\Sigma_f = \{(x,v) \in \mathbb{R}^{m+1} \times \mathbb{R}^k : df_{(x,v)}|\mathbb{R}^k = 0\}$. When the matrix

$$A_{(x,v)} = \underbrace{\left[\frac{\partial^2 f}{\partial x_\ell \partial v_i} \;\middle|\; \frac{\partial^2 f}{\partial v_j \partial v_i} \right]_{(x,v)}}_{\leftarrow\; m+k+1 \;\rightarrow} \begin{matrix} \uparrow \\ k \\ \downarrow \end{matrix}$$

has rank k for all $(x,v) \in \Sigma_f$ we call f a <u>generating function</u> [3]. In this case Σ_f is a smooth $m+1$-dimensional submanifold of $\mathbb{R}^{m+1} \times \mathbb{R}^k$, and the quadratic form H_f on $\mathbb{R}^k$ represented by the right hand block of $A_{(x,v)}$ is the <u>fibre-Hessian</u> of f at $(x,v) \in \Sigma_f$.

Let M be a connected oriented smooth hypersurface (not necessarily compact) in $\mathbb{R}^{m+1}$, and let $g : M \times [0,1] \to \Sigma_f$ be a smooth embedding such that

(i) f is a generating function,

(ii) H_f is positive-definite over $g(M \times \{0\})$ and non-degenerate over $g(M \times \{1\})$.

The following result is a consequence of the argument in [2] §40.

Lemma. Smooth homotopy classes of fibrewise non-degenerate quadratic forms on the trivial bundle $M \times \mathbb{R}^k$ over M correspond bijectively to smooth homotopy classes of inclusions $E \to M \times \mathbb{R}^k$ of orthogonal bundles over M. (We take the Euclidean norm on $\mathbb{R}^k$, and the correspondence is given by $E \to Q_E$ where $Q_E(y,v) = \|v\|^2$ or $-\|v\|^2$ according as $v \in E_y^{\perp}$ or $v \in E_y$.)

So H_f gives rise to an equivalence class $E_{f,g}$ of orthogonal bundles over $M \times \{1\}$.

Theorem. Let $M = N \times \mathbb{R}^{m-n}$ where N is an oriented hypersurface in $\mathbb{R}^{n+1}$. Given an equivalence class E of orthogonal p-plane bundles over M, with $p \leqq m - 2n - 1$, then $E = E_{f,g}$ for some choices of f,g.

Our construction is explicit. In §1 we recall some familiar facts about hypersurfaces in $\mathbb{R}^{m+1}$ and interpret these in our setting. In §2 we prove some routine results about embedding vector bundles. The main work is done in §3 and exploits the geometry of sphere bundles. Applications are reserved for another paper.

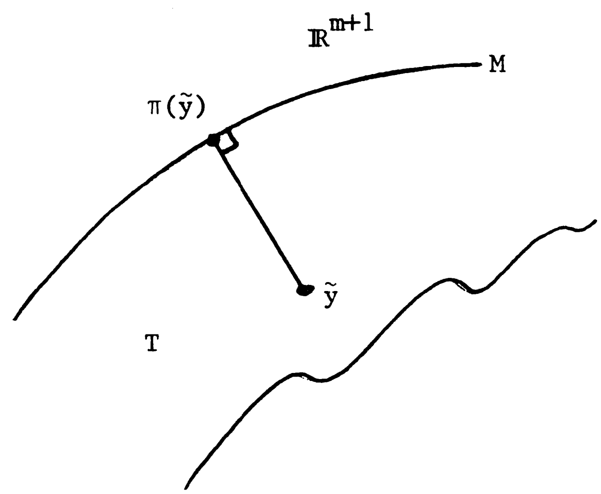
It is not clear whether, by use of a different construction, the stability condition $p \leqq m - 2n - 1$ can always be relaxed. For example, can the nontrivial line bundle over $M = S^1 \times \mathbb{R}^2 \subset \mathbb{R}^4$ represent $E_{f,g}$? Our construction does lead to stronger results in certain cases however.

Example 1. Let ι be the smallest integer such that E is an orthogonal sub-bundle of the trivial bundle $(\iota + p)$ of dimension $\iota + p$ over M. Then $\iota \leqq n$, and it suffices to suppose $p \leqq m - \iota - n - 1$. For example, if E is induced from the tangent bundle of the oriented hypersurface N, then E can represent $E_{f,g}$ provided $m \geqq 2(n + 1)$.

Example 2. If E is induced from the tangent bundle of $N = S^n$ then E can represent $E_{f,g}$ provided only $m \geqq n$.

1. Hypersurfaces

Let ν be a unit normal field on M. Given $y_0 \in M$ let $y : \mathbb{R}^m \to M$ be a diffeomorphism onto an open neighbourhood U of $y_0 = y(\underline{0})$ in M. Define $\tilde{y} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{m+1}$ by $\tilde{y}(u,t) = (y(u), t\nu(y(u)))$. Then, for some open neighbourhood V_{y_0} of $(\underline{0},0)$ in $\mathbb{R}^m \times \mathbb{R}$, we have $\tilde{y}|V_{y_0}$ a diffeomorphism onto an open subset of $\mathbb{R}^{m+1}$. Let $T = \bigcup_{y_0 \in M} V_{y_0}$ and define $\pi : T \to M$ by $\pi(\tilde{y}(u,t)) = y(u)$.



Define $\varepsilon : \mathbb{R}^{m+1} \times T \to \mathbb{R}$ by $\varepsilon(x,\tilde{y}) = \|x - \pi(\tilde{y})\|^2 + \|\tilde{y} - \pi(\tilde{y})\|^2$, where $\| \ \|$ still means the Euclidean norm. Then

$$\frac{\partial \varepsilon}{\partial u_i} = -2 < x - y \, , \, \frac{\partial y}{\partial u_i} > \qquad i = 1,\dots m$$

$$\frac{\partial \varepsilon}{\partial t} = 2t$$

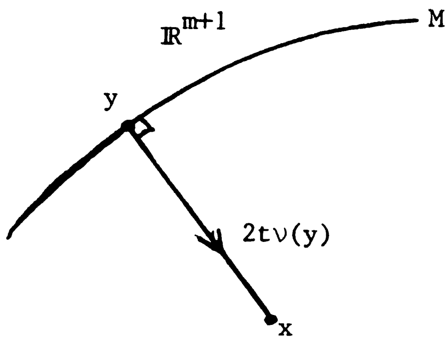
where $y = \pi(\tilde{y})$.

We extend ε to a smooth map $\mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$. Then for (x,y) in $\Sigma_\varepsilon \cap (\mathbb{R}^{m+1} \times T) = \{(x,y) \in \mathbb{R}^{m+1} \times M : x - y \in (TM_y)^\perp\}$ the matrix $A_{(x,y)}$ is

$$\begin{array}{c c}
m & \left[\begin{array}{c|c|c}
-2\,\dfrac{\partial y_\ell}{\partial u_i} & 2 < \dfrac{\partial y}{\partial u_j} \, , \, \dfrac{\partial y}{\partial u_i} > - \;\; 2 < x - y \, , \, \dfrac{\partial^2 y}{\partial u_j \partial u_i} > & 0 \\
\hline
0 & 0 & 2
\end{array}\right] \\
 & \begin{array}{c c} \longleftarrow m \longrightarrow & \longleftarrow \quad m \quad \longrightarrow \end{array}
\end{array}$$

which has rank $m + 1$. So we can suppose that the extension ε is a generating function. Then $\Sigma_\varepsilon \cap (\mathbb{R}^{m+1} \times T)$ is a connected component of Σ_ε .

We have a diffeomorphism $\delta : M \times \mathbb{R} \to \Sigma_\varepsilon \cap (\mathbb{R}^{m+1} \times T)$ given by $\delta(y,t) = (y + 2t\nu(y), y)$.



Then H_ε is positive-definite over $\delta(M \times \{0\})$, and in [1] §6 it is shown that if $\delta(M \times \{1\})$ contains no focal points of M then H_ε is nondegenerate over $\delta(M \times \{1\})$. All this is well-known, and we have a particular case in mind.

2. Vector bundles

Let E be an orthogonal p-plane bundle over $M = N \times \mathbb{R}^{m-n}$, as in our theorem.

Lemma 1. If $p \geqq n + 1$ then E is smoothly equivalent to a Whitney sum $E' \oplus (p - n)$, where E' is an orthogonal n-plane bundle and $(p - n)$ is the trivial bundle of dimension $p - n$.

Proof. We must show that the classifying map χ_E of E lifts to $BO(n)$ in the diagram

$$\begin{tikzcd} & BO(n) \arrow[d] \\ N \times \mathbb{R}^{m-n} \arrow[r, "\chi_E"] \arrow[ur, dashed] & BO(p) \end{tikzcd}$$

But this follows from the fact that the fibre $O(p)/O(n)$ of the vertical map is $n - 1$-connected.

Lemma 2. E embeds as an orthogonal bundle in $(n + p)$.

Proof. We have $E \oplus E' \cong (\ell + p)$ for some orthogonal ℓ-plane bundle E' . If $\ell \leqq n$ then there is nothing to prove.

If $\ell \geqq n + 1$ then by Lemma 1 we have $E' \cong E'' \oplus (\ell - n)$ for some orthogonal n-plane bundle E'' . We have $(E \oplus E'') \oplus (\ell - n) \cong (n + p) \oplus (\ell - n)$, and so the map

$$N \times \mathbb{R}^{m-n} \xrightarrow{\chi_{E\oplus E''}} BO(n + p) \xrightarrow{i} BO(\ell + p)$$

is nulhomotopic. But the homotopy-theoretic fibre of the inclusion i is $O(\ell + p)/O(n + p)$ which is $n + p - 1$-connected. So $\chi_{E\oplus E''}$ is nulhomotopic also.

3. Sphere bundles

In the situation of §2 let $j : N \times \{\underline{0}\} \to N \times \mathbb{R}^{m-n}$ be the usual inclusion, and let $\tilde{E} = j^*E$. Let $\tilde{E}^{\perp}$ be the orthogonal complement of $\tilde{E}$ in the trivial bundle $(n + p)$ over N . (We take the Euclidean inner product on $\mathbb{R}^{n+p}$.)

Let S be the unit p-sphere bundle associated with $\tilde{E} \oplus (1)$, and let s be the cross-section of S given by $s(y) = (y,\underline{0},1)$. We have $S \subseteq \tilde{E} \oplus (1) \subseteq (n+p) \oplus (1) = (n+p+1)$, and fibrewise addition defines an embedding of the fibre product $S \times_N \tilde{E}^{\perp}$ in $(n+p+1)$.

Let P be the induced bundle $\pi^*(S \times_N \tilde{E}^{\perp})$ over $N \times \mathbb{R}^q$ where $\pi : N \times \mathbb{R}^q \to N$ means the usual projection and $q = m - 2n - p$. We have a fibre preserving embedding of P in $(N \times \mathbb{R}^q) \times \mathbb{R}^{n+p+1}$. Now since N is oriented there is a diffeomorphism of $N \times \mathbb{R}$ onto an open subset of $\mathbb{R}^{n+1}$, and so we have a fibre preserving inclusion

$$\begin{array}{ccc} (N \times \mathbb{R}^q) \times \mathbb{R}^{n+p+1} & \longrightarrow & (\mathbb{R}^{n+q}) \times \mathbb{R}^{n+p+1} \\ \downarrow \pi & & \downarrow \pi \\ N \times \mathbb{R}^q & \longrightarrow & \mathbb{R}^{n+q} \end{array}$$

which is the identity on fibres, whereas the inclusion of $N \times \mathbb{R}^q$ in $\mathbb{R}^{n+q}$ is unlikely to be isometric.

We define $\nu(y,t,v_1,v_2) = -v_1$ for $v_1 \in (\pi^*S)_{(y,t)}$, $v_2 \in (\pi^*\tilde{E}^{\perp})_{(y,t)}$. Then ν is a unit normal field on the hypersurface P in $(\mathbb{R}^{n+q}) \times \mathbb{R}^{n+p+1} \cong \mathbb{R}^{m+1}$, and we perform the construction of §1 with P in place of M.

Then $E_{\varepsilon,\delta}$ is represented by the bundle of directions in which P is curved, namely the pullback in the diagram

$$\begin{array}{ccccc} & & & & T \\ & & & & \downarrow \\ P \times \{1\} = P & \longrightarrow & \pi^*S & \longrightarrow & S \end{array}$$

where T is the tangent bundle along the fibres of S .

Let s_1 be the cross-section of π^*S pulled back by π from s , and let $M_1 \subset \pi^*S$ be the bundle over $N \times \mathbb{R}^q$ of open hemispheres whose centre over (y,t) is $s_1(y,t)$. Then M_1 is a fibrewise tubular neighbourhood of $s_1(N \times \mathbb{R}^q)$ in π^*S and is fibrewise diffeomorphic to $\pi^*\tilde{E}$.

Let $M = M_1 \times_{N \times \mathbb{R}^q} \pi^*\tilde{E}^{\perp} \subset P$. Then M is fibrewise diffeomorphic to $(N \times \mathbb{R}^q) \times \mathbb{R}^{n+p}$. We take f to be ε , and g to be the composite $M \times [0,1] \to P \times [0,1] \xrightarrow{\delta} \Sigma_f$. Then $E_{f,g}$ is the restriction of $E_{\varepsilon,\delta}$ to $M \times \{1\}$, namely the pullback in the diagram

$$\begin{tikzcd} & & & T \arrow[d] \\ M \times \{1\} \cong M \arrow[r] & P \arrow[r] & \pi^*S \arrow[r] & S \end{tikzcd} .$$

But the composite

$$N \times \{\underline{0}\} \xrightarrow{j} N \times \mathbb{R}^{m-n} \xrightarrow{\cong} M \longrightarrow P \longrightarrow \pi^*S \longrightarrow S$$

is the cross-section s , and so $j^*E_{f,g}$ is represented by $\pi^*T \cong \tilde{E} = j^*E$. This proves our theorem.

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