RANK INCREMENTATION VIA DIAGONAL PERTURBATIONS

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Introduction. Let $M_n(F)$ denote the set of n-by-n matrices over the field F. We consider the following question: Among matrices $A \in M_n(F)$ with rank $A = k < n$, how many diagonal entries of A must be changed, at worst, in order to guarantee that the rank of A is increased. Our initial motivation arose from an error pointed out in [BOvdD], but we also view this problem as intrinsically important. The simplest example that shows that one entry does not suffice is the familiar Jordan block $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Let

$$J_r = \begin{bmatrix} 0 & 1 & & 0 & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}$$

be the basic nilpotent r-by-r Jordan block. If $A = J_n$, rank $A = n-1$, and all n diagonal entries of A must be changed in order to increase the rank of A to n. More generally, assume that A is a direct sum of such blocks. If rank $A = k$, A is the sum of $n-k$ blocks

$$A = J_{r_1} \oplus J_{r_2} \oplus \cdots \oplus J_{r_{n-k}} \tag{1}$$

in which we take $r_1 \le r_2 \le \cdots \le r_{n-k}$, and $r_1 + r_2 + \cdots + r_{n-k} = n$. To increase the rank of A it is necessary and sufficient to change all the diagonal entries in one block to nonzero entries. Thus, r_1 changes will do. Since

$$(n-k)r_1 \le r_1 + \cdots + r_{n-k} = n$$

we have $r_1 \le \frac{n}{n-k}$ and so $r_1 \le \lfloor \frac{n}{n-k} \rfloor$, in which $\lfloor x \rfloor$, the "floor" function, denotes the greatest integer less than or equal to x. By taking $r_1 = r_2 = \cdots = r_{n-k-1} = \lfloor \frac{n}{n-k} \rfloor$ in equation (1) we see that we cannot hope, in general, to increase the rank of A by fewer than $\lfloor \frac{n}{n-k} \rfloor$ changes. We are therefore led to the statement of our first principal result.

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THEOREM 1. *Let F be any field and let $A \in M_n(F)$ with* rank $A = k < n$. *Then it suffices to change at most $\lfloor \frac{n}{n-k} \rfloor$ diagonal entries of A in order to increase the rank of A.*

We note that $\lfloor \frac{n}{n-k} \rfloor = \lceil \frac{k+1}{n-k} \rceil = 1 + \lfloor \frac{2k}{n} \rfloor + \lfloor \frac{3k}{2n} \rfloor + \lfloor \frac{4k}{3n} \rfloor + \cdots$, each of which suggests an alternate understanding of the quantity calculated in theorem 1; here $\lceil\ \rceil$ denotes the "roof" function.

Although the argument given above shows that theorem 1 is sharp, it is unsatisfactory in one respect. Suppose, for example, that $A = B \oplus 0_{n-k}$, in which B is an invertible matrix in $M_k(F)$. Then rank $A = k$ and changing any diagonal entry in A outside B will increase the rank of A. In this case, the prediction of $\lfloor \frac{n}{n-k} \rfloor$ is overly pessimistic. We can make a slight adjustment to theorem 1 that accomodates such examples.

DEFINITION. For $A \in M_n(F)$, the ***principal rank*** of A is

$$p = p(A) \equiv \max_{\alpha \subseteq N} \{ |\alpha| : A[\alpha] \text{ is invertible } \}$$

Here $N = \{1, 2, \ldots, n\}$, $|\alpha|$ denotes the cardinality of α, and $A[\alpha]$ denotes the principal submatrix of A whose rows and columns lie in α.

Of course, $p(A) \leq k(A) \equiv \text{rank}(A)$, and, by appeal to the presentation of the characteristic polynomial in terms of principal minor sums [HJ], the number of nonzero eigenvalues of A is $\leq p(A)$.

If we now consider $A = I_p \oplus J_{r_1} \oplus \cdots \oplus J_{r_{n-k}}$, so that $r_1 + r_2 + \cdots + r_{n-k} = n - p$, then we realize that we might have to change $\lfloor \frac{n-p}{n-k} \rfloor$ entries of A to increase its rank.

THEOREM 2. *Let $A \in M_n(F)$ with* rank $A = k < n$ *and* principal rank $A = p$. *Then it suffices to change at most $\lfloor \frac{n-p}{n-k} \rfloor$ diagonal entries of A in order to increase the* rank *of A.*

In the case $A = B \oplus 0_{n-k}$ cited above, $p = k$ and theorem 2 now indicates that it suffices to change one diagonal entry.

Note that all aspects of our problem are unchanged by permutation similarity of A, which we are free to use. Arbitrary similarities are not available, however.

We first note that theorem 2 is a consequence of theorem 1.

Proof of Theorem 2. If $p = 0$, there is nothing to prove.

Suppose $p > 0$ and let $\alpha \subseteq N$, $|\alpha| = p$ with $A[\alpha]$ invertible. Then by a permutation similarity, we may assume without loss of generality, that

$$A = \begin{bmatrix} A[\alpha] & B \\ C & D \end{bmatrix}.$$

We use the following elementary fact about the Schur complement of $A[\alpha]$ in A:

$$\text{rank } A = \text{rank } A[\alpha] + \text{rank}(D - C(A[\alpha])^{-1} B).$$

So $\operatorname{rank}(D - C(A[\alpha])^{-1}B) = k - p < n - p$. Now, changing diagonal entries in D corresponds exactly to changing diagonal entries in $D - C(A[\alpha])^{-1}B$. Since $D - C(A[\alpha])^{-1}B \in M_{n-p}(F)$, by Theorem 1 we can increase its rank, and hence increase the rank of A, by changing at most $\lfloor \frac{n-p}{n-p-(k-p)} \rfloor = \lfloor \frac{n-p}{n-k} \rfloor$ diagonal entries. □

COROLLARY. *Let $A \in M_n(F)$, and suppose that the algebraic multiplicity and the geometric multiplicity of zero as an eigenvalue of A are both $n - k > 0$. Then it suffices to change 1 diagonal entry of A in order to increase its rank.*

Proof. In this event $p = k$, as the number of nonzero eigenvalues is k. Thus, $\lfloor \frac{n-p}{n-k} \rfloor = 1$. □

The corollary includes such special cases as diagonalizable, normal and Hermitian matrices. Each requires only one change of a diagonal entry.

We now proceed with the proof of theorem 1. Because of technical difficulties when F is a small finite field, we first prove the following weaker version.

THEOREM 3. *Let F be a field with $|F| \geq 2n$ and let $A \in M_n(F)$ with $\operatorname{rank} A = k < n$. Then it suffices to change at most $\lfloor \frac{n}{n-k} \rfloor$ diagonal entries of A in order to increase the rank of A.*

The proof of theorem 3 relies upon a sequence of observations.

PROPOSITION 1. *Let $A \in M_n(F)$ with $\operatorname{rank} A = k < n$. If there exist $\alpha, \beta \subseteq N$ with $\det A[\alpha|\beta] \neq 0$, $|\alpha| = |\beta| = k$ and $\alpha \cup \beta \neq N$, then the rank of A can be increased by changing one diagonal entry of A.*

Proof. Pick $i \notin \alpha \cup \beta$, let E_i be the matrix in $M_n(F)$ with 1 in the (i,i) position and 0 elsewhere, and let $A' = A + E_i$. Then $\det A'[\alpha \cup \{i\}|\beta \cup \{i\}]$
$= \pm \det A[\alpha|\beta] + \det A[\alpha \cup \{i\}|\beta \cup \{i\}]$. But $\det A[\alpha \cup \{i\}|\beta \cup \{i\}] = 0$ since $\operatorname{rank} A = k$, so $\det A'[\alpha \cup \{i\}|\beta \cup \{i\}] \neq 0$ and the rank of A has been increased by changing one diagonal entry. □

PROPOSITION 2. *Given a field F with $|F| > 2$, let $A \in M_n(F)$ with rank $A = k < n$, and let $A[\alpha|\beta]$, $|\alpha| = |\beta| = k$, be invertible. Assume $\alpha \cup \beta = N$. If there exist $i \in \alpha - \beta$ and $j \in \beta - \alpha$ such that $\det A[\alpha - \{i\}|\beta - \{j\}] \neq 0$, then the rank of A can be increased by changing at most two diagonal entries of A.*

Proof. Let $A'' = A + x(E_i + E_j)$. Assuming that the indices of A occur in the order $\alpha - \beta$, $\alpha \cap \beta$, $\beta - \alpha$, we have

$$A'' = \begin{array}{c} \\ \end{array}\begin{array}{ccc} \alpha-\beta & \alpha\cap\beta & \beta-\alpha \end{array}$$

$$A'' = \left[\begin{array}{c|c|c} \ddots \quad a_{ii}+x \quad \ddots & & \\ \hline & & \\ \hline & & \ddots \quad a_{jj}+x \quad \ddots \end{array}\right] \begin{array}{c} \alpha-\beta \\ \alpha\cap\beta \\ \beta-\alpha \end{array}$$

Then $\det A''[\alpha \cup \{j\}|\beta \cup \{i\}]$ is a polynomial of degree at most 2 in x. Since the coefficient of x^2 is $\pm \det A[\alpha - \{i\}|\beta - \{j\}]$, it has degree exactly 2 and hence has at most 2 roots in F. Since $|F| > 2$, we may choose $x \in F$ so that $\det A''[\alpha \cup \{j\}|\beta \cup \{i\}] \neq 0$. Hence $\operatorname{rank} A'' \geq k+1$ and the rank of A can be increased by changing at most 2 diagonal entries. □

LEMMA. *Let $B \in M_m(F)$ be invertible, and let $\sigma, \tau \subseteq \{1, \ldots, m\}$. Then*

$$\text{nullity } B[\sigma|\tau] = \text{nullity } B^{-1}[\tau^c|\sigma^c].$$

Proof. Various statements equivalent to this fact have arisen in a variety of places, e.g. [BGK], Corollary 3 in [FM], [G], Lemma 4 in [JL]. For completeness we give a simple proof.

Without loss of generality, via permutation equivalence, we may assume $B[\sigma|\tau] = B_{12}$ and $B^{-1}[\tau^c|\sigma^c] = A_{12}$ with

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \text{ and } A = B^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

If $x \in$ null space (B_{12}), then

$$\begin{bmatrix} 0 \\ x \end{bmatrix} = AB \begin{bmatrix} 0 \\ x \end{bmatrix} = A \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} A_{12}y \\ * \end{bmatrix}$$

exhibits an isomorphism between the null spaces of B_{12} and A_{12}. □

Proof of Theorem 3.

Consider first the case $\frac{n}{n-k} < 2$, or equivalently, $2k < n$. Then $|\alpha \cup \beta| \leq |\alpha| + |\beta| = 2k$ so that $\alpha \cup \beta \neq N$. By proposition 1, it suffices to change $1 = \lfloor \frac{n}{n-k} \rfloor$ entry in order to increase $\operatorname{rank} A$.

So from now on assume $\frac{n}{n-k} \geq 2$, or $2k \geq n$. In particular, $n \geq 2$ so $|F| \geq 4$. Also, assume that for any invertible submatrix $A[\alpha|\beta]$ with $|\alpha| = |\beta| = k$ that $\alpha \cup \beta = N$. Otherwise we are done by proposition 1. Since rank $A = k$, there is at least one invertible submatrix $A[\alpha|\beta]$ with $|\alpha| = |\beta| = k$. Let

$$\begin{aligned} B &= A[\alpha|\beta], \\ C &= A[\alpha \cap \beta|\alpha \cap \beta]. \end{aligned} \tag{2}$$

We call B the rank determining block of A. If the condition in Proposition 2 is met, we can increase the rank of A by changing at most two diagonal entries, and since $\frac{n}{n-k} \geq 2$ we are done.

We therefore assume that $\det A[\alpha - \{i\}|\beta - \{j\}] = 0$ for all $i \in \alpha - \beta$ and all $j \in \beta - \alpha$; equivalently, the cofactors B_{ij} in B are zero for $i \in \alpha - \beta$, $j \in \beta - \alpha$. Therefore,

$$B^{-1}[\beta - \alpha|\alpha - \beta] = 0. \tag{3}$$

Now $|\alpha \cap \beta| = |\alpha| + |\beta| - |\alpha \cup \beta| = 2k - n$, so $|\alpha - \beta| = |\alpha| - |\alpha \cap \beta| = k - (2k - n) = n - k$, and $|\beta - \alpha| = n - k$ also. Since $B^{-1} \in M_k(F)$ is invertible, by the Frobenius-König Theorem (see [F] or p. 543 of the survey article [MP]), $|\alpha - \beta| + |\beta - \alpha| \leq k$, so we have $2n \leq 3k$. Since $k \geq \frac{2}{3}n$, $n - k \leq \frac{1}{3}n$, and now $\lfloor \frac{n}{n-k} \rfloor \geq 3$. The matrix $C \in M_{2k-n}(F)$ defined by (2) is a submatrix of B and by the lemma and equation (3)

$$\text{nullity}\, C = \text{nullity}\, B^{-1}[\beta - \alpha, \alpha - \beta] = n - k.$$

Therefore,

$$\text{rank}\, C = 2k - n - (n - k) = 3k - 2n$$

We now proceed by induction on n. Since $2n \leq 3k \leq 3(n-1)$, we have $n \geq 3$, so we know that the theorem holds for $n = 1, 2$. Let $n \geq 3$ be a fixed integer and assume the theorem is true for all positive integers $m < n$. Since $0 < 2k - n < n$, we can increase the rank of C by changing at most s diagonal entries of C where

$$s \leq \left\lfloor \frac{2k - n}{2k - n - (3k - 2n)} \right\rfloor = \left\lfloor \frac{2k - n}{n - k} \right\rfloor.$$

Let $i_1, \ldots, i_s \in \alpha \cap \beta$ be the indices corresponding to the entries which are changed, and let

$$\begin{aligned} \widetilde{A}(x_1, \ldots, x_s) &= A + x_1 E_{i_1} + x_2 E_{i_2} + \cdots + x_s E_{i_s} \\ \widetilde{B}(x_1, \ldots, x_s) &= \widetilde{A}[\alpha|\beta] \\ \widetilde{C}(x_1, \ldots, x_s) &= \widetilde{A}[\alpha \cap \beta|\alpha \cap \beta]. \end{aligned}$$

Then the fact that we can increase the rank of C by changing s diagonal entries is equivalent to the statement that there are nonzero $f_1, \ldots, f_s \in F$ such that $\operatorname{rank} \widetilde{C}(f_1, \ldots, f_s) > 3k - 2n$. However, although $\operatorname{rank} B = \operatorname{rank} \widetilde{B}(0, \ldots, 0) = k$, it may now be the case that $\operatorname{rank} \widetilde{B}(f_1, \ldots, f_s) < k$. We overcome this obstruction by using the fact that $|F|$ is sufficiently large.

Let $\widetilde{C}'(x_1, \ldots, x_s)$ be a submatrix of $\widetilde{C}(x_1, \ldots, x_s)$ of order $3k - 2n + 1$ for which $\widetilde{C}'(f_1, \ldots, f_s)$ is invertible. Since each x_j occurs at most once in $\widetilde{C}'(x_1, \ldots, x_s)$ and $\det \widetilde{C}'(f_1, \ldots, f_s) \neq 0$, then $\det \widetilde{C}'(x_1, \ldots, x_s)$ is a nonzero polynomial in the integral domain $F[x_1, \ldots, x_s]$ with degree at most s.

Since $\det \widetilde{B}(0, \ldots, 0) \neq 0$, a similar argument shows that $\det \widetilde{B}(x_1, \ldots, x_s)$ is a nonzero polynomial in $F[x_1, \ldots, x_s]$ with degree at most s. Since $s \leq \left\lfloor \frac{2(n-1)-n}{n-(n-1)} \right\rfloor = n - 2$, $q(x_1, \ldots, x_s) = \det \widetilde{C}'(x_1, \ldots, x_s) \cdot \det \widetilde{B}(x_1, \ldots, x_s)$ is a nonzero polynomial of degree at most $2n-4$. But $|F| > 2n-4$, so we can choose $g_1, \ldots, g_s \in F$ such that $q(g_1, \ldots, g_s) \neq 0$. Then $\widetilde{C}'(g_1, \ldots g_s)$ and $\widetilde{B}(g_1, \ldots, g_s)$ are invertible so that

$$\begin{aligned} \operatorname{rank} \widetilde{C}(g_1, \ldots, g_s) &> 3k - 2n \\ \operatorname{rank} \widetilde{B}(g_1, \ldots, g_s) &= k. \end{aligned}$$

For the remainder of the proof we abbreviate $\widetilde{A}(g_1, \ldots, g_s)$, $\widetilde{B}(g_1, \ldots, g_s)$, and $\widetilde{C}(g_1, \ldots, g_s)$ by $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$. Now $\widetilde{C} \in M_{2k-n}(F)$ so $\operatorname{nullity} \widetilde{C} < n-k$. Applying Lemma 1 again, along with the invertibility of $\widetilde{B} = \widetilde{A}[\alpha|\beta]$,

$$\operatorname{nullity} \widetilde{B}^{-1}[\beta - \alpha, \alpha - \beta] = \operatorname{nullity} \widetilde{C} < n - k.$$

But $|\alpha - \beta| = n - k$, wo $\widetilde{B}^{-1}[\beta - \alpha, \alpha - \beta]$ is not the zero matrix. Thus, for some $i \in \alpha - \beta$ and some $j \in \beta - \alpha$, $\det \widetilde{B}[\alpha - \{i\}|\beta - \{j\}] \neq 0$ and since $\widetilde{B}$ is a submatrix of $\widetilde{A}$, $\det \widetilde{A}[\alpha - \{i\}|\beta - \{j\}] \neq 0$. By Proposition 2, the rank of $\widetilde{A}$ (which is at least k) can be increased by changing 2 diagonal entries. Therefore the rank of A has been increased by changing at most $s + 2 \leq \lfloor \frac{2k-n}{n-k} \rfloor + 2 = \lfloor \frac{2k-n}{n-k} + 2 \rfloor = \lfloor \frac{n}{n-k} \rfloor$ of its diagonal entries, which completes the proof of Theorem 3. □

We give an example to show that the rank determining block $B = A[\alpha|\beta]$ may become singular as we increase the rank of $C = A[\alpha \cap \beta | \alpha \cap \beta]$, as mentioned in the proof. Suppose $F = GF(2)$, the field of two elements, that $n = 6$, $k = 4$ and that

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $B = A[\{1,2,3,4\},\{3,4,5,6\}]$, $C = A[\{3,4\},\{3,4\}]$ and $\operatorname{rank} C = 3k - 2n = 0$. However, if any $1 (= \lfloor \frac{2k-n}{n-k} \rfloor)$ diagonal entry of C is changed B becomes singular and $\operatorname{rank} A$ drops to 3. It is easy to construct similar examples for other values of n. This shows that no simple repair of the proof of Theorem 3 can yield a proof of Theorem 1. However, there is a construction along with Theorem 3 that gives the result.

Proof of Theorem 1. The only case to consider is $2 \le |F| < 2n$. In this case, embed F in a field K with $|K| \ge 2n$. Then, by Theorem 3, there exist distinct indices $i_1, \ldots, i_s$, $s \le \lfloor \frac{n}{n-k} \rfloor$, and nonzero elements $u_1, \ldots, u_s \in K$ such that $A + u_1 E_{i_1}, + u_2 E_{i_2} + \cdots + u_s E_{i_s}$ has rank greater than k. Let $\widetilde{A}(x_1, \ldots, x_s) = A + x_1 E_{i_1} + \cdots + x_s E_{i_s}$, $\widetilde{B}(x_1, \ldots, x_s)$ be a submatrix of $\widetilde{A}(x_1, \ldots, x_s)$ of order $k+1$ such that $\det \widetilde{B}(u_1, \ldots, u_s) \neq 0$ and let

$$q(x_1, \ldots, x_s) = \det \widetilde{B}(x_1, \ldots, x_s).$$

Without loss of generality, assume that $x_1, \ldots, x_s$ all occur in $\widetilde{B}$.

Now by the definition of q, $p_1(x_1) = q(u_1 + x_1, u_2, \ldots, u_s)$ is either linear in x_1 or a nonzero element of K since $p_1(0) \neq 0$. Therefore p_1 has at most one zero in K. Choose x_1 such that $f_1 = u_1 + x_1 \in F$ and $p_1(x_1) = q(f_1, u_2, \ldots, u_s) \neq 0$. By considering $p_2(x_2) = q(f_1, u_2 + x_2, u_3, \ldots, u_s)$, we similarly obtain $f_2 \in F$ such that $q(f_1, f_2, u_3, \ldots u_s) \neq 0$. Continuing the process yields $q(f_1, f_2, \ldots, f_s) \neq 0$ for $f_1, f_2, \ldots, f_s \in F$. In other words $\widetilde{A}(f_1, \ldots, f_s)$ has a submatrix $\widetilde{B}(f_1, \ldots, f_s)$ of rank $k+1$, and we have increased the rank of A by replacing at most $\lfloor \frac{n}{n-k} \rfloor$ diagonal entries of A by elements of F. This completes the proof. $\square$

We end by stating the following refinement of theorem 2. It may be proven via the same type of argument just used in the proof of theorem 1.

THEOREM 4. *Let F be any field, let g be any nonzero element in F, and let $A \in M_n(F)$ with* $\operatorname{rank} A = k$ *and* principal rank $A = p$. *Then it suffices to add g to at most $\lfloor \frac{n-p}{n-k} \rfloor$ diagonal entries of A in order to increase the* rank *of A.*

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