'A GENERALISED DIFFUSION EQUATION
FOR PHASE SEPARATION OF A MULTI-COMPONENT
MIXTURE WITH INTERFACIAL FREE ENERGY'

By

Charles M. Elliott

and

Stefan Luckhaus

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A generalised diffusion equation for phase separation of a multi-component mixture with interfacial free energy

Charles M. Elliott
School of Mathematical and Physical Sciences
Mathematics Division
University of Sussex
Falmer, Brighton BN1 9QH
UK

and

Stefan Luckhaus
Institut für Angewandte Mathematik
Universität Bonn
Wegelerstrasse 6
5300 Bonn
Germany

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Abstract

A nonlinear multicomponent diffusion equation incorporating uphill diffusion and capillarity effects is studied. In the binary case the problem is the Cahn–Hilliard equation for a regular solution free energy. Global existence is proved. It is shown that the deep quench limit is a parabolic type obstacle problem.
§1 Introduction

This paper is concerned with a system of nonlinear diffusion equations modelling isothermal phase separation of an ideal mixture of \( N \geq 2 \) components occupying an isolated region \( \Omega \subset \mathbb{R}^d \) (\( d = 1,2,3 \)). (Morral and Cahn [1971], Kirkaldy and Young [1987], Purdy [1990]). We begin by deriving the equations in the framework of non-equilibrium thermodynamics. (c.f. de Groot and Mazur [1962], Gurtin [1981].) The basic physical quantities, defined for all \( x \in \Omega \) and all time \( t \), are the mass fraction \( u_i(x,t) \), the mass flux \( \mathbf{j}_i(x,t) \) and the chemical potential \( \mu_i(x,t) \) for each component \( i = 1,2,\ldots,N \) together with the total free energy \( G(x,t) \). Clearly, by definition,

\[
\sum_{i=1}^{N} u_i(x,t) = 1 \quad x \in \Omega, \quad t \geq 0 \quad (1-1a)
\]

and

\[
0 \leq u_i(x,t) \leq 1 \quad x \in \Omega, \quad t \geq 0 \quad (1-1b)
\]

The law of mass conservation is written as, for any subregion \( \mathcal{R} \) of \( \Omega \),

\[
\frac{d}{dt} \int_{\mathcal{R}} u_i(x,t) \, dx = \int_{\partial \mathcal{R}} \mathbf{j}_i \cdot \mathbf{n} \, ds \quad \forall i \quad (1-2)
\]

where \( \mathbf{n} \) denotes the unit outward pointing normal. We use the notation \( \eta \) for \( N \)-vectors, \( \mathbf{Z} \) for \( d \)-vectors and \( \cdot \cdot \cdot \) for
the scalar product of two vectors. It follows from summation of
(1-2) over \( i \) that in order for (1-1a) to hold

\[
\sum_{i=1}^{N} \mathbf{J}_i^0 (x, t) = 0 \quad \text{for} \quad x \in \Omega, \ t > 0
\]  

The homogeneous free energy of the mixture with composition \( \mathbf{u} \) is given by \( \Psi(\mathbf{u}(x, t)) \) where \( \Psi : \mathbb{R}_+^N \to \mathbb{R} \) is a prescribed mapping. In order to model capillarity or interfacial energy associated with large gradients of the composition we follow Cahn & Hilliard [1958] and use the gradient energy \( \frac{1}{2} \Gamma \nabla \mathbf{u} \cdot \nabla \mathbf{u} \) where \( \Gamma = \{ \Gamma_{ij} \}_{i,j=1}^{N} \) is constant positive semi-definite fourth order tensor with \( \Gamma_{ij} \) being \( d \times d \) matrices and

\[
(\Gamma \nabla \mathbf{u})_i = \sum_{j=1}^{N} \Gamma_{ij} \nabla u_j \quad ; \quad \Gamma \nabla \mathbf{u} \cdot \nabla \mathbf{v} = \sum_{i,j} \Gamma_{ij} \nabla u_j \cdot \nabla v_i .
\]

The total free energy is taken to be the sum of the homogeneous free energy and the gradient energy so that

\[
G(x, t) := \Psi(\mathbf{u}(x, t)) + \frac{1}{2} \Gamma \nabla \mathbf{u} \cdot \nabla \mathbf{u} \quad (1-4)
\]

Thus, as in the Cahn–Hilliard model for phase separation in a binary mixture, we have a total free energy functional \( \mathcal{E}(\cdot) \) given by

\[
\mathcal{E}(\mathbf{u}) := \int_{\Omega} \left[ \Psi(\mathbf{u}) + \frac{1}{2} \Gamma \nabla \mathbf{u} \cdot \nabla \mathbf{u} \right] \, dx \quad .
\]  

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In the theory of multi-component diffusion without capillarity the chemical potentials for each component $i$ is given by

$$\mu_i^O := \partial_i \Psi(u)$$  \hspace{1cm} (1-6)

where $\partial_i(\cdot)$ denotes the partial derivative with respect to component $i$. With capillarity effects the vector $\mu$ of chemical potentials is taken to be the functional derivative of $\mathcal{E}(\cdot)$ evaluated at $u \in H^1(\Omega)$ so that

$$\langle \mu, \eta \rangle := \langle D\mathcal{E}(u), \eta \rangle$$

$$\forall \eta \in H^1(\Omega)$$

$$= \left( \Gamma \nabla u, \nabla \eta \right) + \left( \mu^O, \eta \right).$$  \hspace{1cm} (1-7)

Formally it follows that the relationship between $\mu$ and $u$ is given by the boundary value problem

$$\mu = \mu^O - \nabla \left( \Gamma \nabla u \right) \hspace{1cm} x \in \Omega, \ t > 0$$  \hspace{1cm} (1-8a)

$$\left( \Gamma \nabla u \right)_i \cdot \nu = 0 \hspace{1cm} \forall i \hspace{1cm} x \in \partial \Omega, \ t > 0$$  \hspace{1cm} (1-8b)

The constitutive relation for the mass fluxes is assumed to be of the isotropic form

$$\vec{J}_i := - \sum_{j=1}^{N} L_{ij} \nabla \mu_j = - \left( L \nabla \mu \right)_i$$  \hspace{1cm} (1-9)
where \( \mathbf{L} \) is a symmetric \( N \times N \) matrix with constant elements \( L_{ij} \) \((\mathbf{L} = \{L_{ij}\})\) is a fourth order tensor\) which, for (1-3) to hold, is assumed to satisfy

\[
\mathbf{L} \cdot \mathbf{e} = 0 \quad (1-10)
\]

where \( \{\mathbf{e}\}_i = 1 \forall i \). Thus the diffusion equations arising from the mass balance equations (1-2) become

\[
\frac{\partial \mathbf{u}}{\partial t} = \nabla (\mathbf{L} \nabla \mu) \quad x \in \Omega, \ t > 0
\]

(1-11a)

coupled with the no mass flux boundary condition

\[
(\mathbf{L} \nabla \mu)_i \cdot \mathbf{n} = 0 \quad \forall i \quad x \in \partial \Omega, \ t > 0
\]

(1-11b)

In order for this diffusion process to be dissipative we also assume that \( \mathbf{L} \) is positive semi-definite. This yields the property that the total free energy functional is decreasing in time viz.

\[
\frac{d\mathcal{E}(\mathbf{u}(t))}{dt} = \langle \mathbf{D} \mathcal{E}(\mathbf{u}), \mathbf{u}_t \rangle = \langle \mu, \mathbf{u}_t \rangle
\]

\[
= (\mu, \nabla \mathbf{L} \nabla \mu) = (-\mathbf{L} \nabla \mu, \nabla \mu) \leq 0.
\]

Furthermore the following version of the second law of thermodynamics
\[
\frac{d}{dt} \int_{\mathcal{R}} G(x,t) dx + \int_{\partial \mathcal{R}} \left[ \mu \cdot J_v - u_t \cdot (\Gamma \nabla u)_v \right] ds \leq 0 \quad (1-12)
\]

is satisfied for each subregion \( \mathcal{R} \) of \( \Omega \), where we have set \( \{ J_v \}_i = \mathcal{J}_i \cdot \mathcal{v} \) and \( \{(\Gamma \nabla u)_v\}_i = \sum_{j=1}^d \mathcal{J}_{ij} \nabla u_j \cdot \mathcal{v} \). Inequality (1-12) is a generalisation to multi-component diffusion with capillarity of the Clausius-Duhem inequality for binary diffusion with capillarity given by Gurtin [1988]. To see that (1-12) holds, observe that the left hand side can be rewritten using (1.4), (1.8) and integration by parts as

\[
\int_{\mathcal{R}} u_t \cdot \mu + \int_{\partial \mathcal{R}} \mu \cdot J_v
\]

and using (1.9), (1.11a) and an integration by parts we are left with

\[
- \int_{\mathcal{R}} L \nabla \mu \cdot \nabla \mu dx .
\]

Thus the constitutive assumption that \( L \) is positive semi-definite yields the desired inequality.

We now make further constitutive assumptions. First we assume a 'regular solution' for the homogeneous free energy: -

\[
\Psi (u) = \theta \sum_{i=1}^N u_i \ln u_i - \frac{1}{2} u^T A u \quad (1-13)
\]
where $\theta$ is the absolute temperature and $A$ is a constant symmetric $N \times N$ matrix with largest eigenvalue $\lambda_A > 0$. Here we have taken the Boltzmann constant to be 1 so temperature is scaled accordingly. It follows that there exists a critical temperature $\theta_c$ so that for $\theta$ greater (lesser) than $\theta_c$ the homogeneous free energy $\Psi(\cdot)$ is convex (non-convex).

Second we assume that $\Gamma$ is $\gamma I$ so that

$$\mathcal{E}(u) := \int_{\Omega} \left[ \Psi(u) + \frac{\gamma}{2} |\nabla u|^2 \right] dx. \quad (1-14)$$

Third we assume that $L$ is constant, that the kernel of $L$ is one-dimensional and that

$$L \eta \cdot \eta \geq \int_\Omega P \eta \cdot \eta \eta \quad (1-15)$$

where

$$P \eta := \eta - e \sum \eta; \quad \sum \eta = \frac{1}{N} \sum_{i=1}^N \eta_i .$$

It is convenient to introduce the vector of generalised chemical potential differences $w$ defined by

$$w := P \mu . \quad (1-16)$$

The equations (1-7) and (1-10) become
\[ w = P (\Theta \phi (u) - \Lambda u) - \gamma \Delta u \quad x \in \Omega, \ t > 0 \quad (1-17a) \]

\[ \gamma \frac{\partial u}{\partial n} = 0 \quad x \in \partial \Omega, \ t > 0 \quad (1-17b) \]

where \[ \{ \phi (u) \} = \phi (u_i) \equiv \psi'(u_i) - 1; \ \psi (r) = r \ln r, \]

and \[ \frac{\partial u}{\partial t} = \nabla (L \nabla w) \quad x \in \Omega, \ t > 0 \quad (1-18a) \]

\[ (L \nabla w)_{\nu} = 0 \quad x \in \partial \Omega, \ t > 0 \quad (1-18b) \]

Here we have used the facts \[ \sum w = 0 = \sum u - 1 / N. \]

The principal result of this paper is an existence theorem for the system \((1-17, 1-18)\) with the initial condition

\[ u(x, 0) = u_0 \quad (1-19) \]

The major difficulty is that \(\phi (r)\) is singular at \(r = 0\) and \((1-17)\) can have no meaning if \(u_i = 0\) in an open set of non-zero measure. Also there is no maximum principle which precludes this. However it is precisely this form of \(\phi (\cdot)\) that maintains the constraint \((1-1b)\) on the composition. Our result is stated as follows. We use the notation \[ \int \eta = \int_{\Omega} \eta \, dx / |\Omega| \].

**Theorem 1**

Let \( T > 0 \) and \( u_0 \in K = \left\{ \eta \in H^1 (\Omega) : \sum \eta = \frac{1}{N}, \ \eta \geq 0 \right\} \)
Suppose that \( \delta \mathbf{e} \leq \int u_0 \leq (1 - \delta) \mathbf{e} \) then there exists a unique pair \( \{u, w\} \) such that

\[
\begin{aligned}
    u & \in C \left[ 0, T; \left( H^1(\Omega) \right)' \right] \cap L^\infty(0, T; H^1(\Omega)) \\
    \frac{du}{dt} & \in L^2(0, T; \left( H^1(\Omega) \right)') \quad \sqrt{\tau} \frac{du}{dt} \in L^2(0, T; H^1(\Omega)) \\
    \bar{w} &= w - \int w \in L^2(0, T; H^1(\Omega)) \\
    \sqrt{\tau} w & \in L^\infty(0, T; H^1(\Omega)) \\
    \sqrt{\tau} \theta \phi(u) & \in L^\infty(0, T; L^2(\Omega))
\end{aligned}
\]

\[
\begin{aligned}
    u(\cdot, 0) &= u_0 \\
    u(\cdot, t) &\in K \quad \forall \ t > 0 \\
    \int u(\cdot, t) &= \int u_0
\end{aligned}
\]

and for all \( \xi \in C[0, T] \) and \( \eta \in H^1(\Omega) \)

\[
\int_0^T \xi(t) \left\{ \frac{d}{dt} \langle u, \eta \rangle + \left( L \nabla \bar{w}, \nabla \eta \right) \right\} \, dt = 0 \quad (1-20a)
\]

\[
\int_0^T \xi(t) \left\{ \left( \bar{w} - \theta \phi(u) + Au - \varepsilon \sum (\theta \phi(u) - Au), \eta \right) - \gamma \left( \nabla u, \nabla \eta \right) \right\} \, dt = 0 \quad (1-20b)
\]

Based upon this existence theorem it is possible to justify the deep quench limit problem \( \theta \to 0 \) studied by Blowey and Elliott [1991a,b] for binary diffusion with capillarity. See also Oono and Puri [1988].
Theorem 2

Let $T > 0$ and $u_0 \in K$. There exists a unique pair $\{u, w\}$ such that

$$ u \in C \left[ 0, T; \left( H^1(\Omega) \right)' \right] \cap L^\infty \left( 0, T; H^1(\Omega) \right) $$

$$ \frac{du}{dt} \in L^2 \left( 0, T; \left( H^1(\Omega) \right)' \right), \quad \sqrt{t} \frac{du}{dt} \in L^2 \left( 0, T; H^1(\Omega) \right) $$

$$ \bar{w} := w - \int w \in L^2 \left( 0, T; H^1(\Omega) \right), \quad \sqrt{t} w \in L^\infty \left( 0, T; H^1(\Omega) \right) $$

$$ u(\cdot, 0) = u_0 $$

$$ u(\cdot, t) \in K \quad \forall t > 0 $$

and for $\xi \in C \left[ 0, T \right]$ and $\eta \in H^1(\Omega)$

$$ \int_0^T \xi(t) \left\{ \frac{d}{dt} < u, \eta > + \left( L \nabla \bar{w}, \nabla \eta \right) \right\} dt = 0 \quad (1-21a) $$

and for $\xi (\geq 0) \in C \left[ 0, T \right]$ and $\eta \in K$

$$ \int_0^T \xi(t) \left\{ \gamma \left( \nabla u, \nabla \eta - \nabla u \right) - \left( Au - e \sum Au + w, \eta - u \right) \right\} dt \geq 0 \quad (1-21b) $$

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The layout of the paper is as follows. In Section 2 an approximation to (1-10) is studied. Using estimates derived in Section 2, Theorem 1 is proved in Section 3. The proof of Theorem 2 is given in Section 4.
§2 A regularised problem

We shall consider a family of regularised problems parameterised by $\varepsilon$ and obtain the existence result by passing to the limit $\varepsilon = 0$. For each $\varepsilon$ small and positive we define

$$\phi_\varepsilon(r) = \begin{cases} \ln r & r \geq \varepsilon \\ \left( \ln \varepsilon - 1 + \frac{r}{\varepsilon} \right) & r < \varepsilon \end{cases} \quad (2-1)$$

We set

$$\phi_\varepsilon : = \phi_\varepsilon (u^\varepsilon) = \left\{ \phi_\varepsilon (u^\varepsilon) \right\}_i$$

$$q^\varepsilon = \Lambda u^\varepsilon$$

The regularised equations are:

$$\frac{\partial u^\varepsilon}{\partial t} = \nabla (L \nabla w^\varepsilon) \quad (2-2a)$$

$$w^\varepsilon = -\gamma \Delta u^\varepsilon + \theta \phi^\varepsilon - q^\varepsilon + e \sum (q^\varepsilon - \theta \phi^\varepsilon) \quad (2-2b)$$

holding in $\Omega$ for $t > 0$, together with the boundary conditions on $\partial \Omega$

$$\left( L \nabla w^\varepsilon \right)_\nu = 0 \quad \frac{\partial u^\varepsilon}{\partial \nu} = 0 \quad (2-2c)$$

and initial condition

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\( \mathbf{u}^\varepsilon \cdot (\cdot , 0) = \mathbf{u}_0 \quad (2.2d) \)

By using standard arguments based on Galerkin approximations it is easy to show that (2.2) possesses a pair of solutions \( \{ \mathbf{u}^\varepsilon, \mathbf{w}^\varepsilon \} \) such that for each \( T > 0, \)

\[
\mathbf{u}^\varepsilon \in \mathbf{L}^\infty (0, T; \mathbf{H}^1 (\Omega)), \quad \mathbf{d}u^\varepsilon / \mathbf{d}t \in \mathbf{L}^2 (0, T; (\mathbf{H}^1 (\Omega))')
\]

\[
\mathbf{w}^\varepsilon \in \mathbf{L}^2 (0, T; \mathbf{H}^1 (\Omega))
\]

and for a.e. \( t \in (0, T) \) equations (2-2a,b,c) hold in the following weak sense: for all \( \eta \in \mathbf{H}^1 (\Omega) \)

\[
\frac{d}{dt} \left< \mathbf{u}^\varepsilon, \eta \right> + (L \nabla \mathbf{w}^\varepsilon, \nabla \eta) = 0 \quad (2-3a)
\]

\[
\left< \mathbf{w}^\varepsilon, \eta \right> = \gamma \left( \nabla \mathbf{u}^\varepsilon, \nabla \eta \right) + (\theta \rho^\varepsilon - \mathbf{q}^\varepsilon - \mathbf{e} \sum (\theta \rho^\varepsilon - \mathbf{q}^\varepsilon), \eta) \quad (2-3b)
\]

For our purposes we wish to obtain sufficient estimates independent of \( \varepsilon \) in order to pass to the limit.

We define a regularised homogeneous free energy by

\[
\psi^\varepsilon (r) = \begin{cases} 
\frac{r \ln r}{r \geq \varepsilon} \\
\left( \frac{r^2}{2 \varepsilon} + r \ln \varepsilon - \frac{\varepsilon}{2} \right) & \varepsilon < r
\end{cases} \quad (2-4)
\]
and \( \psi^\varepsilon : \mathbb{R}^N \to \mathbb{R} \) by

\[
\psi^\varepsilon (\mathbf{r}) = \theta \sum_{i=1}^{N} \psi_i^\varepsilon (r_i) - \frac{1}{2} \mathbf{r}^T \mathbf{A} \mathbf{r}
\]  

(2-5)

**Lemma 2-1**

There exists an \( \varepsilon_0 > 0 \) and \( k > 0 \) such that for all \( \varepsilon < \varepsilon_0 \)

\[
\psi^\varepsilon (\mathbf{r}) \geq -k \quad \forall \ \mathbf{r} \in \mathbb{R}^N \ \text{such that} \sum r = \frac{1}{N}
\]

(2-6)

**Proof**

Observe that

\[
\psi^\varepsilon (\mathbf{r}) \geq \sum_{i=1}^{N} \left[ \theta \psi_i^\varepsilon (r_i) - \lambda A r_i^2 \frac{1}{2} \right] \quad \forall \ \mathbf{r} \in \mathbb{R}^N
\]

Since for \( \varepsilon_0 < \frac{1}{e} \),

\[
\psi^\varepsilon (\mathbf{r}) \geq - \frac{1}{e}
\]

we need only consider estimating \( \psi^\varepsilon (\mathbf{r}) \) from below for \( \max_i |r_i| > 1 \).

Set

\[
R_m = \min_j r_j, \quad R_M = \max_j r_j
\]
It follows that

$$1 - (N-1)R_M \leq R_m \leq \frac{1-R_M}{(N-1)}$$

and

$$\psi^\varepsilon(r) \geq - \theta (N-1)^/\varepsilon + \theta \left( \frac{R_m^2}{2\varepsilon} + R_m \ln \varepsilon - \frac{\varepsilon}{2} \right) - N \lambda_A (N-1)^2 R_M^2/2.$$ 

Choosing $\varepsilon_0$ sufficiently small (depending on $\theta$, $N$ and $\lambda_A$) gives the result. $\square$

In the next proposition we show that (2.2) possesses natural mass conservation and energy decay properties. We introduce the total regularised energy by

$$\mathcal{E}^\varepsilon(v) = \int_{\Omega} \left[ \frac{\gamma}{2} |\nabla v|^2 + \psi^\varepsilon(v) \right] \, dx \quad (2-7)$$

**Proposition 2-1**

a) *Conservation of Mass*

$$\int \mathbf{u}^\varepsilon (.,t) \, dx = \int \mathbf{u}_0 \quad (2-8a)$$

b) *Conservation of Total Local Mass*

$$\sum \mathbf{u}^\varepsilon (x,t) = \frac{1}{N} \quad x \in \Omega, \, t > 0 \quad (2-8b)$$
c) *Energy Decay*

\[
\frac{d}{dt} \mathcal{E}^\varepsilon(u^\varepsilon) + \int_{\Omega} L \nabla \mathbf{w}^\varepsilon \cdot \nabla \mathbf{w}^\varepsilon \, dx = 0. \quad (2\text{-8c})
\]

d) *Conservation of Total Chemical Potential*

\[
\sum w^\varepsilon(x,t) = 0 \quad (2\text{-8d})
\]

**Proof**

a) Taking \( \eta = e_k = \{ \delta_{ik} \} \) for each \( k \) yields (2-8a) immediately.

b) Setting

\[
U^\varepsilon = \sum_{i=1}^{N} u_i^\varepsilon, \quad \mathbf{W}^\varepsilon = \sum_{i=1}^{N} \mathbf{w}_i^\varepsilon
\]

and taking \( \eta = \eta e_k \) \((k=1, \ldots, N)\) with \( \eta \in H^1(\Omega) \) in (2-2a,b) we obtain after summing,

\[
\langle \frac{dU^\varepsilon}{dt}, \eta \rangle + \left( L \nabla \mathbf{W}^\varepsilon, \nabla \eta \right) = 0
\]

\[
(\mathbf{W}^\varepsilon, \eta) + \gamma \left( \nabla U^\varepsilon, \nabla \eta \right) = 0
\]

Since
\[U^\varepsilon (., 0) = \sum_{i=1}^{N} u_i (., 0) = 1\]

we find that these linear equations have the unique solution

\[U^\varepsilon (x, t) = 1, \quad W^\varepsilon (x, t) = 0\]

which implies (2-8b).

c) By differentiating (2-7) with respect to \( t \) we find that the regularised energy satisfies

\[
\frac{d\varepsilon^\varepsilon}{dt} (u^\varepsilon) = \left( -\gamma \Delta u^\varepsilon + \theta \varphi^\varepsilon (u^\varepsilon) - q^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right)
\]

\[
= \left( w^\varepsilon + e \left( \sum (\theta \varphi^\varepsilon (u^\varepsilon) - q^\varepsilon), \frac{\partial u^\varepsilon}{\partial t} \right) \right)
\]

\[
= \left( w^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right) + \left( \sum (\theta \varphi^\varepsilon - q^\varepsilon), \frac{\partial U^\varepsilon}{\partial t} \right).
\]

Since \( U^\varepsilon \equiv 1 \) and (2-3a) holds we finally obtain (2-8c).

**Proposition 2-2**

There exist constants \( C_j (j=1, 2, 3) \) depending only on the initial data and independent of \( \varepsilon \) so that
\[ \| \nabla u^\varepsilon(t) \|^2 + \int_0^t \| \nabla w^\varepsilon \|^2 \, d\tau \leq C_1 \quad (2-9a) \]

\[ \| u^\varepsilon(t) \|_1 \leq C_2 \quad (2-9b) \]

\[ \sum_{i=1}^N \left\{ \int \left[ - u^\varepsilon_i \right]_+ + \int \left[ u^\varepsilon_i - 1 \right]_+ \right\} \leq C_3 \left/ (\theta \ln \varepsilon) \right. \quad (2-9c) \]

**Proof**

These estimates are consequences of the fact that $\mathcal{E}^\varepsilon(.)$ is a Lyapunov functional for the system. Integrating (2-8c) with respect to $t$ and using (1-15) yields

\[ \gamma \| \nabla u^\varepsilon(t) \|^2 + \int_0^t \| \nabla w^\varepsilon(\tau) \|^2 \, d\tau + \int_{\Omega} \Psi^\varepsilon(u^\varepsilon(t)) \, dx \leq \mathcal{E}^\varepsilon(u_0) \quad (2-10) \]

Inequality (2-9a) follows from Lemma 2-1 and the fact that, since $\{u_0 \}_{i \in [0,1]}$,

\[ \int_{\Omega} \Psi^\varepsilon(u_0) \leq - \frac{1}{2} \langle Au_0, u_0 \rangle. \]

Noting (2-8a) we obtain (2-9b) by a direct use of Poincaré's inequality.
Turning to (2-9c), we first observe that (2-9b) implies that

\[(Au^\varepsilon, u^\varepsilon) \leq C \quad \forall t .\]

Since

\[\int_\Omega \psi^\varepsilon(u^\varepsilon_i) \, dx \geq - \theta |\Omega| / e + \theta \int_{\{u^\varepsilon_i < \varepsilon\}} \psi^\varepsilon(u^\varepsilon_i) \, dx\]

\[\geq - \theta |\Omega| / e + \theta \ln \varepsilon \int_{\{u^\varepsilon_i < \varepsilon\}} u^\varepsilon_i \, dx + \theta \varepsilon \ln \varepsilon |\Omega| - \frac{\theta \varepsilon}{2} |\Omega| ,\]

it follows from the inequality

\[\int_\Omega \psi^\varepsilon(u^\varepsilon) \, dx < C\]

that

\[\sum_{i=1}^{N} \int \left[ - u^\varepsilon_i (\cdot, t) \right]_+ \leq C / (\theta \ln \varepsilon)\]

for \(\varepsilon < \varepsilon_0\) sufficiently small.

Finally we have that, using (2-8b),

- 19 -
\[
\int \left[ u_i^\varepsilon - 1 \right]_+ = \frac{-1}{|\Omega|} \int \frac{\sum_{j \neq i} u_j^\varepsilon (x, t)}{\{ u_i^\varepsilon > 1 \}} \, dx
\]

\[
\leq \frac{1}{|\Omega|} \int \frac{\sum_{j \neq i} [-u_j^\varepsilon]}{\{ u_i^\varepsilon > 1 \}} \, dx
\]

\[
\leq \sum_{j \neq i} \int [-u_j^\varepsilon]_+ .
\]

Proposition 2-3

There exist constants \( C_4 \) and \( C_5 \) depending on the initial data and \( T \) such that

\[
t \| \nabla w^\varepsilon (t) \|^2 + \int_0^T s \| \nabla \frac{du^\varepsilon}{dt} \|^2 \, ds \leq C_4
\]  \( (2-11) \)

\[
\theta t \left( \phi - \int \phi^\varepsilon \right) - e \left( \sum_\phi - \int \sum_\phi^\varepsilon \right) \|^2 \leq C_5
\]  \( (2-12) \)

Proof

Differentiating (2-3b) with respect to \( t \) and taking \( \eta = \frac{du^\varepsilon}{dt} \) yields

- 20 -
\[
\begin{align*}
\left( \frac{dw^\varepsilon}{dt}, \frac{du^\varepsilon}{dt} \right) &= \gamma \| \nabla u^\varepsilon \|^2 + \left( D(u^\varepsilon) \frac{du^\varepsilon}{dt}, \frac{du^\varepsilon}{dt} \right) \\
- \left( A \frac{du^\varepsilon}{dt}, \frac{du^\varepsilon}{dt} \right) + \frac{d}{dt} \left( \left( q^\varepsilon - \theta^\varepsilon \right), \frac{dU^\varepsilon}{dt} \right),
\end{align*}
\]

where \( D(u^\varepsilon) \) is the diagonal matrix with entry \( \{ \theta_{\varepsilon} (u_{i}^\varepsilon) \} \).

Since \( \theta_{\varepsilon} (\cdot) \geq 0 \) and \( U^\varepsilon(x,t) = 1 \), it follows from the above equation that

\[
\frac{1}{2} \frac{d}{dt} \| \nabla w^\varepsilon (t) \|^2 + \gamma \| \nabla \frac{du^\varepsilon}{dt} \|^2 \leq \left( A \frac{du^\varepsilon}{dt}, \frac{du^\varepsilon}{dt} \right).
\]

Since taking \( \eta = A \frac{du^\varepsilon}{dt} \) in (2-3a) yields

\[
\left( A \frac{du^\varepsilon}{dt}, \frac{du^\varepsilon}{dt} \right) = \left( -L \nabla w^\varepsilon, \nabla A \frac{du^\varepsilon}{dt} \right)
\]

\[
\leq C_A \| L \| \| \nabla w^\varepsilon \| \| \nabla \frac{du^\varepsilon}{dt} \|,
\]

we obtain after multiplying by \( t \) that

\[
\frac{d}{dt} \left[ t \| \nabla w^\varepsilon \|^2 \right] + t \| \nabla \frac{du^\varepsilon}{dt} \|^2 \leq C(t+1) \| \nabla w^\varepsilon \|^2.
\]

Inequality (2-11) now follows after integrating with respect to \( t \) and noting (2-9a).
Turning to the proof of estimate (2-12) we set

\[ g^\varepsilon = \rho^\varepsilon - \left( \sum \phi^\varepsilon \right) e \quad \text{and take} \quad \eta = g^\varepsilon - f g^\varepsilon \quad \text{in (2-3b) yielding} \]

\[ \theta \| g^\varepsilon - f g^\varepsilon \|^2 + \gamma (\nabla u^\varepsilon, \nabla \rho^\varepsilon) \]

\[ = (w^\varepsilon - f w^\varepsilon, g^\varepsilon - f g^\varepsilon) + (q^\varepsilon - \sum q^\varepsilon e, g^\varepsilon - f g^\varepsilon) \]

\[ + \gamma (e \nabla U^\varepsilon, \nabla \rho^\varepsilon) / N \]

Therefore it holds that

\[ \theta \| g^\varepsilon - f g^\varepsilon \|^2 \leq C \left( \| w^\varepsilon - f w^\varepsilon \|^2 + \| q^\varepsilon - f q^\varepsilon \|^2 \right) \]

and the estimates (2-9b) and (2-11) together with the Poincaré inequality imply (2-12). □

We are now in a position to state the crucial estimate which will allow us to pass to the limit.

**Proposition 2-4**

There exists a constant \( C_6 \) depending on \( T \), the initial data and \( \theta \) such that for \( \varepsilon_0 \) sufficiently small
\[ \| \varphi \|^2 \leq C_6 t^{-1}. \] (2.13)

**Proof**

Recall that there exists \( \delta \in (0,1) \) such that for each \( i \in \{1, N\} \)

\[ \delta < \int u_i^\varepsilon < 1 - \delta. \] (2-14)

Our estimates will be independent of \( \varepsilon \) but will depend on \( \delta \) and \( \theta \); in particular they require \( \delta \) and \( \theta \) to be positive.

We shall fix \( t > 0 \) and suppress the dependence on \( t \) in the following.

Set

\[ \Omega^\varepsilon := \left\{ x \in \Omega : \max_{1 \leq i \leq N} u_i^\varepsilon > 1 + \sqrt{\frac{C_3}{\theta ||\varepsilon||}} \right\} \] (2-15)

It follows from (2-9c) that

\[
\left( \frac{C_3}{\theta ||\varepsilon||} \right)^{1/2} ||\varepsilon|| < \int_{\Omega^\varepsilon} \left( \max_{1 \leq i \leq N} u_i^\varepsilon - 1 \right) \, dx
\]

\[
\leq \sum_{i=1}^{N} \int_{\Omega} \left[ u_i^\varepsilon - 1 \right]_+ \, dx
\]

\[
< N \frac{C_3 ||\Omega||}{\theta ||\varepsilon||}
\]

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and we have

$$|\Omega^\varepsilon| < K_1 \frac{|\Omega|}{(\varepsilon \ln \varepsilon)} \sqrt[2]{\varepsilon}$$  \hspace{1cm} (2-16)

Set

$$\Omega^\varepsilon_i = \{ x \in \Omega : u^\varepsilon_i > \frac{\delta}{2} \} \setminus \Omega^\varepsilon$$  \hspace{1cm} (2-17)

and assume that $\varepsilon_0$ is sufficiently small so

$$\frac{C_3}{\theta \ln \varepsilon} < \frac{\delta}{4}.$$  \hspace{1cm} (2.18)

Noting (2-14), (2-9c) and (2-18) we find that

$$\int \min \{ u^\varepsilon_i, 1 \} = \int u^\varepsilon_i - \int [u^\varepsilon_i - 1]_+$$

$$> \delta - \frac{C_3}{\theta \ln \varepsilon} > \frac{3}{4} \delta.$$  \hspace{1cm} (2.19)

But also, setting

$$A_1^\varepsilon = \Omega^\varepsilon_i, \quad A_2^\varepsilon = \Omega^\varepsilon \cap \left[u^\varepsilon_i > \frac{\delta}{2}\right],$$

$$A_3^\varepsilon = \left[u^\varepsilon_i < \frac{\delta}{2}\right]$$

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we find that

\[
\min \left\{ u_1^\varepsilon, 1 \right\} < \frac{|A_1^\varepsilon|}{|\Omega|} + \frac{|A_2^\varepsilon|}{|\Omega|} + \frac{\varepsilon}{2} \frac{|A_3^\varepsilon|}{|\Omega|}
\]

\[
\leq \frac{|\Omega_1^\varepsilon|}{|\Omega|} + \frac{|\Omega_2^\varepsilon|}{|\Omega|} + \frac{\varepsilon}{2}.
\]

The above inequalities together with (2-16) imply that

\[
\frac{|\Omega_1^\varepsilon|}{|\Omega|} > \frac{\varepsilon}{8}
\]

(2-19)

provided that \( \varepsilon_0 \) is sufficiently small so that

\[
\frac{K_1}{(\theta|\ln|)^{1/2}} < \frac{\varepsilon}{8}
\]

Since \( \varphi_\varepsilon(\cdot) \) is monotone increasing we have that, using (2-18),

\[
\varphi_\varepsilon(u_1^\varepsilon) \leq \varphi_\varepsilon \left( \max_{1 \leq j \leq N} u_j^\varepsilon \right) \leq \varphi_\varepsilon \left( 1 + \left( \frac{C_3}{\theta|\ln|} \right)^{1/2} \right)
\]

\[
\leq \ln \left( 1 + \sqrt[2]{\varepsilon} \right) + 1
\]

on the complement of \( \Omega^\varepsilon \). It follows that on \( \Omega_i^\varepsilon \),

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\[ g_i^\varepsilon = \phi_\varepsilon(u_i^\varepsilon) - \sum \phi_\varepsilon > \ln\left(\frac{\delta}{2}\right) - \ln\left(1 + \delta^{1/2}\right). \tag{2-21} \]

Let \[ \tilde{z}_j^\varepsilon = \int g_j^\varepsilon \] . If \[ z_i^\varepsilon < 0 \] then from (2-12) and (2-21)

\[
t^{-1} C_S > \int_{\Omega_i^\varepsilon} \left( g_i^\varepsilon - \tilde{z}_i^\varepsilon \right)^2 \, dx \geq |\Omega_i^\varepsilon| \cdot z_i^\varepsilon^2 - 2 z_i^\varepsilon \int_{\Omega_i^\varepsilon} g_i^\varepsilon \\
> |\Omega_i^\varepsilon| \left( z_i^\varepsilon + 2 z_i^\varepsilon \ln \left( \frac{2 + 2\delta^{1/2}}{\delta} \right) \right)
\]

and this implies

\[ |z_i^\varepsilon|^2 < K_2 \, t^{-1}, \tag{2-22} \]

where \( K_2 \) depends on \( \delta \). If \( z_i^\varepsilon > 0 \) then

\[ 0 = \int_1^N \sum_{j=1}^N g_j^\varepsilon \] implies that

\[ 0 < z_i^\varepsilon = \sum_{j \neq i} \int g_j^\varepsilon < - \sum_{\{j: z_j < 0\}} z_j
\]

and by (2-22),

\[ |z_i^\varepsilon|^2 < (N-1)^2 \, K_2 \, t^{-1} \, . \]

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Thus we have shown the existence of $K_3$ such that

$$|\int g_i^\varepsilon |^2 < K_3 t^{-1} \quad i=1, 2, \ldots N. \quad (2-23)$$

It follows from (2-12) that

$$\int_{\Omega} g_i^\varepsilon dx \leq C_5 t^{-1} + |\Omega| \left( \int g_i^\varepsilon \right)^2$$

$$\leq K_4 t^{-1}. \quad (2-24)$$

Set

$$\tilde{\Omega}_i^\varepsilon := \{ x \in \Omega : u_i^\varepsilon = \max_{1 \leq j \leq N} u_j^\varepsilon \}. \quad (2-25)$$

Since $\varphi_\varepsilon (\cdot)$ is monotone we have that on $\tilde{\Omega}_i^\varepsilon$,

$$g_i \geq 0 \quad \text{and} \quad \varphi_\varepsilon (u_i^\varepsilon) \geq \varphi_\varepsilon \left( \frac{1}{N} \right)$$

so

$$g_i \geq \left[ \varphi_\varepsilon \left( \frac{1}{N} \right) - \sum \varphi_\varepsilon \right]_+ \quad \text{on} \quad \tilde{\Omega}_i^\varepsilon,$$

which yields
\[ \int_{\Omega} (g_i^\varepsilon)^2 \ dx \geq \int_{\hat{\Omega}^\varepsilon} (g_i^\varepsilon)^2 \ dx \geq \int_{\hat{\Omega}^\varepsilon} \left[ \phi_\varepsilon \left( \frac{1}{N} \right) - \sum \phi_\varepsilon \right]^2 \]

and summing this inequality over \( i = 1, 2, \ldots, N \), using (2-24),

\[ \int_{\Omega} \left[ \phi_\varepsilon \left( \frac{1}{N} \right) - \sum \phi_\varepsilon \right]^2 \ dx \leq \sum_{i=1}^{N} \int_{\Omega} (g_i^\varepsilon)^2 \ dx \leq K_3 \, t^{-1} \quad (2-26) \]

Furthermore we have for each \( x \in \Omega \),

\[
\max_{1 \leq j \leq N} g_i^\varepsilon = \phi_\varepsilon \left( \max_{1 \leq j \leq N} u_j^\varepsilon \right) - \frac{1}{N} \sum_{j=1}^{N} \phi_\varepsilon (u_j^\varepsilon) \\
= \frac{N-1}{N} \phi_\varepsilon \left( \max_{1 \leq j \leq N} u_j^\varepsilon \right) - \frac{1}{N} \sum_{j \neq m} \phi_\varepsilon (u_j^\varepsilon) \\
+ \frac{1}{N} \left[ \phi_\varepsilon \left( \max_{1 \leq j \leq N} u_j^\varepsilon \right) - \phi_\varepsilon (u_m^\varepsilon) \right]
\]

\[ \geq \frac{1}{N} \left[ \phi_\varepsilon \left( \max_{1 \leq j \leq N} u_j^\varepsilon \right) - \phi_\varepsilon \left( \frac{1}{N} \right) \right] \geq 0 \]

where \( u_m^\varepsilon = \max_{1 \leq j \leq N} u_j^\varepsilon \) and we have used the fact that \( \sum_{j=1}^{N} u_j^\varepsilon = 1 \).

Hence

\[ \int_{\Omega} \left[ \sum \phi_\varepsilon - \phi_\varepsilon \left( \frac{1}{N} \right) \right]^2 \ dx \leq \int_{\Omega} \left( \phi_\varepsilon \left( \max_{1 \leq j \leq N} u_j^\varepsilon \right) - \phi_\varepsilon \left( \frac{1}{N} \right) \right)^2 \]
\[ N^2 \int_{\Omega} \max_{1 \leq j \leq N} (g_j^\varepsilon)^2 \, dx \leq N^2 K_5 t^{-1} \]

and this together with (2-26) yields

\[ \int_{\Omega} \left( \sum \varphi^\varepsilon - \varphi^\varepsilon \left( \frac{1}{N} \right) \right)^2 \, dx \leq K_6 t^{-1} \quad (2-27) \]

Combining (2-24) and (2-27) we obtain

\[ \sum_{i=1}^{N} \| \varphi^\varepsilon (u_i^\varepsilon) \|^2 + \| \sum \varphi^\varepsilon \|^2 \leq K_7 t^{-1} \quad (2-28) \]

which completes the proof of the proposition. \(\square\)
§3 Proof of Theorem 1

It follows from the results of §2 that there exist \( \{u^\varepsilon, w^\varepsilon\} \) uniformly bounded independently of \( \varepsilon \) in the spaces,

\[
u^\varepsilon \in C \left[ 0, T; \left( H^1(\Omega) \right)^\cdot \right] \cap L^\infty \left( 0, T; H^1(\Omega) \right) \tag{3-1a}
\]

\[
\sqrt{\varepsilon} \ |u^\varepsilon|_{L^2(0, T; H^1(\Omega))} \tag{3-1b}
\]

\[
w^\varepsilon = w^\varepsilon - \int_0^T w^\varepsilon \in L^2 \left( 0, T; H^1(\Omega) \right) \tag{3-1c}
\]

\[
\sqrt{\varepsilon} \ w^\varepsilon \in L^\infty \left( 0, T; H^1(\Omega) \right) \tag{3-1d}
\]

such that

\[
\sqrt{\varepsilon} \ \phi^\varepsilon(u^\varepsilon) \in L^\infty \left( 0, T; L^2(\Omega) \right) \tag{3-2}
\]

\[
u^\varepsilon(\cdot, 0) = u_0 \tag{3-3}
\]

and for each \( \xi \in C \left[ 0, T \right] \) and \( \eta \in H^1(\Omega) \),

\[
\int_0^T \xi(t) \left\{ \frac{d}{dt} \langle u^\varepsilon, \eta \rangle + (L \nabla w^\varepsilon, \nabla \eta) \right\} dt = 0 \tag{3-4a}
\]
\[
\int_0^T \xi(t) \left\{ (w^\varepsilon - \theta \varrho^\varepsilon(u^\varepsilon) + Au^\varepsilon - e \sum (\theta \varrho^\varepsilon(u^\varepsilon) - Au^\varepsilon), \eta) \right. \\
- \gamma(\nabla u^\varepsilon, \nabla \eta) \left. \right\} \, dt = 0.
\]  

(3-4b)

Thus passing to the limit \( \varepsilon = 0 \) in (3-4) using (3-1) and (3.2) yields a pair \( \{u, w\} \) satisfying (1-20) provided we can show that

\[
\lim_{\varepsilon \to 0} \int_0^T \xi(t) \left( \varrho^\varepsilon(u^\varepsilon), \eta \right) \, dt = \int_0^T \xi(t) \left( \varrho(u), \eta \right) \, dt
\]  

(3-5)

It follows from (2-9c) that \( u = \lim_{\varepsilon \to 0} u^\varepsilon \) satisfies

\[
\{u\}_i \in [0,1] \quad \forall i
\]  

(3-6)

and from (3-2) that there exists \( \varrho^* \) such that

\[
\sqrt{\varepsilon} \varrho^* \in L^\infty(0, T; L^2(\Omega))
\]

and

\[
\lim_{\varepsilon \to 0} \int_0^T \xi(t) \left( \varrho^\varepsilon(u^\varepsilon), \eta \right) \, dt = \int_0^T \xi(t) \left( \varrho^*, \eta \right) \, dt.
\]

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Hence in order to obtain (3-5) we have to show that

\[ \{ \varphi^* \}_i = \varphi(u_i). \quad (3-7) \]

Since (3.2) holds it follows that for each \( M > 0 \)

\[ t \left| \left| \varphi_\varepsilon (u_i^\varepsilon) \right| > M \right| \leq \frac{c}{M^2}. \quad (3-8) \]

Set

\[ F_M(v) := \max \left\{ -M, \min \{M, v\} \right\}. \quad (3-9) \]

For each \( \tau > 0 \) it holds that, using (3-8),

\[ \left| \int_{\tau}^{T} \xi(t) \left( \varphi_\varepsilon (u_i^\varepsilon) - F_M(\varphi_\varepsilon (u_i^\varepsilon)), \eta \right) dt \right| \]

\[ \leq \left| \int_{\tau}^{T} \xi(t) \int_{\left| \varphi_\varepsilon (u_i^\varepsilon) \right| > M} \left( \left| \varphi_\varepsilon (u_i^\varepsilon) \right| + M \right) |\eta| dx \ dt \right| \]

\[ \leq C(\tau) \| \xi \|_\infty \| \eta \|_\infty / M. \]

Since
\[
\lim_{\varepsilon \to 0} F_M (\phi_\varepsilon (u_{i_1}^\varepsilon)) = \lim_{\varepsilon \to 0} F_M (\phi (u_{i_1}^\varepsilon)) = F_M (\phi (u_{i_1}))
\]

it follows that the left hand side of the above inequality converges to

\[
\left| \int_\tau^T \xi(t) \left( \phi_i^* - F_M (\phi (u_{i_1})), \eta \right) \, dt \right| < c(\tau) \frac{\|\xi\|_\infty \|\eta\|_\infty}{M}
\]

Taking \( \eta = F_M (\phi (u_{i_1})) \) we find that

\[
\int_\tau^T \|F_M (\phi (u_{i_1}))\|^2 \, dt \leq C(\tau) \quad \forall M
\]

which implies

\[
\int_\tau^T \|\phi (u_{i_1})\|^2 \, dt \leq C(\tau)
\]

and

\[
\phi_i^* = \phi (u_{i_1}) \quad \text{on} \quad (\tau, T).
\]

This completes the proof of (3-7) since \( \tau \) is arbitrary.

In order to prove uniqueness we use the idea given in Blowey and Elliott [1991a]. Let \( f = \{f_i\}_{i=1}^N \) where
\[ f_i \in (H^1(\Omega))^\prime, \quad \langle f_i, 1 \rangle = 0 \quad ; \quad \sum_{i=1}^{N} f_i = 0. \quad (3.10) \]

We introduce the Green's operator \( G \) defined by:

\[ G f \in H^1(\Omega), \quad \sum G f = 0, \quad \int G f = 0 \quad (3-11a) \]

\[ (L \nabla G f, \nabla \eta) = \langle f, \eta \rangle \quad \forall \eta \in H^1(\Omega) \quad (3-11b) \]

That (3-11) defines a unique \( G f \) for an \( f \) satisfying (3-10) follows from (1-15) and the Lax-Milgram theorem.

Let \( \{ z^u, z^w \} = \{ u^1 - u^2, w^1 - w^2 \} \) be the difference of two pairs of solutions to (1-20). Using the monotonicity of \( \varphi(\cdot) \) we find from (1-20b) that

\[ \gamma \| \nabla z^u \|^2 \leq (z^u, z^w) + \lambda_A \| z^u \|^2. \]

Since, by (1-20a),

\[ z^w = -G z^u, \]

it follows that

\[ \frac{1}{2} \frac{d}{dt} \| L \nabla G z^u \|^2 + \gamma \| \nabla z^u \|^2 \leq \lambda_A (L \nabla G z^u, \nabla z^u). \]

A standard Gronwall argument yields uniqueness since

\[ z^u(0) = 0. \]
§4 Proof of Theorem 2

Denoting by \( \{u^\theta, \, w^\theta\} \) the solution of (1-20) for fixed \( \theta \), it is clear that from the estimation given in the proof of Theorem 1 that we may pass to the limit.

\[
\{u, \, w\} = \lim_{\theta \to 0} \{u^\theta, \, w^\theta\}
\]

and we need only justify the variational inequality (1-21b) and the uniqueness of the limit.

Let \( \eta^\alpha \in K^+ \) and \( \eta^\alpha \geq \alpha \, e \) for some small positive \( \alpha \). Since \( \sum (\eta^\alpha - u^\theta) = 0 \) we have

\[
0 = \left( \eta^\alpha - u^\theta, \, e \sum v \right) \quad \forall \, v \in L^2(\Omega).
\]

Furthermore \( \varphi(\eta^\alpha) \in L^2(\Omega) \) because \( \eta^\alpha \geq \alpha \, e \). Hence it follows from (1-20b) and the monotonicity of \( \varphi(\cdot) \) that for \( \xi(\geq 0) \in C[0,T] \),

\[
\int_0^T \xi(t) \left\{ \gamma \left( \nabla u^\theta, \nabla \eta^\alpha \right) - \left( w^\theta + Au^\theta, \eta^\alpha - u^\theta \right) \right\} \, dt
\]

\[
= \int_0^T \xi(t) \gamma \left( \nabla u^\theta, \nabla u^\theta \right) \, dt
\]

\[
+ \int_0^T \xi(t) \left\{ \theta \left( \varphi(\eta^\alpha) - \varphi(u^\theta), \eta^\alpha - u^\theta \right) \right\} \, dt
\]

\[
- \int_0^T \xi(t) \theta \left( \varphi(\eta^\alpha), \eta^\alpha - u^\theta \right) \, dt
\]

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\[ \geq \int_0^T \xi(t) \gamma \left( \nabla u^\theta, \nabla u^\theta \right) \, dt - \int_0^T \xi(t) \theta \left( \rho \left( \eta^\alpha \right), \eta^\alpha - u^\theta \right) \, dt. \]

By the weak and strong convergence properties of \( \{u^\theta, w^\theta\} \) as \( \theta \to 0 \), we may pass to the limit and obtain,

\[
\int_0^T \xi(t) \left\{ \gamma \left( \nabla u, \nabla \eta^\alpha \right) - (w + Au, \eta^\alpha - u) \right\} \, dt
\]

\[
= \lim_{\theta \to 0} \int_0^T \xi(t) \left\{ \gamma \left( \nabla u^\theta, \nabla \eta^\alpha \right) - (w^\theta + Au^\theta, \eta^\alpha - u^\theta) \right\} \, dt
\]

\[
\geq \liminf_{\theta \to 0} \int_0^T \xi(t) \gamma \left( \nabla u^\theta, \nabla u^\theta \right) \, dt - \lim_{\theta \to 0} \int_0^T \xi(t) \theta \left( \rho \left( \eta^\alpha \right), \eta^\alpha - u^\theta \right) \, dt
\]

\[
\geq \int_0^T \xi(t) \gamma \left( \nabla u, \nabla u \right) \, dt.
\]

Furthermore, since any \( \eta \in K^+ \) can be approximated by \( \eta^\alpha \in K^+ \). For small \( \alpha \) with \( \eta^\alpha \geq \alpha e \), we may pass to the limit \( \alpha = 0 \) in the left hand side of the above inequality and obtain (1-21b).

Uniqueness is proved in the same way as for the \( \theta > 0 \) problem.
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