A NEKHOROSHEV-LIKE THEORY OF CLASSICAL PARTICLE CHANNELING IN PERFECT CRYSTALS

By

H. Scott Dumas

IMA Preprint Series # 847
August 1991
A NEKHOROSHEV-LIKE THEORY OF CLASSICAL PARTICLE CHANNELING IN PERFECT CRYSTALS

H. SCOTT DUMAS†

Abstract. We apply rigorous, Nekhoroshev-like perturbation theory to a classical Hamiltonian system representing the motion of highly energetic positive particles in perfect cubic crystals. For this system, we prove a channeling theorem which shows that particles with suitable initial conditions execute, over exponentially long times, axial and planar channeling motions as conceived in J. Lindhard’s continuum models. Away from initial conditions for channeling, we find a class of nonchanneling motions which quickly experience “close encounters” with crystal nuclei (by analogy with axial and planar continuum models, we say these motions are governed by a “spatial” continuum model; proofs of some results for the spatial continuum model appear elsewhere).

We begin with brief discussions of the development of Nekhoroshev theory (including the distinction between Hamiltonians with unperturbed part that is convex, as opposed to merely “steep”) and of the physics of particle channeling. Using a simple criterion for identifying channeling motions, we then formulate the channeling problem as a nearly integrable Hamiltonian system, and construct normal forms for it (the construction closely follows the work of Benettin, Galgani and Giorgilli in the convex case). Finally, we describe particle motions in the spatial continuum model, then state and prove the channeling theorem.

Key Words. Nekhoroshev’s theorem, particle channeling in crystals, Hamiltonian perturbation theory

CONTENTS

1. Introduction 1
2. Background 2
3. Formulation of the Channeling Problem 8
4. Construction of the Normal Forms 11
5. The Generalized Continuum Models 30
6. Concluding Remarks 45
   References 48

† Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025
1. INTRODUCTION

Some twenty years ago, N.N. Nekhoroshev announced a theorem in Hamiltonian perturbation theory which may be paraphrased as follows.

Suppose the Hamiltonian
\[ H(I, \theta) = h(I) + f(I, \theta) \] (1.1)
is analytic on \( F \) (a certain complex neighborhood of the real domain \( K \times T^n, \ K \subseteq \mathbb{R}^n, \ T^n = \mathbb{R}^n/\mathbb{Z}^n \)). Suppose further that the perturbation \( f \) is of order \( \epsilon \) (sup \( F \mid f \mid = \epsilon \)), and that the integrable part \( h \) satisfies certain "steepness conditions" (for example \( h \) is convex). Then there exist positive numbers \( a, b, C, \) and \( \epsilon_0 \), depending only on \( F \) and \( h \), such that if \( \epsilon < \epsilon_0 \), then any solution \( (I(t), \theta(t)) \) of Hamilton’s equations
\[ \dot{I} = -\partial_\theta H, \quad \dot{\theta} = \partial_I H \] (1.2)
with arbitrary real initial conditions \( (I(0), \theta(0)) \in K \times T^n \) satisfies
\[ \|I(t) - I(0)\| < e^b \] (1.3)
for all \( t \) in the exponentially long time interval \([0, T], \ T = \exp(C\epsilon^{-a})\).

Nekhoroshev’s theorem has been described as an upper bound on the average speed of Arnold’s diffusion, as a physicist’s KAM theorem, and even as the crowning achievement in classical perturbation theory for nearly integrable Hamiltonian systems. In spite of this, for reasons that will be discussed below, the theorem’s importance was for a long time—and to some extent remains—unappreciated, especially when compared with attention focused on the closely related KAM theorem.

It is the aim of this article to present the complete proof of a Nekhoroshev-like result in the context of a compelling physical application. Mathematically, the classical particle channeling problem treated here provides a good introduction to Nekhoroshev theory in the convex case, as it is one of the simplest problems which retains a full nontrivial resonance structure. The techniques of proof are drawn mainly from the series of vivid articles by G. Benettin, L. Gallgani, G. Gallovvotti and A. Giorgilli, who—with others—have developed and applied Nekhoroshev-like results in the convex case to a wide range of physical problems.

Physically, this work sets up a mathematically rigorous framework for the motion of highly energetic charged particles in crystals, and is an outgrowth of the author’s previous research with J.A. Ellison. The reader familiar with particle channeling will recognize that the treatment here begins with the three dimensional perfect crystal model and proceeds to a simultaneous derivation of generalized versions of J. Lindhard’s continuum models for axial and planar channeling. Using a separate result in geometric number theory (the proof of which appears elsewhere), it is natural to add a “spatial” continuum model to the generalized axial and planar continuum models.
2. BACKGROUND

2.1 Sketch of the Development of Nekhoroshev Theory

Nekhoroshev’s result is the modern culmination of a long history of investigation into the behavior of solutions of “nearly integrable” Hamiltonian systems. The idea of integrability goes back to K.G.J. Jacobi and W.R. Hamilton at the beginning of the nineteenth century and, for a time, it was thought that all or nearly all Hamiltonian systems could be brought into integrable form. In other words, it was thought that for a system with bounded motion governed by an analytic $n$ degree of freedom Hamiltonian of the form $H = H(p, q) ((p, q) \in \mathbb{R}^n \times \mathbb{R}^n)$, there is an open subset $D$ of phase space $\mathbb{R}^n \times \mathbb{R}^n$ and an analytic canonical transformation $(p, q) \mapsto (I, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$ on $D$ under which $H$ takes the integrable form $H' = H'(I)$. That is, $H'$ is independent of $\theta$, so that phase space is foliated into invariant $n$-tori, smoothly parametrized by the actions $I$, with each torus supporting quasiperiodic flow with frequency vector $\partial H'_I(I)$.

These hopes vanished at the turn of the century with Poincaré’s seminal work [42], in which he shows that “most” (we would now say generically) non-degenerate Hamiltonian systems are not integrable. His results fueled hopes for the “ergodic hypothesis” of then-nascent statistical mechanics (a conjecture invalidated for generic Hamiltonians [36]), and they raised questions about the stability of motion for conservative systems and about the convergence of perturbation series used to approximate the motion of nearly integrable systems, especially in celestial mechanics.

Real progress in resolving these questions did not come for more than a half-century, when A.N. Kolmogorov announced [27]—and sketched the proof of—the celebrated Kolmogorov-Arnold-Moser (or KAM) theorem. The content of this surprising result is that, when the integrability of a nondegenerate integrable Hamiltonian is broken in the classical sense by means of a sufficiently small perturbation, a large relative measure of the trajectories of the system retain their integrable behavior. More precisely, tori of the original integrable system supporting quasiperiodic flow with highly nonresonant frequency vectors $\partial_I H(I)$ suffer only small deformations under sufficiently small perturbations of the Hamiltonian, and continue to support quasiperiodic flow.

The activity spawned by Kolmogorov’s announcement has continued up to the present day. Proof of a slightly more general result in the analytic case was given by V.I. Arnold [1] who went on to discuss some of its ramifications [2]; at nearly the same time, J. Moser [37] proved a closely related invariant curve theorem for perturbations of finitely differentiable area-preserving twist maps of the plane. The result for planar twist mappings is especially important for Hamiltonian systems with $n = 2$ (two degrees of freedom), as it ensures the existence of nested, two-dimensional invariant tori that partition each three-dimensional energy surface into narrow toroidal compartments to which nonintegrable tra-
jectories are confined. In this way the global (or topological) stability of nearly integrable systems with two degrees of freedom is assured under sufficiently small perturbations away from integrability.

There is however a fundamental aspect of the case $n > 2$ (more than two degrees of freedom) which the elegant results on invariant tori fail to address. For $n > 2$, the $n$-dimensional KAM tori no longer partition the $2n - 1$-dimensional energy surfaces, so that trajectories not initially residing on invariant tori may “leak out” through the connected complement of the tori and suffer large deviations in action over finite times. That this leakage actually occurs was first demonstrated for a specific system with “$2\frac{1}{2}$” degrees of freedom by Arnold[3] shortly after his proof of the KAM theorem, and the phenomenon has since been known as “Arnold diffusion.” Despite examples for systems with any finite number of degrees of freedom [26], detailed and insightful discussions of the mechanisms generating it [14], [15], [16], [31], and nonconstructive proofs of its genericity [15], [16], Arnold diffusion remains poorly understood. It is not entirely clear why this type of instability behaves like diffusion; its existence has not been rigorously demonstrated in many important examples where it is suspected; and above all, its “speed” has not been gauged accurately.

Nonetheless, for analytic nearly integrable systems, an upper bound on the average rate of diffusion was suspected early on. In the mid 1960’s, Arnold conjectured [4] that generically in such systems, the timescale required to observe an $O(1)$-displacement in the action variables was beyond all orders in inverse powers of the small parameter. Nekhoroshev’s theorem affirms this conjecture, and shows further that the timescale required for $O(1)$-displacement is exponentially long in an inverse power of the small parameter.

Nekhoroshev’s theorem and the KAM theorem are complementary results; each represents a slightly different response to the question: In what sense is integrable behavior continuous with respect to perturbation? KAM theory assures the persistence of a Cantor set of invariant tori and the continuity of their measure under perturbation; Nekhoroshev’s theorem takes the more prosaic but physically practical view of integrability as stability of the action variables, showing strong continuity with respect to perturbation—for all initial conditions—of the timescale of stability.

Though it relies partly on technology from Arnold’s proof of the KAM theorem, Nekhoroshev’s proof of his own result is more classical, more geometric, and more complex. Proof of the $C^\infty$-genericity of “steepness” in the class of analytic functions [39] came two years after the theorem’s announcement [38]; the theorem’s proof, excepting certain technical results, appeared three years later [40], while the final technical estimates were published two years after that [41].

No doubt the complexity and length of the proof contributed to relative ignorance of the result among both physicists and mathematicians. The complexity was however more than justified by Nekhoroshev’s achievement of the
optimal geometric hypothesis (steepness—a generalization of convexity) on the unperturbed part of the Hamiltonian; the subsequent focus on the convex case in applications to particular problems (as in this article) confronts a much less daunting task geometrically.

In the earliest announcement of his result [38], Nekhoroshev formulated the steepness property for functions on a domain \( K \subset \mathbb{R}^n \) as follows. Assume that the function \( f \) has nonvanishing gradient on \( K \), and denote by \( \{\lambda^r(x)\} \) the set of hyperplanes of dimension \( r \), passing through \( x \in K \) and perpendicular to \( \text{grad} f(x) \). The function \( f \) is steep on \( K \) if for each \( r = 1, \ldots, n-1 \), there are positive constants \( C_r, \delta_r \) (called steepness coefficients), and \( \alpha_r \geq 1 \) (the steepness indices) such that for all \( x \in K \), all \( \lambda^r(x) \in \{\lambda^r(x)\} \) and all \( \xi \in (0, \delta_r) \),

\[
\max_{0 \leq \eta \leq \xi} \min_{y \in \lambda^r(x) \cap K \atop \|x-y\| = \eta} \|\text{grad}(f|_{\lambda^r(x)})(y)\| > C_r \xi^{\alpha_r}.
\]

(2.1)

Conditions similar to steepness may be traced back to J. Glimm’s work [24] on the formal stability of Hamiltonian systems; but it was in [39] that Nekhoroshev first showed steepness to be a strongly generic property of analytic functions, in the sense that the coefficients of the Taylor series of a nonsteep function satisfy an infinite number of algebraic conditions. For detailed discussions of steepness, its generalizations, and its role in the stability of Hamiltonian systems (including its relation to convexity and various nondegeneracy conditions), the reader is referred to [32], [34], [39] and [40]. Here we limit ourselves to the following remarks.

In the proof of Nekhoroshev’s theorem, steepness assures that the contact between resonant surfaces and the so-called planes of fast drift (the hyperplanes in action space where motion is unconstrained by a given resonant normal form; cf. Section 5.1 below) is weakly transverse; this allows for bounds on the size of the intersection of zones surrounding the resonant surfaces and planes of fast drift. In this way steepness controls the passage of trajectories through resonance; without it, trajectories may quickly suffer large deviations in action as they move unimpeded along resonances. On the other hand, the complicated passage through resonance phenomena may be largely eliminated by strengthening the hypothesis on the unperturbed part of the Hamiltonian from steepness to convexity (convex functions have unit steepness indices and are consequently the “steepest”). In that case it may be shown that trajectories remain trapped, to a certain level of approximation, in a single resonance for exponentially long times. This stronger result is essential in the problem treated below, as the channeling phenomenon is itself interpreted as motion at resonance in the model system.

As his remarks indicate [40], Nekhoroshev was aware of the possibility of a simplified proof when \( h \) in (1.1) is convex or quasiconvex (i.e., has convex level sets). In the 1980’s a group of Italian mathematicians and physicists
began to exploit the simplifications afforded by convexity (or by cases where the unperturbed part consists of uncoupled harmonic oscillators, which is simpler still; cf. [11]), and they proved estimates for a number of physical models. Their results are extremely interesting, as they address delicate stability questions in physics that had not previously received rigorous treatment. These include estimates of the size of the region surrounding the Lagrange points of the Sun-Jupiter system which enjoy stability times on the order of the age of the universe [13], [23], investigations into the problem of holonomic constraint in mechanical systems [10], and a vindication and extension of some of the ideas of L. Boltzmann and J. Jeans concerning very slow relaxation times for high frequency degrees of freedom in statistical systems (as related, for example, to the ultraviolet catastrophe in classical physics and the problems concerning equipartition of energy and the behavior of classical polyatomic gases with various internal degrees of freedom) [9].

In spite of the novelty of these results, and the digestability of the simpler proofs (only somewhat more complex than proofs of KAM), there remained the problem of physically unrealistic values for the order constants of the theorems; for example, in early versions of Nekhoroshev-like theorems the “threshold of validity” $\epsilon_0$ (from Section 1) is typically more than a dozen orders of magnitude smaller than what might be expected physically, and the stability exponents $a$ and $b$ are dismaying far from optimal. (The dependence of $b$ on the number of degrees of freedom $n$ is especially critical in applications to statistical mechanics.)

As work on the long-time stability of conservative systems moves forward, much effort has gone toward making the estimates realistic for specific model systems [13], [23], [35], [43]. A significant improvement in the general convex case may be found in the recent work of P. Lochak [32], [33], who has produced nearly optimal stability exponents (the optimal exponents are very likely $a = 1/2n$, $b = 1/2$) using an entirely different method of proof. Lochak replaces the usual approach of Fourier series and estimates of small divisors with a study of the behavior of trajectories near periodic orbits of the unperturbed problem. Inspired partly by Lochak’s work, J. Pöschel has produced slightly sharper estimates of the exponents by reformulating only the geometric part of the original proof, using a new efficient method of partitioning action space into resonant blocks [43].

Finally, exponential estimates have recently been obtained for symplectic maps [7]; these were spurred by the need for a theory to support the numerical technique of symplectic integration, and by the need for estimates of the stability times of particle beams orbiting in accelerator storage rings. The application presented in this article treats the motion of charged particles as they emerge from accelerator storage rings or linear accelerators and impinge upon crystalline targets; the physics of the problem is sketched in the next subsection.
2.2 Brief Description of the Physics of Particle Channeling in Crystals

When a beam of energetic positive particles (e.g., MeV protons or GeV positrons) is directed at a crystalline target in a random direction, the beam and crystal interact strongly: beam particles are backscattered, matter is ejected from the crystal; nuclear reactions may even take place. Radiative and collisional energy loss eventually bring many particles to rest, so that beam matter is implanted in the crystal. But if the crystal is now repositioned so that the beam is incident in a “non-random” direction—in the direction of a low-order crystal axis or plane—the results observed are very different. The average depth of penetration into the crystal is greatly increased, and the rate of particle backscattering may decrease by as much as two orders of magnitude. The interaction of the beam with the crystal in this way is called channeling, and for beams with positive charge, it is not entirely naïve to imagine particles streaming through the channels between planes or rows of crystal nuclei, with soft collisions with these planes or rows guiding particles away from nuclei.

Channeling has proved to be a useful tool for understanding the properties of solids, and has had numerous technological applications. It has been used as a material analysis tool to study crystal defects, surfaces and interfaces, and to determine the location of crystal impurities. It has been used to measure nuclear lifetimes, to study the strain in “strained-layer superlattices,” and to deflect high energy particle beams. These and many other applications of channeling—including more speculative possibilities such as monoenergetic gamma ray sources and cosmic-ray telescopes—are discussed in a large body of literature, much of which is cited in the bibliographies of [12] and [21], and in the excellent if by now somewhat early review article [22].

Since the early 1960’s, theoretical investigations of channeling have relied on the so-called continuum model, in which channeling is described as the motion of particles moving in a continuum potential obtained by averaging the crystal potential over the axis or plane with which the particles’ incident direction is most nearly parallel. This model was introduced independently by a number of theoreticians, but the most convincing arguments for it were given by J. Lindhard. His and his coworkers’ papers [28], [29], [30] include detailed physical arguments; by translating the simplest classical cases considered in these articles into the language of Hamiltonian systems, we may paraphrase that part of the theory which ignores electron multiple scattering as follows. We begin by considering the perfect crystal model, in which particle motion is governed by the three degree of freedom Hamiltonian

\[ \mathcal{H} = \frac{1}{2m} p^2 + V(q) , \text{ where } p,q \in \mathbb{R}^3 , \text{ and } p^2 = \sum p_i^2 . \]  

Here \( V \) is the periodic crystal potential, usually expressed as a sum of thermally averaged screened Coulomb atomic potentials at the lattice sites. The relevance
of this model in channeling physics and some of the effects it ignores are discussed briefly in the next section. Lindhard argued that for incident particle directions nearly parallel to low-index axes, the motions of $\mathcal{H}$ are well described by the so-called impulse-momentum approximation, wherein, for grazing angles, the influence of a string of atoms on a particle may be approximated by a sequence of soft collisions with individual atoms in the string. He went on to show that, because particles experience a large number of soft collisions with atoms in a given string, another approximation is justified in which each discrete string of atoms is replaced by a “continuum string.” In terms of the Hamiltonian (2.2), this amounts to replacing the crystal potential $V$ by the axial continuum potential $V_A$ obtained from $V$ by averaging along the direction of the strings considered. This greatly simplifies (2.2); the position $q_3'$ along the axial direction is now a cyclic coordinate, so the momentum $p_3'$ in this direction is constant, and the motion in the plane transverse to the axial direction is governed by the two degree of freedom axial continuum Hamiltonian

$$\mathcal{H}_A = \frac{1}{2m} (p_1'^2 + p_2'^2) + V_A(q_1', q_2').$$  

(2.3)

Though Lindhard did not express his result in this way originally, and though he did not transform system (2.2) into system (2.3) nor give rigorous estimates of how solutions of (2.3) approximate those of (2.2), it is clear that he did establish a strong link between the two systems, and that this link is one of the cornerstone achievements in channeling theory. It is not clear however, that a similarly convincing connection has ever been established between (2.2) and the one degree of freedom planar continuum Hamiltonian

$$\mathcal{H}_P = \frac{1}{2m} p_1''^2 + V_P(q_1''),$$  

(2.4)

in which the conjugate coordinates $p_1''$ and $q_1''$ are momentum and displacement in the direction transverse to the planes in question, and the planar continuum potential $V_P$ is the average of $V$ parallel to the planes. Although this planar continuum model shares the intuitive appeal of the axial continuum model and has gained the same basic currency among channeling theorists, the arguments used to derive it from the perfect crystal model have not been wholly convincing. The problem is that the impulse-momentum approximation breaks down, because, while it is clear how particles can undergo successive grazing collisions with a segment of collinear atoms, it is not clear how they can do so with a region of coplanar atoms. This criticism does not detract from the planar continuum model, the usefulness of which has been borne out through repeated experiment, but it does point out a basic theoretical difficulty in the way it is descended from the perfect crystal model.

In the 1970’s, J.A. Ellison noticed that the equations of motion for the perfect crystal model could be scaled so as to appear in standard form for the
method of averaging for ODE's; this led to new mathematical results in channeling theory, including general derivations of axial continuum models from perfect crystal models [20]. Inasmuch as Nekhoroshev-type perturbation theory may be viewed as a specialized averaging method for Hamiltonian systems, the results given here extend that work to a derivation of axial, planar, and “spatial” continuum models (the methods here are however not suited to practical calculation as are the averaging methods; see §5 of [20] for a brief discussion).

3. FORMULATION OF THE PROBLEM

This section formulates the basic mathematical problem considered in the sequel. The perfect crystal model is introduced, and a criterion is given for identifying channeling motions in the model. This criterion leads naturally to a nearly integrable form of the perfect crystal model, and it is argued that this system is a good model of particle motion whenever the channeling criterion is satisfied.

3.1 The Perfect Crystal Model

Our approach to the problem of classical, positively charged particle motion in perfect cubic crystals begins with a classical Hamiltonian system with three degrees of freedom and with a periodic potential having finitely many (usually only a few) smooth repulsive sites in each lattice cell. In fact, throughout this work the crystal potential is assumed to be analytic, which makes possible the necessary technical estimates. The requirement of analyticity is not as severe as it might seem, as some of the best available perfect crystal potentials are analytic, essentially because of thermal averaging (cf. [30]).

In physical variables, the perfect crystal Hamiltonian represents the total energy of a point particle with mass \( m \) moving under the influence of the potential \( V(q) \), and may be written

\[
\mathcal{H}(p, q) = \frac{1}{2m} p^2 + V(q)
\]  

where \( p, q \in \mathbb{R}^3 \) are the momenta and position of the particle under consideration. Here \( V \) has period \( d \) in each component of \( q \), along with a small number of repulsive sites in each periodic cell. More precisely, \( V \) is constructed by summing screened, thermally averaged Coulombic atomic potentials centered on each lattice site (see Appendix 2 of [17] for a more complete discussion of the potential \( V \)).

Reducing the channeling problem to the study of (3.1) neglects a number of physical effects, among which are the following:
(i) Quantum and relativistic effects
(ii) Dynamic effects of thermal crystal lattice vibrations (Thermal effects are included in the potential in an average way only.)
(iii) Effects of close encounters with nuclei (The hard Coulombic singularities have been smoothed out of the potential, so close encounter processes, e.g. Rutherford scattering, are not represented.)
(iv) Effects of individual electrons (Their influence is instead incorporated into $V$ by means of a screening function, which essentially smears each electron over its spatial probability distribution.)
(v) Energy loss effects, which arise, for example, through radiation emission and electron multiple scattering

The neglect of effects (i) is known to be approximately valid over a certain range of particle compositions and particle energies (see, e.g., [22], [28], [29]). On the other hand, it is hard to imagine circumstances in which a theory of particle channeling neglecting effects (ii) through (v) could be called complete. The reader may therefore be hesitant about the viability of (3.1) as a model for positive particle channeling. But the arguments for its relevance are strong. In fact, under conditions where effects (i) may be neglected, theories accounting for the remaining effects, however detailed, may be broadly described as perturbations to the perfect crystal model—it is the foundation for theories of particle channeling. In spite of this, no completely satisfactory account of the model has yet been given. This paper is meant to remedy this situation by providing a single, coherent mathematical framework in which to view the basic channeling phenomena. In retrospect, the fidelity with which the solutions of system (3.1) reproduce channeling effects is surprising, and is perhaps the most pleasing aspect of this work.

Nevertheless, the neglect of effect (iii) requires special consideration. Physicists are apt to say that the potential $V$ used here is a good classical approximation to the true potential away from lattice sites, and a poor approximation in their close vicinity (cf. [30]). In this work, the preceding statement will be formalized by singling out neighborhoods of lattice sites inside of which the potential is “untrustworthy.” The precise criterion is given in the next subsection.

### 3.2 The Channeling Criterion

We now introduce a simple criterion, key to much of what follows, for distinguishing channeling from nonchanneling trajectories of (3.1). We first assume that the potential $V$ has been adjusted so that its minimum value is zero and its maximum value over $\mathbb{R}^3$ is $\mathcal{E}_M$, so that for energies $\mathcal{E}_\perp (0 \leq \mathcal{E}_\perp \leq \mathcal{E}_M)$, we may consider the subsets of configuration space

$$
\mathcal{B}(\mathcal{E}_\perp) = \{ q \in \mathbb{R}^3 \mid V(q) \geq \mathcal{E}_\perp \}.
$$

(3.2)
If, as is assumed here, the potential governs the motion of positively charged particles, then clearly for sufficiently large \( \mathcal{E}_\perp < \mathcal{E}_M \), the set \( \mathcal{B}(\mathcal{E}_\perp) \) is the disjoint union of (slightly deformed) balls centered on the lattice sites. By choosing a physically suitable value for \( \mathcal{E}_\perp \), we may distinguish particle trajectories which come too close to nuclei to be governed by the thermally averaged potential as those which enter \( \mathcal{B}(\mathcal{E}_\perp) \). More precisely, fix \( \mathcal{E}_\perp \), and consider a solution \( (p(\tau), q(\tau)) \) of the equations of motion corresponding to (3.1). Such a solution is a channeling solution on the time interval \( I \) provided

\[
q(\tau) \notin \mathcal{B}(\mathcal{E}_\perp), \quad \text{or equivalently} \quad V(q(\tau)) < \mathcal{E}_\perp \quad \forall \tau \in I. \quad (3.3)
\]

This is the channeling criterion, and it is assumed that the perfect crystal model is a good approximation for particle trajectories that satisfy it. A trajectory which first fails to satisfy this criterion at time \( t_1 \) is assumed to suffer a "close encounter" with a nucleus, and is not viewed as a good approximation to an actual particle trajectory for subsequent times \( t > t_1 \). While the channeling criterion is independent of the incident energy \( \mathcal{E} \), channeling motions are such that \( \mathcal{E} \gg \mathcal{E}_\perp \), and this defines the fundamental perturbation parameter in the analysis below.

### 3.3 Transformation to Nearly Integrable Form

The Hamiltonian (3.1) may be transformed to nondimensional, nearly integrable form as follows. Restricting attention to particles of fixed energy \( \mathcal{E} \), i.e., to trajectories \( (p(\tau), q(\tau)) \) satisfying \( \mathcal{H}(p(\tau), q(\tau)) = \mathcal{E} \), we define the scaled momentum (actions) \( I \in \mathbb{R}^3 \), the scaled position (angles) \( \theta \in \mathbb{T}^3 \), the scaled potential \( W \) and the scaled time \( t \) by

\[
I = (m\mathcal{E})^{-1/2} p, \quad \theta = \frac{1}{d} q, \quad W(\theta) = \frac{1}{\mathcal{E}_\perp} V(\theta d), \quad t = \frac{1}{\tau_0} \tau. \quad (3.4)
\]

Here \( \tau_0 = d \sqrt{m/\mathcal{E}} \) is the time required for a particle to travel a distance \( \sqrt{2}d \) in the potential-free case. The choice of scaling is motivated by the assumption \( \mathcal{E} \gg \mathcal{E}_\perp \), so that trajectories satisfying the channeling criterion maintain a kinetic energy of approximately \( \mathcal{E} \). The transformed Hamiltonian \( H \) now reads \( H(I, \theta) = \frac{1}{2} I^2 + (\mathcal{E}_\perp/\mathcal{E}) W(\theta) \), or, writing \( \epsilon = \mathcal{E}_\perp/\mathcal{E} \ll 1 \), this becomes

\[
H(I, \theta) = \frac{1}{2} I^2 + \epsilon W(\theta), \quad \text{where} \quad W \in C^\omega(\mathbb{T}^3), \quad (3.5)
\]

and \( 0 \leq W(\theta) \leq \mathcal{E}_M/\mathcal{E}_\perp \equiv E_o > 1 \),

and we are interested in the behavior of solutions on the surface \( H = 1 \) for small \( \epsilon \) and long times.
It is no surprise that the scaled Hamiltonian appears in action-angle form, since this scaling views the potential as a perturbation of rectilinear motion in the lattice $T^3$. System (3.5) is called the “nearly integrable form of the perfect crystal model” or the “scaled perfect crystal model.”

The region of close encounter $B(\mathcal{E}_\perp)$ described in (3.2) is immediately reformulated in terms of the scaled variables as

$$\mathcal{C}(1) = \{\theta \in T^3 \mid W(\theta) \geq 1\}, \quad (3.7)$$

and so the criterion (3.3) for a solution $(I(t), \theta(t))$ of Hamilton’s equations corresponding to (3.5) to be a channeling trajectory on the time interval $\mathcal{I}$ becomes

$$\theta(t) \notin \mathcal{C}(1), \text{ or equivalently } W(\theta(t)) < 1 \quad \forall \ t \in \mathcal{I}. \quad (3.8)$$

Finally, we note that for fixed $\epsilon$ (i.e., for fixed values of $\mathcal{E}_\perp/\mathcal{E}$), it follows from energy conservation that for all $t$, real-valued motions $(I(t), \theta(t))$ of (3.5) are confined to a spherical shell in action space of thickness $O(\epsilon)$ given by

$$S^{\sqrt{\epsilon}} = \{I \in \mathbb{R}^3 \mid 2 - 2\epsilon E_o \leq \|I(t)\|^2 \leq 2\}, \quad (3.9)$$

where $\|I\|^2 = I_1^2 + I_2^2 + I_3^2$ is the Euclidean norm, and $E_o \equiv \mathcal{E}_o/\mathcal{E}_\perp > 1$. We will always assume that $\epsilon$ is small enough to ensure that $I \in S^{\sqrt{\epsilon}} \Rightarrow \|I\| \geq \sqrt{2} - \epsilon E_o$; this fact will be useful in proving the stability of channeling motions in Section 5.

4. CONSTRUCTION OF THE NORMAL FORMS

In this section we construct the normal forms on which the generalized continuum models are based. This construction corresponds to what is called the analytic lemma in the proof of Nekhoroshev’s theorem, and the methods used here follow those used in proving the analytic lemma in [8]. The result obtained is however closer in spirit to the analytic lemma in [11]. The use of KAM-type resonant zones here is apparently unique in a Nekhoroshev-like result; use of these zones is not warranted in Nekhoroshev’s theorem itself, as the aim is to obtain uniform estimates of the deviation of actions from their initial values. The goal here is instead to obtain separate descriptions of particle motions in regions of phase space with distinct resonance properties, and the KAM zones
adhere to the observed decrease in size of planar and axial channeling zones with increasing order.

This rather lengthy and detailed section begins with a description of the methods to be used, and is followed by subsections devoted to notation, and to the Lie method for canonical transformations. The main result—the analytic lemma—is stated in §4.4; its proof requires the iterative lemma, whose statement and proof occupy §4.5 (with a few technical results relegated to §4.6). The analytic lemma is proved in §4.7.

4.1 Description of Methods

As discussed in Section 1, it is generally impossible to transform the nearly integrable Hamiltonian

$$H(I, \theta, \epsilon) = h(I) + \epsilon f(I, \theta, \epsilon)$$  \hspace{1cm} (4.1)

to completely integrable form $H'(I', \epsilon)$ on an open subset of phase space. Various alternate approaches are possible; the approach used here, which yields exponentially long stability times, is classical, in that it requires transformations to be defined on subsets of phase space with nonempty interior. In this way (4.1) is brought into a resonant normal form (in this case Gustafson normal form)

$$H'(J, \phi) = h(J) + G(J, \phi^*) + R(J, \phi)$$  \hspace{1cm} (4.2)

consisting of an effective Hamiltonian $h + G$ together with a small remainder $R$ of order $O(\epsilon^L)$. The dependence of the normal form on $\epsilon$ is intentionally suppressed because $\epsilon$ must be fixed before the transformation takes place; the transformation is analytic in $(I, \theta)$ and $(J, \phi)$, but is not even continuous in $\epsilon$. This is because, in order to limit the encroachment of small divisors into the domain of the transformation, we truncate the Fourier series of $f$ at some finite order and push the tail series into the remainder $R$ before effecting the transformation; the truncation order must depend on $\epsilon$ to ensure that $R = O(\epsilon^L)$, which introduces a discontinuous dependence of the transformation’s domain on $\epsilon$. (This “ultraviolet cutoff” technique was introduced by Arnol’d in his proof of the KAM theorem [1].) The remainder $R$ is minimized by letting the exponent $L$ depend on $\epsilon$, and by balancing the need to make $L$ as large as possible against the need to keep the small divisors from intruding too far into the domain. This balance is achieved for $L$ equal to an inverse power of $\epsilon$, so that $R$ is indeed exponentially small.

The $\phi^*$ appearing in (4.2) represents the so-called “resonant variables”; these span a subspace of lower dimension than the original configuration variables $\theta$, or in other words, the effective Hamiltonian is cyclic in one or more linear combinations of the transformed configuration variables $\phi$. The precise composition of the resonant variables $\phi^*$ depends on a subset (called a resonant
block; cf. Section 5.1) of the region of action space where the transformation is defined. Roughly speaking, these blocks are constructed by excluding from action space those points for which the gradient \( Dh \) is nearly orthogonal to the \( k \in \mathbb{Z}^3 \) (with norm less than the cutoff) which do not belong to a particular submodule \( \mathcal{M} \) of \( \mathbb{Z}^3 \) (the modular structure of \( \mathbb{Z}^3 \) is recalled in the next subsection). These points must be excluded to control the (finitely many) small divisors arising in the homological equation defining the transformation. The resonant variables \( \phi^* \) then belong to \( \text{span} \mathcal{M} \). Given a nontrivial \( q \)-dimensional submodule \( \mathcal{M} \), we thus have a normal form consisting of a \( q \) degree of freedom effective (or resonant) Hamiltonian and a small remainder. If \( \mathcal{M} \) is the trivial submodule \( \{0\} \), then the resonant variables are absent and the effective (nonresonant) Hamiltonian is completely integrable. The normal forms corresponding to nontrivial submodules of dimension 1 then govern motion inside a portion of the zones excluded from the domain of definition of the nonresonant Hamiltonian; 2 degree of freedom normal forms (i.e., with 2 degree of freedom effective Hamiltonians) govern motion inside zones excluded from 1 degree of freedom normal forms, and so on.

### 4.2. Notation

In this section we will always use the max norm \( \|x\|_\infty = \max_i |x_i| \) for vectors \( x \) in \( \mathbb{R}^3 \) or in \( \mathbb{C}^3 \), while for integer vectors \( k \in \mathbb{Z}^3 \) we use the taxicab norm \( |k| = |k_1| + |k_2| + |k_3| \). In succeeding sections, the Euclidean norm \( \|x\|_2 = (\sum_i x_i^2)^{1/2} \) will sometimes be used, both for real and for integer vectors. The action-angle variables \( (I, \theta) \) on which (3.5) depends initially belong to \( K \times T^3 \), where \( K \) is a compact subset of \( \mathbb{R}^3 \), and where the notation \( \theta \in T^3 \) is used to indicate that any function depending on \( \theta \) has unit periodicity in the real part of each of the components \( (\theta_1, \theta_2, \theta_3) \).

We will take advantage of the fact that the Hamiltonian (3.5) is analytic by extending its domain of definition \( K \times T^3 \) to complex phase space, which in turn allows us to use standard Cauchy estimates. Following [8] and [34], given \( K \subset \mathbb{R}^3 \) we define

\[
F(K, \rho, \sigma) =
\{(I, \theta) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid \text{Re} I \in K, \|\text{Im} I\|_\infty \leq \rho, \text{Re} \theta \in T^3, \|\text{Im} \theta\|_\infty \leq \sigma\}.
\] (4.3)

Following Nekhoroshev, for \( K \subset \mathbb{R}^3 \) and \( \delta \geq 0 \) we next define \( K - \delta \) as the set of centers of closed, max norm \( \delta \)-neighborhoods contained in \( K \), and \( K + \delta \) as the union of \( \delta \)-neighborhoods with centers in \( K \). The following simple properties of these operations are immediately verified:

\[
(K + \delta_1) + \delta_2 = K + (\delta_1 + \delta_2), \quad (K - \delta_1) - \delta_2 = K - (\delta_1 + \delta_2) \quad (4.4a)
\]
\[
(K - \delta_1) + \delta_2 \subset K - (\delta_1 - \delta_2), \quad K + (\delta_1 - \delta_2) \subset (K + \delta_1) - \delta_2, \quad \delta_1 \geq \delta_2 \quad (4.4b)
\]
\[(K - \delta) \cap (L - \delta) = (K \cap L) - \delta, \quad (K \cap L) + \delta \subset (K + \delta) \cap (L + \delta) \quad (4.4c)\]

\[K + \delta \subset L \iff K \subset L - \delta \quad (4.4d)\]

For complex domains \(D = F(K, \rho, \sigma)\) and \(0 < \delta < \rho, \ 0 < \sigma < \xi\) we set \(D - (\delta, \xi) = F(K - \delta, \rho - \delta, \sigma - \xi)\).

For \(f\) analytic on \(D\), we use \(\|f\|_D\) to denote \(\sup_D |f(I, \theta)|\) if \(f\) is scalar-valued, and \(\max_i \sup_D |f_i(I, \theta)|\) if \(f\) is vector-valued. For nonnegative integers \(\alpha, \beta, \|\partial^{\alpha + \beta} f / \partial I^\alpha \partial \theta^\beta\|_D\) will denote \(\max_{|r| = |\alpha|, |s| = |\beta|} \|\partial^{r+s} f / \partial I^r \partial \theta^s\|_D\), where \(r = (r_1, r_2, r_3)\) and \(s = (s_1, s_2, s_3)\) are multiindices with nonnegative integer entries, and where \(I^r = I_1^{r_1} I_2^{r_2} I_3^{r_3}\) and \(\theta^s = \theta_1^{s_1} \theta_2^{s_2} \theta_3^{s_3}\).

In order to discuss resonances, we recall that a finite subset \(\{k^{(i)}\}_{i=1}^m\) of \(\mathbb{Z}^3\) generates a submodule of \(\mathbb{Z}^3\) defined as \(\{z \in \mathbb{Z}^3 | z = \sum n_i k^{(i)}, n_i \in \mathbb{Z}\}\). Each submodule of \(\mathbb{Z}^3\) has dimension 0, 1, 2, or 3, in the sense that the smallest vector subspace of \(\mathbb{R}^3\) containing the submodule has that dimension. If \(\{k^{(i)}\}_{i=1}^m\) generates a submodule of dimension \(n\), then the maximal submodule generated by \(\{k^{(i)}\}_{i=1}^m\) is the largest submodule of dimension \(n\) containing \(\{k^{(i)}\}_{i=1}^m\). In the remainder of this article, the symbol \(\mathcal{M}\) will refer to a 0-, 1- or 2-dimensional maximal submodule of \(\mathbb{Z}^3\). Not surprisingly, every \(\mathcal{M}\) is generated respectively by integer combinations of a 1- or 2-element basis in \(\mathbb{Z}^3\) (cf. Appendix 1 of [17]); the \textit{order} \(|\mathcal{M}|\) of a maximal submodule is the smallest nonnegative integer \(r\) such that \(\mathcal{M}\) admits a basis of vectors with norm less than or equal to \(r\).

Finally, for functions \(f\) on \(D\) with Fourier coefficients \(f_k(I)\), we introduce the notation

\[\Pi_{\mathcal{M}} f = \sum_{k \in \mathcal{M}} f_k(I)e^{2\pi i k \cdot \theta}\quad (4.5)\]

for the resonant subseries of \(f\) corresponding to the maximal submodule \(\mathcal{M}\). \(\Pi_{\mathcal{M}}\) is a projection operator, as the notation suggests, and, most important for the channeling problem, the subseries \(\Pi_{\mathcal{M}} W\) is precisely the continuum potential \(\overline{W}\) obtained by averaging \(W\) over its period(s) in the orthogonal complement (in \(\mathbb{R}^3\)) of \(\mathcal{M}\). Thus when \(\dim \mathcal{M} = 1\) or 2, \(\Pi_{\mathcal{M}} W\) is a planar or axial continuum potential, respectively. The proofs of these facts are nearly obvious; full details may be found in Appendix 2 of [17].

4.3 Near Identity Canonical Transformations via the Lie Method

The following lemma expresses facts known about canonical transformations since the time of Sophus Lie, and it (or some variant) is proved in [8] and in [34]. Because of its importance, we also give a proof which references some standard results but is more detailed than the proofs just cited.

\textbf{Lemma 4.1.} (Lie method for canonical transformations) Given a compact set \(K \subset \mathbb{R}^3\) and positive numbers \(\rho\) and \(\sigma\), consider the domain \(D = F(K, \rho, \sigma)\) and an analytic function \(\chi : D \to \mathbb{C}\) which is real for real arguments; i.e., for
\((I, \theta) \in K \times \mathbb{T}^3\). Let \((I, \theta)(\tau) = (T_\tau^I, T_\tau^\theta) = T^\tau(J, \phi)\) denote the flow which takes the initial condition \((J, \phi) \in D\) to the solution \((I, \theta)\) at time \(\tau\) of the Hamiltonian initial value problem

\[
\frac{d I}{d \tau} = -\frac{\partial \chi}{\partial \theta}, \quad \frac{d \theta}{d \tau} = \frac{\partial \chi}{\partial I},
\]

\((I, \theta)(0) = (J, \phi).\) (4.6a)

Define \(T = T^1\) to be the time-one map corresponding to the flow \(T^\tau\). If \(A, B, \delta, \xi\) are positive numbers satisfying

\[
A < \delta < \rho \quad \text{and} \quad B < \xi < \sigma, \quad \text{and if}
\]

\[
\|\frac{\partial \chi}{\partial I}\|_{D - (\delta/2, \xi/2)} \leq \frac{B}{2}, \quad \|\frac{\partial \chi}{\partial \theta}\|_{D - (\delta/2, \xi/2)} \leq \frac{A}{2},
\]

then for \(0 \leq \tau \leq 1\), the flow \(T^\tau\) is a well-defined analytic function of \((J, \phi)\) and \(\tau\) on a suitable \((I, \theta)\)-domain \(D' = F(K', \rho - \delta, \sigma - \xi)\) satisfying

\[
D - (\delta, \xi) \subset D' \subset D - (\delta/2, \xi/2).
\]

Moreover for fixed \(\tau \in [0, 1]\), the map \(T^\tau : D' \to T(D')\) is a bijective canonical transformation taking real sets onto real sets, with a canonical analytic inverse \((T^\tau)^{-1} : T(D') \to D'\) which coincides with \(T^{-\tau}\). Each of these maps is close to the identity in the sense that

\[
K - \delta \subset K' \subset K - \delta/2, \quad \text{(4.10a)}
\]

\[
K - \delta \subset T_I^\tau(K') \subset K - \delta/2, \quad \text{and}
\]

\[
\|I - J\|_{D'} \leq \frac{A}{2}, \quad \|\theta - \phi\|_{D'} \leq \frac{B}{2}.
\]

(4.11)

Finally, given any function \(f\) analytic on \(D\), the analytic function \(f' = f \circ T : D' \to \mathbb{C}\) satisfies

\[
\|f'\|_{D'} \leq \|f\|_D \quad \text{and}
\]

\[
\|f' - f - \{f, \chi\}\|_{D'} \leq \frac{1}{2} \|\{f, \chi\} \|_{D - (\delta/2, \xi/2)},
\]

(4.13)

where \(\{,\}\) is the Poisson bracket

\[
\{f, \chi\} = \sum_i (\frac{\partial f}{\partial \theta_i} \frac{\partial \chi}{\partial I_i} - \frac{\partial f}{\partial I_i} \frac{\partial \chi}{\partial \theta_i}).
\]

(4.14)

**Proof.** From the standard existence-uniqueness theory for ODE’s (see, e.g., [5], §32.5) we know that, given \((J, \phi) \in \text{Int}(D)\), there is an open interval \((a, b)\)
containing 0 for which the flow \( (I(\tau), \theta(\tau)) = T^{r}(J, \phi) \) is a one-to-one analytic function of \( \tau \) and \( (J, \phi) \) for all \( \tau \in (a, b) \). Since \( \chi \) is real for real arguments, if \( (J, \phi) \) is real then \( T^{r}(J, \phi) \) is real for all \( \tau \). By well-known continuation arguments (cf. [25], §1.2), the existence interval extends to a maximal existence interval \( (a', b') \) such that \( T^{r}(J, \phi) \to \partial D \) as \( \tau \to a' \) or \( \tau \to b' \). If the partial derivatives of \( \chi \) are uniformly bounded on \( D \) or some appropriate subset, it is a simple matter to estimate the maximal existence time \( \min \{|a'|, |b'|\} \) in terms of the distance from \( (J, \phi) \) to \( \partial D \). This observation may be turned around to estimate the size of \( D' \) such that \( T^{r} \) is defined and \( T^{r}(D') \subset D - (\delta/2, \xi/2) \) for all \( \tau \in [0, 1] \). More precisely, set \( D'' = D - (\delta/2, \xi/2) \) and \( D' = \{ (J, \phi) \in D'' \mid T^{r}(J, \phi) \in D'' \text{ for } 0 \leq \tau \leq 1 \} \). Then \( D - (\delta, \xi) \subset D' \) is easily seen as follows. Let \( (J, \phi) \in D - (\delta, \xi) \subset D'' \) and suppose \( (I, \theta) = T^{r}(J, \phi) \notin D'' \) for some \( \tau \in [0, 1] \). Set \( \tau_{1} = \inf_{0 \leq \tau \leq 1} \{ T^{r}(J, \phi) \in \partial D'' \} \). (\( \tau_{1} \) exists in \([0, 1]\) by continuity of the flow in \( D'' \).) By definition \( \| T^{r}_{I_{1}}(J, \phi) - J \|_{\infty} \geq \delta/2 \); on the other hand, by (4.6a), \( T^{r}_{I_{1}}(J, \phi) - J = -\int_{0}^{\tau_{1}} (\partial \chi / \partial \theta) d\tau' \), and so using (4.7) and (4.8), we have \( \| T^{r}_{I_{1}}(J, \phi) - J \|_{\infty} < \delta/2 \). The contradiction shows that \( T^{r}(J, \phi) \in D'' \), which proves (4.9) and (4.10a) for \( 0 \leq \tau \leq 1 \). A very similar argument also shows that \( T^{r}(J, \phi) \in D'' \) for \( -1 \leq \tau \leq 0 \); we thus have

\[
T^{r}(D - (\delta, \xi)) \subset D - (\delta/2, \xi/2) \quad \text{for all } \tau \in [-1, 1].
\]

(4.15)

To prove (4.10b), let \( \tau \in [0, 1] \). By definition \( T^{r}(D') \subset D - (\delta/2, \xi/2) \); now let \( (I, \theta) \in D - (\delta, \xi) \subset D' \). Then (4.15) \( \Rightarrow \)

\[
T^{-\tau}(I, \theta) \in D''.
\]

(4.16)

For every \( \eta \in [0, 1] \), \( |\eta - \tau| \leq 1 \) \( \Rightarrow \)

\[
T^{\eta}T^{-\tau}(I, \theta) = T^{\eta-\tau}(I, \theta) \in D''.
\]

(4.17)

From (4.16) and (4.17), it follows that \( T^{-\tau}(I, \theta) \in D' \), so \( T^{-\tau}(D - (\delta, \xi)) \subset D' \), or \( (D - (\delta, \xi)) \subset T^{r}(D') \), from which (4.10b) follows.

To establish the conclusions of the lemma concerning the inverse transformation, note that for any \( \tau \in [0, 1] \) we now have a domain \( D' = F(K', \rho - \delta, \sigma - \xi) \) for \( T^{r} \) where \( K' \) satisfies (3.7a). Since \( T^{r} : D' \to T^{r}(D') \) is bijective, it has an inverse \( (T^{r})^{-1} \). On the other hand, given \( (I, \theta) \in T^{r}(D') \), suppose \( (T^{r})^{-1}(I, \theta) = (J, \phi) \). Because the system (3.3) is autonomous we also have \( T^{-\tau}(I, \theta) = (J, \phi) \). Therefore \( (T^{r})^{-1} = T^{-\tau} \), which is analytic.

It has long been known that a well-defined flow such as \( T^{r} \) given by an autonomous Hamiltonian system (4.6) provides a canonical transformation of its domain onto its range for fixed \( \tau \), and that its inverse does the same. By canonical, we mean that the symplectic form \( dI \wedge d\theta \) is preserved (\( dI \wedge d\theta = dJ \wedge d\phi \)), and that any set of Hamilton’s equations written in terms of the
phase variables \((I, \theta)\) and the Hamiltonian \(H\) retain their form in terms of the variables \((J, \phi)\) with new Hamiltonian \(H \circ T^r\). Proofs of this fact abound; the reader may consult [6], §38A and §44E for a modern treatment; detailed proofs in canonical coordinates may be found in older texts.

To complete the proof, note that the bounds (4.11) now clearly follow from (4.6), (4.7) and (4.8), while (4.12) is immediate, since \(f' = f \circ T^r\) and \(T(D') \subset D\). As for (4.13), using the familiar relation \(\frac{d}{dT^r}(f \circ T^r) = \{f, \chi\} \circ T^r\) (see, e.g., [5], §40A), we have by Taylor’s formula at second order: \(f' = f \circ T^0 + \{f, \chi\} \circ T^0 + \frac{1}{2}\{\{f, \chi\}, \chi\} \circ T^r\) for some \(\tau \in (0, 1) \Rightarrow \|f' - f - \{f, \chi\}\|_{D'} = \frac{1}{2}\|\{\{f, \chi\}, \chi\} \circ T^r\|_{D'}\). But \((J, \phi) \in D'\) and \(\tau \in (0, 1) \Rightarrow T^r(J, \phi) \in D - (\delta/2, \xi/2)\), so this last quantity is less than or equal to \(\frac{1}{2}\|\{\{f, \chi\}, \chi\}\|_{D - (\delta/2, \xi/2)}\). □

4.4 Statement of the Analytic Lemma

The original version of the following result was called the “analytic lemma” by Nekhoroshev, because it is simply a statement about the transformation of a Hamiltonian on certain parts of phase space; no mention is made of the corresponding solutions to Hamilton’s equations. Showing that solutions remain in regions where the transformation to normal form is valid is a separate issue to be discussed in the next section.

Using the notation of §4.2, we now state

**Lemma 4.2.** (analytic lemma) Let \(\alpha, \tau, d, c > 1/2, p > 4,\) and \(a \equiv \alpha + (p + 1)\tau\) be positive numbers satisfying the consistency relations

\[
1 - 2a - (p + 5)\tau/2 - d > 0, \quad (4.18a)
\]

\[
1 - a - (p + 5)\tau/2 - c > 0, \quad (4.18b)
\]

\[
1 - 4a - (p + 6)\tau > 0. \quad (4.18c)
\]

(An example of such a set of numbers is \(\alpha = 1/8, \tau = 1/72, d = 1/2, c = 5/8, p = 5, a = 5/24\).)

Let \(0 < \sigma, A \leq 1, E\) be positive numbers and let \(\epsilon > 0\) be a small parameter satisfying the following restrictions (no attempt is made to combine them):

\[
\epsilon \leq \epsilon^{-1/\tau}, \quad (4.19a)
\]

\[
\epsilon \leq \left(\frac{A^2 \sigma^{p+4} E}{2^{2p+13} C_1 E} \right)^{\frac{1}{1 - a - (p + 5)\tau/2 - c}}, \quad (4.19b)
\]

\[
\epsilon \leq \left(\frac{A^2 \sigma^{p+4} E}{2^{2p+12} C_1 E} \right)^{\frac{1}{1 - a - (p + 5)\tau/2 - c}}, \quad (4.19c)
\]
\[ \epsilon \leq \left( \frac{A^2 \sigma^{2p+8}}{C_2 \epsilon^{2p+22}} \right)^{1 - \frac{3}{2(p+6)}}, \quad (4.19d) \]

\[ \epsilon^{-\tau/2} \geq \frac{8}{\pi \sigma} \left( 1 - \log \left( \frac{(\pi \sigma)^3}{213[2(p+2)]p+2} \right) - \frac{5\tau}{2} \log \epsilon \right), \quad (4.19e) \]

where

\[ C_1 = \frac{2^7 [2(p+4)]^{p+4}}{\pi^{p+4} \epsilon_0}, \quad \text{and} \]

\[ C_2 = 4C_1 \left( \frac{E}{A} \right)^3 \left[ 9A + 8C_1 \left( \frac{E}{A} \right)^2 \right]. \quad (4.20b) \]

Set \( N = \epsilon^{-\tau} \) and let \( M \neq \mathbb{Z}^3 \) be a maximal submodule of \( \mathbb{Z}^3 \). Define \( h(I) = \frac{1}{2} I^2 \), and suppose \( H(I, \theta) = h(I) + f(I, \theta) \) is analytic on \( D = F(K, A \epsilon, \sigma) \) where \( K \subseteq \mathbb{R}^3 \) is a compact set for which

\[ I \in K \Rightarrow |k \cdot I| \geq \frac{3}{2} A \epsilon_0 |k|^{-p} \quad \text{for all} \quad k \notin M \quad \text{with} \quad |k| \leq N. \quad (4.21) \]

Suppose that

\[ \|H\|_D \leq E \quad \text{and} \]

\[ \|f\|_D \leq \epsilon E \quad (4.22b) \]

(in general, \( E \geq E_0 \), where \( E_0 \) is defined in (3.6)). Then there exists a bijective transformation \( T : D_{\infty} \to T(D_{\infty}) \) taking real sets onto real sets, with both \( T \) and \( T^{-1} \) canonical and analytic, and with

\[ D - \left( \frac{A}{2} \epsilon, \sigma \right) \subseteq D_{\infty} \subseteq D - \left( \frac{A}{8} \epsilon, \sigma \right) \quad \text{and} \]

\[ D - \left( \frac{A}{2} \epsilon, \sigma \right) \subseteq T(D_{\infty}) \subseteq D - \left( \frac{A}{8} \epsilon, \sigma \right). \quad (4.23b) \]

On \( D_{\infty} \), \( T \) transforms \( H \) to the form \( H' = H \circ T \), which may be written

\[ H'(J, \phi) = h(J) + G(J, \phi) + R(J, \phi), \quad \text{where} \]

\[ G(J, \phi) = \sum_{k \in M} G_k(J) e^{2\pi i k \cdot \phi}, \quad \text{and} \]

\[ R(J, \phi) = \sum_{k \notin M} R_k(J) e^{2\pi i k \cdot \phi}. \quad (4.25b) \]

Furthermore,

\[ \|R\|_{D_{\infty}} \leq 2\epsilon E \exp(-\epsilon^{-\tau/4}), \quad \text{and} \]

\[ \|G - \Pi_M f\|_{D_{\infty}} \leq E \epsilon^{1+\tau}. \quad (4.27) \]
Finally, given \((J, \phi) \in D_\infty\) and setting \((I, \theta) = T(J, \phi)\), we have

\[
\|I - J\|_\infty \leq \frac{A}{4} \epsilon^c, \quad \text{and} \quad \|\theta - \phi\|_\infty \leq \frac{\sigma}{4} \epsilon^d. \tag{4.28a}
\]

Proof of the analytic lemma is deferred to the end of the section; it is first necessary to set up the “iterative lemma,” which allows for a single step in the formation of the composite transformation \(T\) appearing above.

4.5 The Iterative Lemma

**Lemma 4.3.** (iterative lemma) Assume the hypotheses of Lemma 4.2, with the exception of (4.22b). Write

\[
H(I, \theta) = h(I) + G(I, \theta) + R(I, \theta), \tag{4.29}
\]

where \(G = \Pi_M f\) and \(R = (1 - \Pi_M)f\). Assume that

\[
\|G + R\|_D \leq \gamma E \leq 2\epsilon E \quad \text{and} \quad \|R\|_D \leq \eta E \leq 2\epsilon E, \tag{4.30}
\]

where \(\gamma, \eta\) satisfy

\[
0 < \eta, \gamma \leq 2\epsilon \quad \text{and} \quad \eta \leq 2\gamma. \tag{4.32}
\]

Then given \(A'\) and \(B'\) satisfying \(\frac{1}{4} A \epsilon^{r/2} \leq A' < A\) and \(\frac{1}{4} \sigma \epsilon^{r/2} \leq B' < \sigma\), there is a bijective transformation \(T' : D' \rightarrow T'(D')\) with both \(T'\) and \((T')^{-1}\) canonical and analytic and with

\[
D - (A' \epsilon^a, B') \subset D' \subset D - \left(\frac{A' \epsilon^a}{2}, \frac{B'}{2}\right), \tag{4.33}
\]

\[
D - (A' \epsilon^a, B') \subset T'(D') \subset D - \left(\frac{A' \epsilon^a}{2}, \frac{B'}{2}\right), \tag{4.34}
\]

such that if \(T'(J, \phi) = (I, \theta)\) then

\[
\|I - J\|_{D'} \leq \frac{A'}{2} \epsilon^c, \quad \text{and} \quad \|\theta - \phi\|_{D'} \leq \frac{B'}{2} \epsilon^d, \tag{4.35}
\]

\[
\|\theta - \phi\|_{D'} \leq \frac{B'}{2} \epsilon^d, \tag{4.36}
\]
and such that the transformed Hamiltonian $H' = H \circ T'$ takes the normal form

$$H'(J, \phi) = h(J) + G'(J, \phi) + R'(J, \phi)$$  \hspace{1cm} (4.37)

on $D'$, where

$$\Pi_{\mathcal{M}} G' = G', \quad \Pi_{\mathcal{M}} R' = 0,$$  \hspace{1cm} (4.38)

$$\|H'\|_{D'} \leq E,$$  \hspace{1cm} (4.39)

$$\|G' + R'\|_{D'} \leq \gamma' E,$$  \hspace{1cm} (4.40)

$$\|R'\|_{D'} \leq \eta' E = (\eta'_1 + \eta'_2) E,$$  \hspace{1cm} (4.41)

$$\gamma' = \gamma + \frac{1}{2} \eta' = \gamma + \frac{1}{2}(\eta'_1 + \eta'_2),$$  \hspace{1cm} (4.42)

$$\eta'_1 = \frac{2^5[2(p+2)]^{p+2} \eta e^{-N \pi B'/2}}{(\pi B')^3}$$  \hspace{1cm} (4.43)

$$\eta'_2 = C_2 \frac{\gamma \eta}{(A')^2 (B')^2 p^3 \epsilon^{4a}}$$  \hspace{1cm} (4.44)

$$C_1 = \frac{2^7[2(p+4)]^{p+4}}{\pi^{p+4} \epsilon^{p}}$$  \hspace{1cm} (4.45)

$$C_2 = 4C_1 \left( \frac{E}{A} \right) \left[ 9A + 88C_1 \left( \frac{E}{A} \right) \right].$$  \hspace{1cm} (4.46)

**Proof.** Proceeding according to Lemma 4.1, we construct the desired transformation by means of a Lie generating function $\chi : D \rightarrow \mathbb{C}$. Assume tentatively that $\chi$ is analytic and define $T'$ as in Lemma 4.1. Setting $H' = H \circ T'$ and using (4.29), the identity $H' = H' + H - H + \{H, \chi\} - \{H, \chi\}$ may be written

$$H' = h + G + R + \{h, \chi\} + \{G + R, \chi\} + H' - H - \{H, \chi\}$$

$$= h + G + R^\leq + \{h, \chi\} + S,$$  \hspace{1cm} (4.47)

where we have introduced the cut off part of $R$ (recall $N = \epsilon^{-r}$):

$$R^\leq(J, \phi) = \sum_{\substack{k \in \mathcal{M} \cap \mathbb{Z}^4 \atop |k| \leq N}} R_k(J)e^{2\pi ik \cdot \phi},$$  \hspace{1cm} (4.48)

and the $N$-tail of $R$:

$$R^\geq = R - R^\leq,$$  \hspace{1cm} (4.49)

$$S = R^\geq + \{G + R, \chi\} + H' - H - \{H, \chi\}.$$  \hspace{1cm} (4.50)
This suggests choosing \( \chi \) so as to eliminate \( R^\leq \). We should then be able to estimate \( S \) by means of (4.13), (4.30) and the fact that \( R^\geq \) is small. Proceeding this way, we are led to consider the homological equation \( R^\leq + \{h, \chi\} = 0 \), or explicitly

\[
\sum_{k \in \mathcal{M}, |k| \leq N} R_k(J)e^{2\pi i k \cdot \phi} - J \cdot \frac{\partial \chi}{\partial \phi} = 0.
\] (4.51)

Expanding \( \chi \) in its Fourier series \( \chi = \sum_k \chi_k(J)e^{2\pi i k \cdot \phi} \), we find that (4.51) will be satisfied if

\[
\chi_k(J) = -\frac{i R_k(J)}{2\pi (k \cdot J)} \quad \text{for} \quad k \notin \mathcal{M}, \quad |k| \leq N,
\] (4.52)

\( \chi_k(J) = 0 \) otherwise.

\( \chi \) defined in this way is analytic on \( D \), since it is a finite trigonometric sum and its coefficients \( \chi_k \) are well-behaved analytic functions of \( J \) on \( D \), by (4.21) and Proposition 4.5. (Note: Propositions 4.5 through 4.7 are in §4.6.) With a view to using Lemma 4.1, let us estimate the partial derivatives of \( \chi \). First, using the defining relation (4.52) together with the nonresonance condition (4.21) and Proposition 4.5, we find

\[
\|\chi_k\|_D \leq \frac{|k|^p}{A \pi \epsilon^a} \|R_k\|_D \leq \frac{|k|^p}{A \pi \epsilon^a} \|R\|_D.
\] (4.53)

Because \( R \) is analytic on \( D \), it is a simple matter to show (see, e.g., Appendix 2.3 of [17]) that its Fourier coefficients decrease exponentially as

\[
\|R_k\|_D \leq e^{-2\pi \sigma |k|} \|R\|_D,
\] (4.54)

and so combining the last two inequalities gives

\[
\|\chi_k\|_D \leq \frac{|k|^p}{A \pi \epsilon^a} e^{-2\pi \sigma |k|} \|R\|_D.
\] (4.55)

Proposition 4.6 and (4.31), (4.45), and (4.55) then give

\[
\|\chi\|_{D - (0, B')} \leq \left( \frac{2^5 |2(p + 2)|^{p+2}}{\pi^{p+4} \epsilon^{p+2 - \pi B'/2}} \right) \frac{\eta E}{A(B')^{p+3} \epsilon^a} \leq C_1 \frac{\eta E}{A(B')^{p+3} \epsilon^a}.
\] (4.56)

A standard Cauchy estimate gives

\[
\|\frac{\partial \chi}{\partial J}\|_{D - (A' \epsilon^a, B')} \leq \frac{2}{A' \epsilon^a} \|\chi\|_{D - (0, B')} \leq 2C_1 \frac{\eta E}{AA'(B')^{p+3} \epsilon^{2a}},
\] (4.57)
which, since \( \eta \leq 2\epsilon \) and \( A' \geq \frac{1}{4} A \epsilon^{\tau/2}, \ B' \geq \frac{1}{4} \sigma \epsilon^{\tau/2} \), is bounded by

\[
\frac{B'}{2} \left( \frac{2^{2p+13} C_1 E}{A^2 \sigma p^4} \right) \epsilon^{1-2a-(p+5)\tau/2} \leq \frac{B'}{2} \epsilon^d,
\]

where the last inequality follows from the restriction (3.16b).

The other partial derivative may be estimated with the help of (4.45), (4.55) and Proposition 4.7:

\[
\begin{align*}
\| \frac{\partial \chi}{\partial \phi} \|_{D-(0, \frac{B'}{2})} & \leq \frac{2^6 \| R \|_D [2(p+3)]^{p+3}}{A \epsilon^a (\pi B')^{p+4}} e^{\pi B'/2-p-3} \leq \\
& \left( \frac{2^6 [2(p+3)]^{p+3}}{\pi p^4 \epsilon^p} \right) \frac{\eta E}{A(B')^{p+4} \epsilon^a} \leq C_1 \frac{\eta E}{A(B')^{p+4} \epsilon^a}. \tag{4.59}
\end{align*}
\]

Again using \( A' \geq \frac{1}{4} A \epsilon^{\tau/2}, \ B' \geq \frac{1}{4} \sigma \epsilon^{\tau/2} \), and \( D-(\frac{A' \epsilon^a, B'}{2}) \subset D-(0, \frac{B'}{2}) \), we find that

\[
\begin{align*}
\| \frac{\partial \chi}{\partial \phi} \|_{D-(\frac{A' \epsilon^a, B'}{2})} & \leq \| \frac{\partial \chi}{\partial \phi} \|_{D-(0, \frac{B'}{2})} \leq \\
& \frac{A'}{2} \left( \frac{2^{2p+12} C_1 E}{A^2 \sigma p^4} \right) \epsilon^{1-a-(p+5)\tau/2} \leq \frac{A'}{2} \epsilon^c, \tag{4.60}
\end{align*}
\]

where restriction (4.19c) was used in the last inequality.

Lemma 4.1 on canonical transformations now ensures that the transformation \( T' \) defined as the time-one map of the Hamiltonian flow generated by \( \chi \) given in (4.52) is a well-defined, analytic canonical transformation, with a canonical analytic inverse, on a domain \( D' \) satisfying (4.33) and (4.34). The lemma also assures that \( T' \) satisfies (4.35) and (4.36).

On \( D' \), we have by construction (see 4.47)

\[
H' = H \circ T' = h + G + S, \tag{4.61}
\]

where \( S \) is given by (4.50). Let us estimate the terms comprising \( S \). First, since \( D' \subset D-(0, \frac{B'}{2}) \),

\[
\| R^\gamma \|_{D'} \leq \| R^\gamma \|_{D-(0, \frac{B'}{2})} \leq \\
\frac{2^5 [2(p+2)]^{p+2}}{(\pi B')^3} \| R \|_{D - N \pi B'/2} e^{-N \pi B'/2} \leq \frac{2^5 [2(p+2)]^{p+2}}{(\pi B')^3} \eta e^{-N \pi B'/2} E = \frac{1}{2} \eta_1 E. \tag{4.62}
\]

The second inequality derives from (4.54) and Proposition 4.6; the third inequality derives from (4.31). (Proposition 4.6 applies to \( R^\gamma \) as well as to \( R^\delta \).)
To estimate the next term of $S$, we abbreviate $D_2 = D - (\frac{A'}{2} \varepsilon^a, \frac{B'}{2})$ and write
\[
\| \{G + R, \chi\}\|_{D'} \leq \| \{G + R, \chi\}\|_{D_2} \leq \sum_{j=1}^{3} \left( \| \frac{\partial (G + R)}{\partial \phi_j} \|_{D_2} \| \frac{\partial \chi}{\partial J_j} \|_{D_2} + \| \frac{\partial (G + R)}{\partial J_j} \|_{D_2} \| \frac{\partial \chi}{\partial \phi_j} \|_{D_2} \right) \leq 3 \left( \frac{2}{B'} \| G + R \|_{D} \| \frac{\partial \chi}{\partial J} \|_{D - (0, \eta'_1)} + \frac{2}{A' \varepsilon^a} \| G + R \|_{D} \| \frac{\partial \chi}{\partial \phi} \|_{D - (0, \eta'_1)} \right),
\]
(4.63)
where the last inequality was obtained with Cauchy estimates. By (4.30), (4.57) and (4.59), we then have
\[
\| \{G + R, \chi\}\|_{D'} \leq 18C_1 \left( \frac{E}{A} \right) \frac{E \gamma \eta}{A'(B')^p + 4 \varepsilon^2 a}.
\]
(4.64)
We now proceed to the last term of $S$, which is estimated here by brute force. Using (4.13) with $\delta = A \varepsilon^a$, $\xi = \sigma$, we have
\[
\| H' - H - \{H, \chi\}\|_{D'} \leq \frac{1}{2} \| \{H, \chi\}, \chi\|_{D_2}.
\]
(4.65)
Expanding the double Poisson bracket and collecting terms, this is
\[
\leq \frac{9}{2} \left( \| \frac{\partial \chi}{\partial J} \|_{D_2} \| \frac{\partial H}{\partial \phi} \|_{D_2} \| \frac{\partial^2 \chi}{\partial J \partial \phi} \|_{D_2} + \| \frac{\partial \chi}{\partial J} \|_{D_2} \| \frac{\partial^2 \chi}{\partial \phi \partial \phi} \|_{D_2} \| \frac{\partial H}{\partial \phi} \|_{D_2} + \| \frac{\partial \chi}{\partial J} \|_{D_2} \| \frac{\partial^2 H}{\partial J \partial \phi} \|_{D_2} \| \frac{\partial \chi}{\partial \phi} \|_{D_2} + \| \frac{\partial \chi}{\partial J} \|_{D_2} \| \frac{\partial^2 \chi}{\partial J \partial \phi} \|_{D_2} + \| \frac{\partial \chi}{\partial \phi} \|_{D_2} \| \frac{\partial^2 \chi}{\partial J \partial \phi} \|_{D_2} \| \frac{\partial H}{\partial \phi} \|_{D_2} \right).
\]
(4.66)
Most of the work needed to gauge the factors in the preceding expression has already been done. $\| \frac{\partial \chi}{\partial J} \|_{D_2}$ and $\| \frac{\partial \chi}{\partial \phi} \|_{D_2}$ are estimated via (4.57) and (4.59); these inequalities together with Cauchy estimates then give
\[
\| \frac{\partial^2 \chi}{\partial J \partial \phi} \|_{D_2} \leq \frac{2C_1 \eta E}{A A'(B')^p + 4 \varepsilon^2 a},
\]
(4.67)
\[
\| \frac{\partial^2 \chi}{\partial J^2} \|_{D_2} \leq \frac{8C_1 \eta E}{A(A')^2 (B')^p + 3 \varepsilon^3 a},
\]
(4.68)
while
\[
\| \frac{\partial^2 \chi}{\partial \phi^2} \|_{D_2} \leq \left( \frac{2^7 [2(p + 4)]^{p+4}}{\pi^{p+4} \varepsilon^{p+2}} \right) \frac{\eta E}{A(B')^p + 5 \varepsilon^a} \leq C_1 \frac{\eta E}{A(B')^p + 5 \varepsilon^a}.
\]
(4.69)
follows from Proposition 4.7 and (4.31) and (4.55). Finally, the partial derivatives of $H$ are all bounded using Cauchy estimates:

$$\left\| \frac{\partial H}{\partial J} \right\|_{D_2} \leq \frac{2E}{A'e^a}, \quad \left\| \frac{\partial H}{\partial \phi} \right\|_{D_2} \leq \frac{2E}{B'}$$  \hspace{1cm} (4.70)

$$\left\| \frac{\partial^2 H}{\partial J^2} \right\|_{D_2} \leq \frac{8E}{(A')^2 e^{2a}}, \quad \left\| \frac{\partial^2 H}{\partial \phi^2} \right\|_{D_2} \leq \frac{8E}{(B')^2}$$  \hspace{1cm} (4.71)

$$\left\| \frac{\partial^2 H}{\partial J \partial \phi} \right\|_{D_2} \leq \frac{4E}{A'B'e^a}$$  \hspace{1cm} (4.72)

Inserting these expressions into (4.66) and simplifying yields the required estimate of the last term in $S$:

$$\left\| H' - H - \{H, \chi\} \right\|_{D'} \leq 88 \left( \frac{C_1 E}{A} \right)^2 \frac{\eta^2 E}{(A')^2(B')^{2p+8e^4a}}.$$

Now combining (4.64) and (4.73) results in

$$\left\| \{G + R, \chi\} \right\|_{D'} + \left\| H' - H - \{H, \chi\} \right\|_{D'} \leq 2C_1 \left( \frac{E}{A} \right) \left[ 9A'(B')^{p+4}e^{2a} + 44C_1 \left( \frac{E}{A} \right) \frac{\eta}{\gamma} \right] \frac{\gamma \eta E}{(A')^2(B')^{2p+8e^4a}}.$$  \hspace{1cm} (4.74)

Using $A' < A$, $B' < 1$, $\epsilon \leq 1$ and $\eta \leq 2\gamma$, the last expression is

$$\leq 2C_1 \left( \frac{E}{A} \right) \left[ 9A + 88C_1 \left( \frac{E}{A} \right) \right] \frac{\gamma \eta E}{(A')^2(B')^{2p+8e^4a}} = \frac{1}{2} C_2 \frac{\gamma \eta E}{(A')^2(B')^{2p+8e^4a}} = \frac{1}{2} \eta'_1 E.$$  \hspace{1cm} (4.75)

Putting together (4.62) and (4.75) gives the final bound on $S$:

$$\left\| S \right\|_{D'} \leq \left\| R' \right\|_{D'} + \left\| \{G + R, \chi\} \right\|_{D'} + \left\| H' - H - \{H, \chi\} \right\|_{D'} \leq \frac{1}{2} \eta'_1 E + \frac{1}{2} \eta'_2 E = \frac{1}{2} \eta'E.$$  \hspace{1cm} (4.76)

The iterative lemma may now be concluded by identifying the correct expressions for $G'$ and $R'$ and by verifying the final estimates. We take

$$G' = G + \Pi_{\mathcal{M}} S \quad \text{and} \quad R' = (1 - \Pi_{\mathcal{M}}) S,$$  \hspace{1cm} (4.77)
so that (4.37) and (4.38) are clearly satisfied, and
\[ \|G' + R'\|_{D'} = \|G + S\|_{D'} \leq \|G\|_{D'} + \|S\|_{D'} \leq \gamma E + \frac{1}{2} \eta' E = \gamma' E, \]  
(4.78)
which proves (4.40). On the other hand,
\[ \|R'\|_{D'} \leq \|S\|_{D'} + \|\Pi_M S\|_{D'} \leq 2\|S\|_{D'} \leq \eta' E \]  
(4.79)
proves (4.41). Finally,
\[ \|H'\|_{D'} = \|H \circ T'\|_{D'} \leq \|H\|_D \leq E, \]  
(4.80)
since \( T'(D') \subset D \), by (4.34). The iterative lemma is proved. □

4.6 Technical Estimates

This subsection establishes some of the inequalities used in the proof of the iterative lemma. We begin by showing that a real subset of action space which is nonresonant (to a certain order and with respect to a particular submodule) remains nonresonant when complexified.

**Proposition 4.5.** Given a maximal submodule \( \mathcal{M} \) of \( \mathbb{Z}^3 \) and an ultraviolet cutoff \( N = \epsilon^{-\tau} \), suppose \( K \subset \mathbb{R}^3 \) is such that \( I \in K \Rightarrow |k \cdot I| \geq \frac{3}{2} A \epsilon^\alpha |k|^{-p} \) for some \( \alpha, p > 0 \) and for all \( k \notin \mathcal{M} \) with \( |k| \leq N \). If \( a = \alpha + (p + 1)\tau, \sigma \geq 0 \), and \( D = F(K, A \epsilon^a, \sigma) \), then given \( (I, \theta) \in D \), the complex action variable \( I \) satisfies
\[ |k \cdot I| \geq \frac{A}{2} \epsilon^\alpha |k|^{-p} \quad \text{for} \quad k \notin \mathcal{M}, |k| \leq N. \]  
(4.81)

**Proof.** \( |k \cdot I| \geq |k \cdot \text{Re}I| - |k \cdot \text{Im}I| \geq \frac{3}{2} A \epsilon^\alpha |k|^{-p} - |k| A \epsilon^a \geq \frac{3}{2} \epsilon^\alpha |k|^{-p} + A \epsilon^{a+\tau} - A \epsilon^{a-\tau} \geq \frac{A}{2} \epsilon^\alpha |k|^{-p}. \) □

The following all-purpose proposition is adapted from [34], and estimates the tail of a Fourier series whose coefficients satisfy a generalized exponential decay law.

**Proposition 4.6.** Given numbers \( N \geq 0, p \geq 0, \) and \( 0 < \beta < \sigma \leq 1 \), let \( f(I, \theta) = \sum_k f_k(I) e^{2\pi i k \cdot \theta} \) be analytic on the domain \( D = F(K, \rho, \sigma) \) defined in (3.1), and suppose that \( \|f_k\|_D \leq C |k|^p e^{-2\pi \sigma |k|} \). If the \( N \)-tail of \( f \) is given by \( f^\geq(I, \theta) = \sum_{|k| \geq N} f_k(I) e^{2\pi i k \cdot \theta} \), then
\[ \|f^\geq\|_{D-(0,\theta)} \leq \frac{16C (p + 2)^p + 2 e^{2\pi \beta - p - 2}}{\pi \beta} e^{-\pi \beta N}. \]  
(4.82a)
In particular, for $N = 0$ we recover $f^\geq = f$, and
\[
\|f\|_{D-(0,\beta)} \leq \frac{16C(p+2)^{p+2}}{(\pi \beta)^{p+3}} e^{\pi \beta - p - 2}.
\] (4.82b)

**Proof.** We have
\[
\|f^\geq\|_{D-(0,\beta)} \leq \sum_{|k| \geq N} \|f_k\|_{D-(0,\beta)} \|e^{2\pi ik \cdot \theta}\|_{D-(0,\beta)}.
\] (4.83)

Now since $D-(0,\beta) \subset D$ and since $(I, \theta) \in D-(0,\beta) \Rightarrow \|\text{Im}\theta\|_{\infty} \leq \sigma - \beta$, the expression above is less than or equal to
\[
\sum_{|k| \geq N} C|k|^p e^{-2\pi \sigma |k|} e^{2\pi (\sigma - \beta) |k|} = C \sum_{|k| \geq N} |k|^p e^{-2\pi \beta |k|}.
\] (4.84)

Using an elementary counting procedure, it is easy to see that the number of elements $k$ in $\mathbb{Z}^3$ with index norm $|k| = j$ is precisely $4j^2 + 2$ for $j \geq 1$. For any nonnegative integer $j$, we may thus overestimate the number of elements in $\mathbb{Z}^3$ with norm $j$ by $(j + 1)^2$. The desired sum is therefore bounded by
\[
C \sum_{j = N}^{\infty} 4(j + 1)^2 j^p e^{-2\pi \beta j} \leq 4C \sum_{j = N}^{\infty} (j + 1)^{2+p} e^{-2\pi \beta j}.
\] (4.85)

Using $xe \leq e^x$ to write $[\pi \beta (j+1)/p+2]e \leq e^{\pi \beta (j+1)/p+2} \Rightarrow (j+1)^{p+2} \leq \left(\frac{p+2}{\pi \beta e}\right)^{p+2} e^{\pi \beta (j+1)}$, we see that the last sum is less than or equal to
\[
4C \left(\frac{p+2}{\pi \beta e}\right)^{p+2} e^{\pi \beta} \sum_{j = N}^{\infty} e^{-\pi \beta j} = 4C \left(\frac{p+2}{\pi \beta e}\right)^{p+2} e^{\pi \beta} \left(\frac{e^{-\pi \beta N}}{1 - e^{-\pi \beta}}\right).
\] (4.86)

Finally, using $\pi \beta /4 \leq 1 - e^{-\pi \beta}$ for $0 \leq \beta < 1$, we arrive to the desired estimate (4.82a), from which (4.82b) follows in turn. \qed

The preceding proposition is now used to prove

**Proposition 4.7.** Given $0 < \beta < \sigma \leq 1$, let $f(I, \theta) = \sum_k f_k(I) e^{2\pi ik \cdot \theta}$ be analytic on the domain $D = F(K, \rho, \sigma)$ defined in (3.1), and suppose that $\|f_k\|_D \leq C|k|^p e^{-2\pi \sigma |k|}$. Then for any positive integer $s$,
\[
\|\partial_s^s f\|_{D-(0,\beta)} \leq 16(2\pi)^s C(s + p + 2)^{(s+p+2)}(\pi \beta)^{-s-p-3} e^{\pi \beta - s - p - 2}.
\] (4.87)
Proof. Given any multiindex \( r \), the Fourier coefficient of the partial derivative 
\[
\left( \frac{\partial^{\vert r \vert} f}{\partial \theta^r} \right)_{k(I)} = (2\pi i)^{\vert r \vert} k^r f_k(I),
\]  
(4.88)

where \( k^r = k_1^{r_1} k_2^{r_2} k_3^{r_3} \).

Thus \( \| \left( \frac{\partial^{\vert r \vert} f}{\partial \theta^r} \right) \|_D \leq (2\pi)^{\vert r \vert} \| k^r \|_D \| f_k \|_D \leq (2\pi)^{\vert r \vert} C \| k \|^{\vert r \vert + p} e^{-2\pi \sigma \| k \|} \). Now by Proposition 4.6,

\[
\| \frac{\partial^{\vert r \vert} f}{\partial \theta^r} \|_{D_{-0,\beta}} \leq 16(2\pi)^{\vert r \vert} C (\| r \| + p + 2)^{\| r \| + p + 2} (\pi \beta)^{-\| r \| - 3} e^{\pi \beta - \| r \| - p - 2},
\]  
(4.89)

and the result follows. \( \square \)

4.7. Proof of the Analytic Lemma

Proof of the analytic lemma is of course accomplished through repeated application of the iterative lemma. We will show under the present hypotheses that the iterative lemma may be applied \( L = [\epsilon^{-\nu/4}] \) times, where \([\epsilon^{-\nu/4}]\) denotes the greatest integer in \( \epsilon^{-\nu/4} \).

We first show that the iterative lemma may be applied once. In accordance with (4.29), we prepare the original Hamiltonian \( H = h + f \) for the first application of the iterative lemma by writing \( H = H^{(0)} = h + G^{(0)} + R^{(0)} \), where \( G^{(0)} = \Pi_{\mathcal{M}} f \) and \( R^{(0)} = (1 - \Pi_{\mathcal{M}}) f \). We then have \( \| G^{(0)} + R^{(0)} \|_D = \| f \|_D \leq \epsilon E \) and \( \| R^{(0)} \|_D \leq \| f \|_D + \| \Pi_{\mathcal{M}} f \|_D \leq 2 \| f \|_D \leq 2\epsilon E \), so that (4.30), (4.31) and (4.32) are satisfied with \( \gamma_0 = \epsilon \) and \( \eta_0 = 2\epsilon \). We then take \( A' = A = A/4 \) and \( B' = B = B/4 \). Under the additional hypotheses of the analytic lemma, the iterative lemma may now be applied to give a canonical transformation \( T^{(1)} : D^{(1)} \to T^{(1)}(D^{(1)}) \) and a corresponding transformed Hamiltonian \( H^{(1)} = h + G^{(1)} + R^{(1)} \) satisfying (4.33) through (4.34), where each of the primed parameters now carries a subscript of 1.

Now define \( A_j = A/(4j^2) \) and \( B_j = \sigma/(4j^2) \) (note that \( \frac{1}{4} A \epsilon^{\nu/2} \leq A_j < A \) and \( \frac{1}{4} \sigma \epsilon^{\nu/2} \leq B_j < \sigma \) for \( j \leq L \)). Assume that the iterative lemma has been successfully applied \( l - 1 \) times \( (l \leq L) \), where at the \( j^{th} \) application we have used \( A' = A_j \) and \( B' = B_j \). We will show that it can be applied again. After the \( l - 1^{st} \) transformation, the Hamiltonian \( H^{(l-1)} = H^{(0)} \circ T^{(1)} \circ \cdots \circ T^{(l-1)} \) has the form \( H^{(l-1)} = h + G^{(l-1)} + R^{(l-1)} \) and is analytic on the domain \( D^{(l-1)} \) satisfying

\[
D - \left( \sum_{j=1}^{l-1} A_j \epsilon^a, \sum_{j=1}^{l-1} B_j \right) \subset D^{(l-1)} \subset D - \left( \frac{1}{2} \sum_{j=1}^{l-1} A_j \epsilon^a, \frac{1}{2} \sum_{j=1}^{l-1} B_j \right).
\]  
(4.90)
From (4.42) we also have

\[ \gamma^{(l-1)} = \gamma^{(0)} + \frac{1}{2} \sum_{j=1}^{l-1} \eta^{(j)}, \]  

(4.91)

and the remaining conclusions of the iterative lemma hold, where the primed parameters now carry the subscript \((l - 1)\). To apply the iterative lemma once more, the only hypothesis which is not immediately verified is (4.32). We therefore assume provisionally that

\[ \eta^{(j)} \leq 2\epsilon^{1+\tau} e^{-j} \quad \text{for } 1 \leq j \leq l - 1; \]  

(4.92)

this assumption will also be justified as part of the finite induction. With this assumption, we find that \(\eta^{(l-1)} \leq 2\epsilon^{1+\tau} e^{l-1} < 2\epsilon = 2\gamma^{(0)} < 2\gamma^{(l-1)}\), while (4.91) with \(\gamma^{(0)} = \epsilon\) ensures that

\[ \gamma^{(l-1)} \leq \gamma^{(0)} + \epsilon^{1+\tau} \sum_{j=1}^{l-1} e^{-j} < 2\epsilon. \]  

(4.93)

The remaining hypothesis (4.32) is thus verified, and the iterative lemma may be applied again.

Let us now prove (4.92) for \(j = l\) (we skip the proof for the starting value \(j = 1\), since it is nearly identical, assuming \(\gamma^{(0)} = \epsilon\) and \(\eta^{(0)} = 2\epsilon\)). After the \(l^{th}\) application of the iterative lemma, we have \(\eta^{(l)} = \eta_{1}^{(l)} + \eta_{2}^{(l)}\), where, by (4.43),

\[ \eta_{1}^{(l)} \leq \frac{2^{6r}[2(p + 2)]^{p+2} \eta^{(l-1)}}{(\pi \sigma)^{3}} e^{-N \pi B_{l}/2}. \]  

(4.94)

Using \(N = \epsilon^{-\tau}, B_{l} \geq \frac{1}{4} \sigma \epsilon^{\tau/2}\), and \(\eta^{(l-1)} \leq 2\epsilon \epsilon^{l-1}\), this expression is bounded by

\[ \left( \frac{2^{13}[2(p + 2)]^{p+2}}{(\pi \sigma)^{3}} e^{-5 \tau/2 \epsilon_{1-\frac{1}{2} \kappa \sigma \epsilon^{-\tau/2}}} \right) \epsilon^{1+\tau} \epsilon^{-l} \leq \epsilon^{1+\tau} \epsilon^{-l}, \]  

(4.95)

the last inequality being a consequence of restriction (4.19e). As for \(\eta_{2}^{(l)}\), by (4.44)

\[ \eta_{2}^{(l)} = C_{2} \frac{\gamma^{(l-1)} \eta^{(l-1)}}{(A_{1})^{2}(B_{l})^{2p+8} \epsilon^{4a}}. \]  

(4.96)

Now \(N = \epsilon^{-\tau}, A_{l} \geq \frac{1}{4} A_{1} \epsilon^{\tau/2}, B_{l} \geq \frac{1}{4} \sigma \epsilon^{\tau/2}, \gamma^{(l-1)} \leq 2\epsilon\), and \(\eta^{(l-1)} \leq 2\epsilon \epsilon^{l-1}\) imply that this is less than or equal to

\[ \frac{4 C_{2} \epsilon^{1-\tau} \epsilon^{2a-4a-(p+5)\tau}}{(A/4)^{2}(\sigma/4)^{2p+8}} = \left( \frac{C_{2} \epsilon^{24p+22 \epsilon_{1-4a-(p+6)\tau}}}{A^{2} \sigma^{2p+8}} \right) \epsilon^{1+\tau} \epsilon^{-l} \leq \epsilon^{1+\tau} \epsilon^{-l}, \]  

(4.97)
where the final inequality follows from (4.19d).

We have succeeded in showing that the iterative lemma may be applied \( l \) times in succession, for any \( l \leq L \), and for such \( l \) we have \( \eta^{(l)} = \eta_1^{(l)} + \eta_2^{(l)} \leq 2\epsilon^{1+\epsilon} \epsilon^{-l} \), by (4.95) and (4.97). We construct the transformation \( T: D_\infty \to T(D_\infty) \) by setting \( T = T^{(1)} \circ T^{(2)} \circ \cdots \circ T^{(L)} \) and \( D_\infty = D^{(L)} \). Since \( \sum_{j=1}^L A_j = (A/4) \sum_{j=1}^L j^{-2} < (A/4)(\pi^2/6) < A/2 \) and \( \sum_{j=1}^L A_j > A/4 \) (Similarly \( \sigma/4 < \sum_{j=1}^L B_j < \sigma/2 \)), we see that (4.23a) and (4.23b) follow from (4.90). On the other hand, setting \( T(J, \phi) = (I, \theta) \) for \( (J, \phi) \in D_\infty \) and applying (4.35) and (4.36) recursively to the definition of \( T \) gives \( ||I - J||_{\infty} \leq \frac{\epsilon^d}{2} \sum_{j=1}^L A_j < \frac{A}{4} \epsilon^d \) and \( ||\theta - \phi||_{\infty} \leq \frac{\epsilon^d}{2} \sum_{j=1}^L B_j < \frac{\sigma}{4} \epsilon^d \), which establishes (4.28a) and (4.28b).

Next, we take \( H' = H \circ T \) and \( G = G^{(L)} \), \( R = R^{(L)} \), so the transformed Hamiltonian is properly separated into its resonant and nonresonant parts as prescribed by (4.24), (4.25a) and (4.25b). The norm of \( R \) is exponentially small as required, since

\[
\|R\|_{D_{\infty}} = \|R^{(L)}\|_{D^{(L)}} \leq E\eta^{(L)} \leq 2E\epsilon^{1+\epsilon} \epsilon^{-L} = 2E\epsilon^{1+\epsilon} \epsilon^{-L} < 2E\epsilon e^{-\epsilon^d}, \quad (4.98)
\]

the last inequality following from (4.19a). This proves (4.26).

For the remaining inequality (4.27), note that by construction,

\[
G - \Pi_{\mathcal{M}} f = \sum_{j=1}^L \Pi_{\mathcal{M}} S_j, \quad (4.99)
\]

where \( S_j \) is given by (4.50) for the \( j \)th application of the iterative lemma. Thus

\[
\|G - \Pi_{\mathcal{M}} f\|_{D_{\infty}} = \| \sum_{j=1}^L \Pi_{\mathcal{M}} S_j \|_{D_{\infty}} \leq \sum_{j=1}^L \| \Pi_{\mathcal{M}} S_j \|_{D_{\infty}} \leq \sum_{j=1}^L \|S_j\|_{D_{\infty}} \leq E \sum_{j=1}^L \eta^{(j)} < E\epsilon^{1+\epsilon} \sum_{j=1}^L \epsilon^{-j} < E\epsilon^{1+\epsilon}, \quad (4.100)
\]

where the third to last inequality follows from (4.76). This establishes (4.27), and the analytic lemma is proved. \square
5. THE GENERALIZED CONTINUUM MODELS

The article culminates in this section with the description of channeling (and certain nonchanneling) motions of particles in classical perfect crystals. This is accomplished by dividing phase space into three kinds of nonoverlapping regions, or "blocks," where trajectories are governed by the normal forms constructed in Section 4. For initial conditions in the degenerate (or nonresonant) block, an auxiliary result in geometric number theory shows that trajectories are initially nearly rectilinear but quickly fail the channeling criterion (this is the "spatial continuum model"). For initial conditions in the two other kinds of resonant blocks, energy conservation and convexity of the unperturbed Hamiltonian are combined with analytic results to show that trajectories remain trapped at resonance for exponentially long times, during which they execute channeling motions as conceived in Lindhard’s continuum models.

We begin with terminology concerning the geometry of the resonances.

5.1 Resonant Zones and Resonant Blocks

To each maximal submodule \( \mathcal{M} \) of \( \mathbb{Z}^3 \) corresponds a resonance in action space, namely the subspace of all \( J \) orthogonal to each \( k \in \mathcal{M} \). Actions in the neighborhood of a low-order resonance (and sufficiently far from other low-order resonances) are channeling directions; this is formalized by first defining the resonant zone corresponding to \( \mathcal{M} \) by

\[
Z_{\mathcal{M}}(C, \alpha) = \{ J \in \mathbb{R}^3 \mid |k \cdot J| \leq C \epsilon^\alpha |k|^{-p} \quad \forall k \in \mathcal{M}, 0 < |k| \leq |\mathcal{M}| \} \quad (5.1)
\]

for appropriate positive values of \( C, \alpha \) and \( p \). In the case where \( \mathcal{M} \) is one-dimensional, the zone \( Z_{\mathcal{M}}(C, \alpha) \) will also be denoted \( Z_k(C, \alpha) \), where \( k \) is the unique generator (up to inversion) of \( \mathcal{M} \). The case \( \mathcal{M} = \{0\} \) also deserves comment; since \( |\{0\}| = 0 \), no \( k \in \mathbb{Z}^3 \) satisfy \( 0 < |k| \leq |\{0\}| \), and the "zone" \( Z_{\{0\}}(C, \alpha) \) degenerates to \( \mathbb{R}^3 \) for any \( C, \alpha \).

Next, given \( \mathcal{M} \), its associated resonant block of order \( N \geq |\mathcal{M}| \) is defined as the subset of action space

\[
\hat{Z}_{\mathcal{M}, \alpha}^N(N, C_1, \alpha_1) = Z_{\mathcal{M}}(C_1, \alpha_1) \setminus \bigcup_{\substack{k \in \mathcal{M} \\ |k| \leq N}} \text{Int} \ Z_k(C, \alpha), \quad (5.2)
\]

where \( \text{Int} \) denotes interior. These actions correspond to channeling directions; physically, it is interesting to note that when \( \dim \mathcal{M} = 1 \) (corresponding to a planar direction), low-order "axial" zones are removed, as they should be. In the degenerate case \( \mathcal{M} = \{0\} \), the corresponding block is called a "nonresonant block," and since the underlying zone is all of \( \mathbb{R}^3 \), it is denoted \( \hat{Z}_0^{C, \alpha}(N) \) without
reference to $C_1$ or $\alpha_1$. Explicitly,

$$\hat{Z}^{C,\alpha}_0(N) = \mathbb{R}^3 \setminus \bigcup_{0<|k| \leq N} \text{Int} \ Z_k(C,\alpha). \quad (5.3)$$

Of course, the resonant blocks may be empty in certain parameter regimes; however, because they are made up of intersections of elementary geometric objects (planar slabs) in $\mathbb{R}^3$, it is not difficult to show that the blocks are nonempty as $\epsilon \to 0^+$. In fact, we may estimate the volume occupied by the resonant blocks inside the spherical shell $S_{/2}^\epsilon = \{ I \in \mathbb{R}^3 \mid 2 - 2\epsilon E_0 \leq \|I(t)\|^2 \leq 2 \}$ of admissible actions for the Hamiltonian (3.5); for $p > 4$ and for positive $C$, $\alpha$, $C_1$, $\alpha_1$, it is easy to show that

$$\text{Vol}(\hat{Z}^{C,\alpha}_M(N,C_1,\alpha_1) \cap S_{/2}^\epsilon) \to \text{Vol}(Z_M(C_1,\alpha_1) \cap S_{/2}^\epsilon) \quad (5.4)$$

as $\epsilon \to 0^+$, independently of $N$. In other words, for the parameter values of interest, as $\epsilon \to 0^+$, the relative volume of a block approaches the relative volume of the zone on which it is based.

In Section A1.5 of Appendix 1 of [17], explicit (but crude) estimates are given for the threshold $\epsilon^* > 0$ such that $0 < \epsilon \leq \epsilon^* \Rightarrow \text{Vol}(\hat{Z}^{C,\alpha}_M(N,C_1,\alpha_1) \cap S_{/2}^\epsilon) > 0$. Separate thresholds are given for each of the cases $\text{dim} M = 0$, 1, or 2, and for $\text{dim} M = 2$, it is shown that, not only does (5.4) hold in the small $\epsilon$ limit, but there is in fact an $\epsilon$-threshold below which $\hat{Z}^{C,\alpha}_M(N,C_1,\alpha_1) \cap S_{/2}^\epsilon = Z_M(C_1,\alpha_1) \cap S_{/2}^\epsilon$. In other words, an axial resonant block coincides with the zone on which it is based for small enough $\epsilon$.

The explicit thresholds $\epsilon^*$ are not repeated here since they constitute much weaker restrictions on $\epsilon$ than the restrictions (4.19).

### 5.2 Geometric Considerations

Following [8], we now prove a result analogous to Nekhoroshev’s “geometric lemma,” which says that as long as trajectories remain inside the domain of a particular normal form, they very nearly move in an affine subspace in action space which is parallel to the submodule $\mathcal{M}$ defining the normal form.

First, some terminology concerning the transformed Hamiltonian (4.24). By removing the remainder $R$ we obtain the so-called effective Hamiltonian

$$\tilde{H}(J,\phi) = h(J) + G(J,\phi) = \frac{1}{2} J^2 + \sum_{k \in \mathcal{M}} G_k(J) e^{2\pi i k \cdot \phi}. \quad (5.5)$$

which is cyclic in certain linear combinations of $\phi$. In other words, $\tilde{H}$ is independent of any $\phi$ in the orthogonal complement of $\mathcal{M}$. In particular, if $\mathcal{M} = \{0\}$ is the trivial submodule, its orthogonal complement is all of $\mathbb{R}^3$ and
(5.5) is completely cyclic, or independent of $\phi$. In general, to emphasize the
dependence of $G$ on only those angle variables lying in span$\mathcal{M}$, we will often
write $G(J, \phi) = G(J, \phi^*)$, where $\phi^*$ denotes the projection of $\phi$ onto span$\mathcal{M}$
(these “resonant coordinates” are used in the explicit proof that $\Pi_{\mathcal{M}} W$ is the
continuum potential $\overline{W}$ corresponding to $\mathcal{M}$; cf Appendix 2 of [17]). This cyclic
or partly cyclic property is nearly true of (4.24), and we may prove the following

**Proposition 5.1.** (geometric proposition) Assume the hypotheses of the
analytic lemma (Lemma 4.2). Let $(J_0, \phi_0) \in K_\infty \times T^3$ be any initial condition
in the real part $K_\infty \times T^3$ of the domain $D_\infty$ of the transformed Hamiltonian
(4.24) and let $(J, \phi)$ be the solution of Hamilton’s equations corresponding to
(4.24). Denote by $T_0$ the solution’s (possibly infinite) first time of escape from
$K_\infty \times T^3$. If $\mathcal{M}$ is the maximal submodule to which (4.24) is adapted, denote
by $J^*$ the projection of $J$ onto span$\mathcal{M}$ and set $\tilde{J} = J - J^*$ so that we have the
orthogonal splitting $J = J^* + \tilde{J}$. Then for $0 \leq t \leq T_0$,

$$
\|\tilde{J}(t) - \tilde{J}_0\|_\infty \leq \frac{4E}{\sigma} t e^{-\epsilon/4}.
$$

(5.6)

**Proof.** By the analytic lemma, for $0 \leq t \leq T_0$ we have

$$
J(t) = \tilde{J}(t) - \int_0^t \frac{\partial R}{\partial \phi}(J(t'), \phi(t')) dt',
$$

(5.7)

where

$$
\tilde{J}(t) \equiv J_0 - \int_0^t \frac{\partial G}{\partial \phi}(J(t'), \phi(t')) dt' =
$$

$$
J_0 - 2\pi i \int_0^t \sum_{k \in \mathcal{M}} k G_k(J(t')) e^{2\pi i k \cdot \phi(t')} dt' =
$$

$$
\tilde{J}_0 + J^*_0 - 2\pi i \sum_{k \in \mathcal{M}} k \int_0^t G_k(J(t')) e^{2\pi i k \cdot \phi(t')} dt'.
$$

(5.8)

The last term belongs to span$\mathcal{M}$, and we see that the motion generated by
(5.8) takes place entirely in the affine subspace parallel to $\mathcal{M}$ passing through
$J_0$:

$$
\tilde{J}(t) = \tilde{J}_0 + (\tilde{J}(t))^*.
$$

(5.9)

From the equation

$$
J(t) = \tilde{J}(t) + J^*(t) = \tilde{J}_0 + (\tilde{J}(t))^* - \left(\int_0^t \frac{\partial R}{\partial \phi}(t') dt'\right)^* - \left(\int_0^t \frac{\partial R}{\partial \phi}(t') dt'\right)^*,
$$

(5.10)
we identify

$$\tilde{J}(t) = \tilde{J}_0 - \left( \int_0^t \frac{\partial R}{\partial \phi}(t')dt' \right)$$

and

$$J^*(t) = (\tilde{J}(t))^* - \left( \int_0^t \frac{\partial R}{\partial \phi}(t')dt' \right)^*.$$  \hfill (5.11)

Now because the initial condition \((J_0, \phi_0)\) is real, the solution \((J, \phi)\) is real for all \(t\). Thus for \(0 \leq t \leq T_0\),

$$\|\tilde{J}(t) - \tilde{J}_0\|_\infty \leq \left\| \left( \int_0^t \frac{\partial R}{\partial \phi}(t')dt' \right) \right\|_{K_\infty \times T^3} \leq \left\| \int_0^t \frac{\partial R}{\partial \phi}(t')dt' \right\|_{K_\infty \times T^3} \leq \left( \int_0^\infty \|\frac{\partial R}{\partial \phi}\|_{K_\infty \times T^3} \right) \leq \frac{2t}{\sigma} \|R\|_{D_\infty} \leq \frac{4tE}{\sigma} e^{-\epsilon^{-r/4}}, \hfill (5.13)$$

where the second to last inequality follows from a Cauchy's estimate and (4.23a), and the last inequality uses (4.26). \(\Box\)

Using this or a similar proposition, it is possible to prove simultaneously the confinement of motions of (4.24) within blocks corresponding to any submodule \(M\) (cf. [8]). Our purpose here however is to examine these motions separately, and for the sake of the channeling problem, the nonresonant case \(M = \{0\}\) is treated separately in the next subsection.

5.3 The Spatial Continuum Model

In the case \(M = \{0\}\), every vector \(J \in \mathbb{R}^3\) is orthogonal to \(M\) and the splitting \(J = J^* + \tilde{J}\) described in Proposition 5.1 degenerates to \(J = \tilde{J}\). Therefore for any initial condition \((J_0, \phi_0)\) lying in the real part \(K_\infty \times T^3\) of the domain \(D_\infty\) of (4.24), by Proposition 5.1 the real solution \((J, \phi)\) of (4.24) satisfies

$$\|J(t) - J_0\|_\infty \leq \frac{4tE}{\sigma} e^{-\epsilon^{-r/4}} \hfill (5.14)$$

for as long as it remains in \(K_\infty \times T^3\).

This near constancy of the transformed actions will be exploited by way of a "bootstrap argument" to show that solutions of (4.24) beginning with actions well inside a nonresonant block \(\hat{K}_0\) remain inside the block for exponentially long times. It is first necessary to show that points beginning inside a block remain nonresonant when allowed to wander slightly outside the block in \(\mathbb{R}^3\). We use the abbreviated neighborhood notation introduced in §4.2, along with the manipulations listed in (4.4).
Proposition 5.2a. Let \( \mathcal{M} \neq \mathbb{Z}^3 \) be a maximal submodule of \( \mathbb{Z}^3 \). Suppose \( N = \epsilon^{-r} \) and \( A, \alpha, c, a \), and \( r \) satisfy the hypotheses of the analytic lemma, and \( C_1, \alpha_1 \) are any positive numbers (or are absent, if \( \mathcal{M} = \{0\} \)). Then

\[
\hat{\mathcal{Z}}^\frac{5}{2}_\mathcal{M}^A,^\alpha(N, C_1, \alpha_1) + Ae^a \subset \mathbb{R}^3 \setminus \bigcup_{k \in \mathcal{M}, |k| \leq N} \text{Int} \mathcal{Z}_k(\frac{3}{2}A, \alpha).
\]  

(5.15)

In other words, if \( I_0 \in \hat{\mathcal{Z}}^\frac{5}{2}_0^A,^\alpha(N) \) and if \( \|I_0 - J_0\|_\infty \leq Ae^a \), then for every \( k \notin \mathcal{M} \) with \( |k| \leq N \), \( |k \cdot J_0| \geq \frac{3}{2}Ae^a \).

Proof. Let \( I_0 \in \hat{\mathcal{Z}}^\frac{5}{2}_0^A,^\alpha(N) \) and suppose \( \|I_0 - J_0\|_\infty \leq Ae^a \). For any \( k \notin \mathcal{M} \) with \( |k| \leq N \), we have

\[
|k \cdot J_0| \geq |k \cdot I_0| - |k \cdot (I_0 - J_0)| \geq \frac{5}{2}Ae^a |k|^{-p} - |k| \|I_0 - J_0\|_\infty \geq \frac{5}{2}Ae^a |k|^{-p} - |k|^{p+1} |k|^{-p} Ae^a \geq \frac{5}{2}Ae^a |k|^{-p} - Ae^a^{-(p+1)} |k|^{-p} = \frac{3}{2}Ae^a |k|^{-p}. \square
\]

The following corollary simply restates Proposition 5.2a as it is used in the upcoming proofs; it follows from properties (4.4).

Corollary 5.2b. Under the same hypotheses,

\[
\hat{\mathcal{Z}}^\frac{5}{2}_0^A,^\alpha(N) \subset \hat{\mathcal{Z}}^\frac{5}{2}_0^A,^\alpha(N) - Ae^a = \left( \mathbb{R}^3 \setminus \bigcup_{k \notin 0, |k| \leq N} \text{Int} \mathcal{Z}_k(\frac{3}{2}A, \alpha) \right) - Ae^a \text{ and }
\]

(5.16)

\[
\hat{\mathcal{Z}}^\frac{5}{2}_\mathcal{M}^A,^\alpha(N, C_1, \alpha_1) \subset \left( \mathbb{R}^3 \setminus \bigcup_{k \in \mathcal{M}, |k| \leq N} \text{Int} \mathcal{Z}_k(\frac{3}{2}A, \alpha) \right) - Ae^a.
\]

(5.17)

We may now prove
Theorem 5.3. (rectilinear trajectories) Assume the hypotheses of the analytic lemma with $H$ given by (3.5), $\mathcal{M} = \{0\}$, and with

$$K = \hat{\mathcal{Z}_0}^{3A,\alpha}(N) \cap (S_{\sqrt{3}} + Ae^a),$$

which is clearly compact and satisfies (4.21). Let

$$(I_0, \theta_0) \in \left(\hat{\mathcal{Z}_0}^{3A,\alpha}(N) \cap S_{\sqrt{3}}^\epsilon\right) \times T^3$$

be an initial condition for (3.5) which is nonresonant to order $N = \epsilon^{-r}$. Then for $0 \leq t \leq (A\sigma/16E)\epsilon^{c-1}e^{-r/4}$, the solution $(I, \theta)$ of (3.5) with initial condition $(I_0, \theta_0)$ satisfies

$$\|I(t) - I_0\|_\infty \leq \frac{3}{4}A\epsilon^c$$

and

$$\|\theta(t) - \theta_0 - tI_0\|_\infty \leq \frac{3}{4}At\epsilon^c.$$  

Proof. Let $(J_0, \phi_0) = T^{-1}(I_0, \theta_0)$. To see that $(J_0, \phi_0)$ belongs to the real part $K_\infty \times T^3$ of $D_\infty$ (and so is an admissible initial condition for (4.24)), first note that by (5.16a) and (5.19), we have

$$I_0 \in \left(\hat{\mathcal{Z}_0}^{3A,\alpha}(N) - Ae^a\right) \cap S_{\sqrt{3}}^\epsilon \subset \left(\hat{\mathcal{Z}_0}^{3A,\alpha}(N) \cap (S_{\sqrt{3}}^\epsilon + Ae^a)\right) - Ae^a$$

$$= K - Ae^a.$$  

Since $\theta_0 \in T^3$, it follows from (4.23b) that $(I_0, \theta_0) \in T(D_\infty)$ (the domain of $T^{-1}$), and so $(J_0, \phi_0)$ is well-defined and real, since $(I_0, \theta_0)$ is real. Furthermore, by (4.28a), $\|I_0 - J_0\|_\infty \leq \frac{A}{4}\epsilon^c \leq \frac{A}{4}\epsilon^a$ (c > a by (4.18b) and c > 1/2), and so $I_0 \in K - Ae^a \Rightarrow J_0 \in K - \frac{3}{4}Ae^a \subset K_\infty - \frac{A}{4}\epsilon^a$ (by (4.23a)). Therefore $(J_0, \phi_0) \in D_\infty$ is in the domain of the transformed Hamiltonian $H'$ given by (4.24).

Now suppose the solution $(J, \phi)$ of (4.24) with initial condition $(J_0, \phi_0)$ first leaves $D_\infty$ at $T_0 < (A\sigma/16E)\epsilon^{c-1}e^{-r/4}$. Since $(J, \phi)$ is real for all $t$, $\phi$ cannot exit $T^3$ and we must have $J(T_0) \in \partial K_\infty$. But by (5.14), $\|J(T_0) - J_0\|_\infty \leq (4E/\sigma)T_0e^{-r/4} < \frac{A}{4}\epsilon^c < \frac{A}{4}\epsilon^a$, which contradicts $J(T_0) \in \partial K_\infty$, since then $J_0 \in K_\infty - \frac{A}{4}\epsilon^a$ would imply $\|J(T_0) - J_0\|_\infty \geq \frac{A}{4}\epsilon^a$. Thus $T_0 \geq (A\sigma/16E)\epsilon^{c-1}e^{-r/4}$, and so $(J(t), \phi(t)) \in K_\infty \times T^3$ for $t \in [0, (A\sigma/16E)\epsilon^{c-1}e^{-r/4}]$; and the inequality (5.14) also holds on this time interval.

On this same time interval, we have the solution $(I(t), \theta(t)) = T(J(t), \phi(t))$ of (3.5) satisfying $(I(0), \theta(0)) = T(J_0, \phi_0) = TT^{-1}(I_0, \theta_0) = (I_0, \theta_0)$. Using
(4.28a), (5.14) and the identity \( I - I_0 = I - I_0 + J - J_0 - J_0 \) we have, for any \( t \in [0, (A\sigma/16E)e^{c-1}e^{-r'/4}] \),

\[
\|I(t) - I_0\|_\infty \leq \|J_0 - I_0\|_\infty + \|J(t) - J_0\|_\infty + \|I(t) - J(t)\|_\infty \leq \frac{A}{4}e^c + \frac{4E}{\sigma}te^{c-e^{-r'/4}} + \frac{A}{4}e^c \leq \frac{3A}{4}e^c,
\]

which proves (5.20).

On the other hand, from (3.5)

\[
\theta(t) = \theta_0 + \int_0^t I(t')dt' = \theta_0 + \int_0^t I_0 dt' + \int_0^t (I(t') - I_0)dt' = \theta_0 + tI_0 + \int_0^t (I(t') - I_0)dt' \implies \\
\|\theta(t) - \theta_0 - tI_0\|_\infty \leq \int_0^t \|I(t') - I_0\|_\infty dt' \leq \frac{3Ate^c}{4}
\]

for \( t \in [0, (A\sigma/16E)e^{c-1}e^{-r'/4}] \), as claimed in (5.21). \( \square \)

Theorem 5.3 says that for initial conditions which are nonresonant to a certain order \( N \), motions of (3.5) have nearly constant action \( I \simeq I_0 \) on an exponentially long time interval, while the configuration or angle variable \( \theta \) is approximated by the rectilinear motion

\[
\dot{\theta}(t) = \theta_0 + tI_0
\]

on any time interval \([0, Ce^{-b}]\) with \( C > 0, 0 < b < c \). This Nekhoroshev-like result is close to the KAM theorem, and complements it by showing that trajectories beginning near invariant tori remain near them and exhibit integrable-like behavior for very long times.

Mathematically, Theorem 5.3 characterizes the behavior of trajectories of (3.5) for small \( \epsilon \) and for initial conditions in the nonresonant block. However, in terms of the channeling problem, more can be shown by recalling the postulate that solutions of (3.5) model particle motions in crystals only in so far as they satisfy the channeling criterion (3.8).

Now it is well known that linear flow \( \dot{\theta}(t) = \theta_0 + tI_0 \) with nonresonant direction vector \( I_0 \) densely fills \( T^3 \), and it is reasonable to suspect that linear flow with “nearly” highly nonresonant direction vector \( I_0 \in \hat{Z}_{0\alpha}^c(N) \) will quickly fill “enough” of \( T^3 \) to fail the channeling criterion. The following two theorems (the proofs of which appear elsewhere) confirm this suspicion.
Theorem 5.4. (Close encounters) Assume the hypotheses of Theorem 5.3; i.e., assume the hypotheses of the analytic lemma with $H$ given by (3.5), $\mathcal{M} = \{0\}$, and with $K = \hat{Z}_0^{\frac{
u}{2}} \cap (S_{\nu} + Ae^a)$. Let $R > 0$ be the radius of the largest ball $B_R$ contained in the region of close encounter $C(1) \subset T^3$ described in (3.7). Then for $\epsilon > 0$ sufficiently small, the initial actions (or initial directions)

$$I_0 \in \hat{Z}_0^{\frac{
u}{2}} \cap S_{\nu}^\epsilon$$

(5.26)

are nonchanneling directions in the sense that given any initial condition

$$(I_0, \theta_0) \in \left(\hat{Z}_0^{\frac{
u}{2}} \cap S_{\nu}^\epsilon\right) \times T^3,$$

(5.27)

the corresponding solution $(I, \theta)$ of (3.5) fails to satisfy the channeling criterion (3.8) at some $t'$ in the (brief) time interval

$$[0, C_3 \epsilon^{-\alpha}], \quad \text{where} \quad C_3 = \frac{2(2\sqrt{3})^p \|V_*\|_\Delta}{AR^{\psi_3/2}},$$

(5.28a, b)

and the constant $\|V_*\|_\Delta$ is described following Theorem 5.5.

The proof of this theorem depends on the rate at which linear flow fills the torus when subject to truncated Diophantine conditions. In order to state a theorem which estimates this rate, we introduce the following terminology.

For vectors $\alpha \in R^3$ with unit length (denoted $\alpha \in S^{n-1}$), we first consider the one-parameter family of translation maps $\alpha_t : T^3 \to T^3$ defined by $\alpha_t(\theta) = \theta + t\alpha$ (the vector arithmetic takes place on the torus, i.e., is first performed in the universal covering space $R^3$, then projected onto $T^3$). A rectilinear orbit of $T^3$ with direction vector $\alpha$ and initial condition $\theta$ is then the image of $\theta$ under the flow defined by $\alpha_t$ over some closed interval in $R$:

$$\bigcup_{a \leq t \leq b} \alpha_t(\theta).$$

(5.29)

Next, given $R > 0$, the direction vector $\alpha \in S^{n-1}$ is said to *ergodize $T^3$ to within $R$ after time $T$* if no point in $T^3$ is more than $R$-distant (Euclidean norm) from any rectilinear orbit $\bigcup_{b \leq t \leq T} \alpha_t(\theta)$.

Finally, for convenience we define the set $\mathcal{D}(p, C, N)$ of direction vectors satisfying a truncated Diophantine condition of order $N$ as

$$\mathcal{D}(p, C, N) = \{\alpha \in S^{n-1} \mid |\alpha \cdot k| > \frac{C}{|k|^p}, \forall k \in \mathbb{Z}^n, 0 < |k| \leq N\}.$$

(5.30)

37
It is intuitively clear that, for fixed $R > 0$, rectilinear orbits arising from resonant direction vectors will ergodize the torus, provided they are resonant at sufficiently high order. Such orbits are indeed degenerate (i.e., fill only lower dimensional tori), but at high enough order, they build a "web" with holes no larger than $R$. In fact, given $R$ and given a Diophantine condition, it is possible to show that for $N > N^*$, where

$$N^* \equiv \kappa(p)R^{-\left(\frac{p+2/2}{p-3/2}\right)} + 3/2,$$  \hspace{1cm} (5.31)

when the Diophantine condition is truncated at order $N$, the ergodization time may be explicitly estimated. (Here $\kappa(p)$ is a constant depending on $p$; it is given explicitly in [18].) This result is stated as

**Theorem 5.5.** Let $0 < R \leq 1$, and choose $N > N^*$. Then given $\alpha \in \mathcal{D}(p,C,N)$, $T^n$ will be ergodized to within $R$ by rectilinear orbits with direction vector $\alpha$ after time

$$T = \frac{2(3p/2)||V_*||_\Delta}{C\pi R^{p+3/2}} \left[ 1 - \left( \frac{N^* - 3/2}{N - 3/2} \right)^{p-\frac{2}{p}} \right]^{-1/2}.$$ \hspace{1cm} (5.32)

Here the constant $||V_*||_\Delta$ is a Sobolev norm of a certain maximally smooth test function $V_*$ on the unit ball in $\mathbb{R}^3$; it depends only on $p$ and is discussed further in [18], where complete proofs of this and related results appear. The important feature of this theorem for our purposes is the appearance of $C$ in the denominator of the estimate of the ergodization time $T$.

It is now straightforward to prove Theorem 5.4 from Theorem 5.5 by first normalizing the initial action vector $I_0$ and the time $t$, taking $C = (5/2\sqrt{2})A\epsilon^\alpha$, $N = \epsilon^{-\tau}$, and by using the rectilinear motion (5.25) to approximate the true motion on the time interval $[0, C_3\epsilon^{-\alpha}]$. Because the ergodization time is less than the length of this interval, it is possible to produce a $t' \in [0, C_3\epsilon^{-\alpha}]$ at which the angle $\theta$ belongs to $\mathcal{B}_R \subset \mathcal{C}(1)$. The solution $(I, \theta)$ therefore fails the channeling criterion at time $t'$. A proof along these lines (but with less wieldy constants) may be found in [19].

Theorems 5.3 and 5.4 taken together provide the picture of the "spatial continuum model" mentioned in the introduction. Namely, sufficiently energetic particles injected into a classical crystal in the nonresonant (or nonchanneling) directions $I_0 \in \hat{\mathcal{S}}_{\alpha}^A(N) \cap S_{1/2}$ will follow roughly straight paths to their first collisions with a region of close encounter $\mathcal{C}(1)$, and these collisions happen quickly, at times no later than $C_3\epsilon^{-\alpha}$ from the time of injection. Moreover as mentioned in §5.1, with increasing particle energy, the volume in action space of the admissible nonresonant actions $\hat{\mathcal{S}}_{\alpha}^A(N) \cap S_{1/2}$ approaches the volume of the set of all admissible actions $S_{1/2}$ for the perfect crystal Hamiltonian.
(3.5). Finally, it should be mentioned that although the emphasis in this article is on positively charged particles, Theorems 5.3 and 5.4 make no use of the nonnegativity of $W$; they hold irrespective of particle charge.

5.3 Channeling

With the addition of a few definitions and a physically motivated assumption about the potential $W$, we will be in position to prove the existence of solutions of the scaled perfect crystal Hamiltonian (3.5) which satisfy the channeling criterion (3.8) on exponentially long time intervals. These solutions have characteristics which physicists recognize as the traits of axial or planar channeling; namely, they have a low probability of encountering nuclei, their "transverse energy" and "longitudinal momenta" are nearly conserved, and they are approximated in some sense by solutions of an appropriate generalized (axial or planar) continuum Hamiltonian, which in turn is the ordinary (axial or planar) continuum Hamiltonian to leading order.

To begin with, we recall that given a 1- or 2-dimensional maximal submodule $\mathcal{M}$ of $\mathbb{Z}^3$ and given $\theta \in T^3$, we write $\theta = \theta^* + \hat{\theta}$, where $\theta^*$ and $\hat{\theta}$ are the projections onto $\text{span} \mathcal{M}$ and $(\text{span} \mathcal{M})^\perp$ in $\mathbb{R}^3$; we also recall that the resonant subseries $\Pi_{\mathcal{M}} W$ of the potential $W$ depends only on $\theta^*$ and is equal to the function $\overline{W}$ obtained by averaging over $\hat{\theta}$. This function $\Pi_{\mathcal{M}} W(\theta) = \Pi_{\mathcal{M}} W(\theta^*) = \overline{W}(\theta^*)$ is the (scaled) continuum potential corresponding to $\mathcal{M}$; if $\text{dim} \mathcal{M} = 1$ it is a planar continuum potential, if $\text{dim} \mathcal{M} = 2$ it is an axial continuum potential.

In order to state the needed assumption concerning $W$, we consider regions in configuration space bounded by equipotential surfaces of the continuum potentials. Given $\mathcal{M}$, $\text{dim} \mathcal{M} = 1$ or 2, and any number $Q \geq 0$, define the closed subset $\mathcal{A}_\mathcal{M}(Q)$ as

$$\mathcal{A}_\mathcal{M}(Q) = \{ \theta \in T^3 | \Pi_{\mathcal{M}} W(\theta^*) \leq Q \}. \quad (5.33)$$

Our assumption about $W$ is then the following:

**Assumption 5.6.** There exists a critical order $M^* \geq 1$ such that for any 1- or 2-dimensional maximal submodule $\mathcal{M} \subset \mathbb{Z}^3$ with order $|\mathcal{M}| \leq M^*$, there exist numbers $Q'$ and $Q^*$, $0 \leq Q' < Q^* < 1$ with the property that given any $Q \in [Q', Q^*]$, $\mathcal{A}_\mathcal{M}(Q) \neq \emptyset$ and $\mathcal{A}_\mathcal{M}(Q) \cap C(1) = \emptyset$, where $C(1) = \{ \theta \in T^3 | W(\theta) \geq 1 \}$ is the region of close encounter described in (3.7).

This physical assumption says that at sufficiently low order, there are equipotential surfaces of the continuum potential $\Pi_{\mathcal{M}} W$ which do not intersect the restricted region $C(1)$. If $\text{dim} \mathcal{M} = 1$, these surfaces are planes; if $\text{dim} \mathcal{M} = 2$, the surfaces are cylindrical sheets or tubes. In other words, at sufficiently low
order, there are clear planar or axial pathways through the crystal which do not meet the close encounter region $C(1)$. We will not examine this assumption further, since a calculation of $M^*$ requires specific knowledge about the location of lattice sites in the crystal which we do not assume here. We simply remark that the assumption is satisfied by any physical cubic crystal for reasonable values of $\mathcal{E}_\perp$, which ultimately defines the size of $C(1)$.

We next define the transverse energy of a trajectory with respect to a particular continuum potential by

$$E_\perp(I, \theta) = \frac{1}{2}(I^*)^2 + \epsilon \Pi_{\mathcal{M}} W(\theta^*), \quad (5.34)$$

where again $I^*$ and $\theta^*$ denote the projections of $I$ and $\theta$ onto $\text{span} \mathcal{M}$. (It should be pointed out that if $(I, \theta)$ is a solution corresponding to (3.5) then $(I^*, \theta^*)$ is a pair of conjugate variables for (3.5), while if $(J, \phi)$ is a solution corresponding to (4.24), $(J^*, \phi^*)$ is not (quite) a conjugate pair; this will not be troublesome.)

It should also be pointed out that the set $\{(I, \theta) \mid E_\perp(I, \theta) \leq Q\epsilon\}$ is not a Cartesian product of a subset of $\mathbb{R}^3$ with a subset of $\mathbb{T}^3$, but it is a subset of the product $\mathcal{Z}_\mathcal{M}(|\mathcal{M}|^{p+1}(2Q/3)^{1/2}, 1/2) \times \mathcal{A}_\mathcal{M}(Q)$, where the zone $\mathcal{Z}_\mathcal{M}(C, \alpha)$ is defined in §5.1. In fact, every $\theta \in \mathcal{A}_\mathcal{M}(Q)$ is the second factor of the point $(0, \theta) \in \{(I, \theta) \mid E_\perp(I, \theta) \leq Q\epsilon\}$.

We may now state the theorem which gives this article its name,

**Theorem 5.7.** (channeling) Let $\mathcal{M}$ be a 1- or 2-dimensional submodule of $\mathbb{Z}^3$ with order $|\mathcal{M}| \leq M^*$. Assume the hypothesis of the analytic lemma with

$$K = (S_{\sqrt{3}}^3 + A\epsilon^a) \setminus \bigcup_{k \not\equiv 0 \mod N} \text{Int} \mathcal{Z}(\frac{3}{2}A, \alpha). \quad (5.35)$$

Fix the maximum initial (scaled) transverse energy $Q \geq Q'$ and the maximum change in (scaled) transverse energy $\delta > 0$ such that $Q + \delta \leq Q^*$, where $Q' < Q^* < 1$ are defined in Assumption 5.6; set $C = |\mathcal{M}|^{p+1}(2Q/3)^{1/2}$. In addition to the restrictions (4.19) on $\epsilon$, assume the (possibly weaker) further restrictions

$$\epsilon \leq \frac{\delta}{3} \left( \frac{3}{32} A^2 + \frac{3\sqrt{2}}{4} A + \frac{C_4 \sigma}{4} \right)^{-1/b} \quad (b = \min\{c - 1/2, d\}), \quad (5.36a)$$

$$\epsilon \leq \min \left\{ \left( \frac{1}{E} \right)^2, \left( \frac{2}{A} \right)^{1/c}, \left( \frac{\delta}{6} \right)^{1/\tau}, \left( \frac{A}{12E} \right)^{1/\tau} \right\}, \quad (5.36b)$$

40
where $C_4$ is the Lipschitz constant for $\Pi_{\mathcal{M}}W$ with respect to the sup norm:

$$|\Pi_{\mathcal{M}}W(\theta^*) - \Pi_{\mathcal{M}}W(\phi^*)| \leq C_4\|\theta^* - \phi^*\|_{T^3} \leq C_4\|\theta - \phi\|_{T^3}. \quad (5.37)$$

Then any real initial condition $(I_0, \theta_0)$ for (3.5) with initial transverse energy

$$E_\bot(I_0, \theta_0) \leq \epsilon Q \quad (5.38)$$

and with suitable initial direction

$$I_0 \in \tilde{\mathcal{Z}}_{\mathcal{M}}^{\frac{1}{2}, A, \alpha}(N, C, 1/2) \quad (5.39)$$

gives rise to a solution $(I, \theta)$ of (3.5) which satisfies the channeling criterion (3.8) on the exponentially long time interval $[0, T_0]$, where

$$T_0 = \frac{\sigma}{24} e^{r_0 e^{-r_0/4}} - 1. \quad (5.40)$$

This solution is approximated by a “generalized continuum model” solution in the sense that $(I, \theta) = T(J, \phi)$, where $T$ is the near-identity transformation defined in the analytic lemma and $(J, \phi)$ is the solution to (4.24) with initial condition $(J_0, \phi_0) = T^{-1}(I_0, \theta_0)$. Furthermore, on the interval $[0, T_0]$, the longitudinal momentum $\hat{I}(t)$ is nearly constant:

$$\|\hat{I}(t) - \hat{I}_0\|_\infty \leq \frac{3}{4} A \epsilon^c, \quad (5.41)$$

as is the transverse energy:

$$|E_\bot(I(t), \theta(t)) - E_\bot(I_0, \theta_0)| \leq \epsilon \delta. \quad (5.42)$$

**Proof.** We first verify that $(I_0, \theta_0) \in T(D_{\infty})$, so that $(J_0, \phi_0)$ may be defined as claimed. By (5.39) we have $I_0 \in \tilde{\mathcal{Z}}_{\mathcal{M}}^{\frac{1}{2}, A, \alpha}(N, C, 1/2)$ which implies, using (5.17), that

$$I_0 \in \left[ \left( \mathbb{R}^3 \setminus \bigcup_{k \in \mathcal{M}, |k| \leq N} \text{Int} \mathcal{Z}_k(\frac{3}{2} A, \alpha) \right) - A \epsilon^a \right] \cap S_{\sqrt{2}}^c. \quad (5.43)$$

Using the properties (4.4), this is in turn contained in

$$I_0 \in \left[ \left( \mathbb{R}^3 \setminus \bigcup_{k \in \mathcal{M}, |k| \leq N} \text{Int} \mathcal{Z}_k(\frac{3}{2} A, \alpha) \right) \cap (S_{\sqrt{2}}^c + A \epsilon^a) \right] - A \epsilon^a =$$
\[ K - A\epsilon^a = (K - \frac{A}{2} \epsilon^a) - \frac{A}{2} \epsilon^a \subset K_\infty - \frac{A}{2} \epsilon^a, \quad (5.44) \]

where \( K_\infty \times \mathbb{T}^3 \) is the real part of the domain \( D_\infty \), and where \( K - \frac{A}{2} \epsilon^a \subset K_\infty \) by (4.23a). Since by (4.23b), \( K - \frac{A}{2} \epsilon^a \times \mathbb{T}^3 \subset T(D_\infty) \), we see that \((I_0, \theta_0) \in T(D_\infty) \), and we may apply \( T^{-1} \) to \((I_0, \theta_0)\) to obtain \((J_0, \phi_0) = T^{-1}(I_0, \theta_0)\). Since \((I_0, \theta_0)\) is real, \((J_0, \phi_0)\) is real, and we need only check that \( J_0 \in K_\infty \) to see that \((J_0, \phi_0)\) belongs to the domain \( D_\infty \) of (4.24). But by (4.28a),

\[ \|I_0 - J_0\|_\infty \leq \frac{A}{4} \epsilon^c \leq \frac{A}{4} \epsilon^a \quad (c > a) \] by (4.18b) and because \( c > 1/2 \), and so using (5.44), we have \( J_0 \in K_\infty - \frac{A}{4} \epsilon^a \), and \((J_0, \phi_0) \in (K_\infty - \frac{A}{4} \epsilon^a) \times \mathbb{T}^3 \subset K_\infty \times \mathbb{T}^3 \) as required.

If \((J, \phi)\) is the solution of (4.24) with initial condition \((J_0, \phi_0)\), let \( T^* \) be the time of first exit of \((J, \phi)\) from \( K_\infty \times \mathbb{T}^3 \). By Proposition 5.1,

\[ \|\hat{J}(t) - \hat{J}_0\|_\infty \leq \frac{4E}{\sigma} t e^{-\epsilon^{-r/4}} \quad \text{for } 0 \leq t \leq T^*. \quad (5.45) \]

We proceed to estimate the change in the transformed transverse energy \( E_\perp(J, \phi) \) by first noting that by (5.34),

\[ \frac{1}{2} (I^*_0)^2 = E_\perp(I_0, \theta_0) - \epsilon \Pi_\mathcal{M} W(\theta_0^*). \]

But \( \Pi_\mathcal{M} W \) is nonnegative, since it is the average of \( W \) over \( \hat{\phi} = \phi - \phi^* \), and \( W \) is nonnegative by construction. Therefore \( \frac{1}{2} (I^*_0)^2 \) is bounded by \( E_\perp(I_0, \theta_0) \leq \epsilon Q < \epsilon \), by (5.38) and \( Q < 1 \). Thus \( \|I^*_0\|_\infty^2 \leq \|I^*_0\|_\infty^2 < 2\epsilon \Rightarrow \|I^*_0\|_\infty < \sqrt{2\epsilon^{1/2}} \), and so

\[ \|J^*_0\|_\infty \leq \|J^*_0 - I^*_0\|_\infty + \|I^*_0\|_\infty = \|(J_0 - I_0)^*\|_\infty + \|I^*_0\|_\infty \leq \]

\[ \|J_0 - I_0\|_\infty + \|I^*_0\|_\infty \leq \frac{A}{4} \epsilon^c + \sqrt{2\epsilon^{1/2}}, \quad (5.46) \]

and therefore

\[ |E_\perp(J_0, \phi_0) - E_\perp(I_0, \theta_0)| = \frac{1}{2} ((J^*_0)^2 - (I^*_0)^2) + \epsilon \left( \Pi_\mathcal{M} W(\phi_0^*) - \Pi_\mathcal{M} W(\theta_0^*) \right) \leq \]

\[ \frac{1}{2} ((J^*_0 - I^*_0) \cdot (J^*_0 + I^*_0)) + \epsilon \left| \Pi_\mathcal{M} W(\phi_0^*) - \Pi_\mathcal{M} W(\theta_0^*) \right| \leq \]

\[ \frac{3}{2} \|J^*_0 - I^*_0\|_\infty \|J^*_0 + I^*_0\|_\infty + C_4 \epsilon \|\phi_0^* - \theta_0^*\|_\infty \leq \]

\[ \frac{3}{2} \left( \frac{A}{4} \epsilon^c \right) \left( \|J^*_0\|_\infty + \|I^*_0\|_\infty \right) + C_4 \epsilon \|\phi_0 - \theta_0\|_\infty \leq \]

\[ \frac{3A}{8} \epsilon^c \left( \frac{A}{4} \epsilon^c + 2\sqrt{2\epsilon^{1/2}} \right) + C_4 \epsilon \sigma \frac{\epsilon^d}{4} \leq \]

\[ \epsilon^{1+b} \left( \frac{3}{32} A^2 + \frac{3\sqrt{2}}{4} A + \frac{C_4 \sigma}{4} \right) \leq \epsilon^\delta, \quad (5.47) \]
where \( b = \min\{c - 1/2, d\} \) and where the last inequality follows by restriction (5.36a). For \( 0 \leq t \leq \min\{T_0, T^*\} \), we now estimate

\[
|E_\perp(J(t), \phi(t)) - E_\perp(J_0, \phi_0)| =
\frac{1}{2} ((J^*(t))^2 - (J_0^*)^2) + \epsilon (\Pi_M W(\phi^*(t)) - \Pi_M W(\phi_0^*))
\]

(5.48)

Using (4.24) and \( J^2 = (J^*)^2 + \hat{J}^2 \), we see that this is equal to

\[
\frac{1}{2} (\hat{J}_0^2 - (\hat{J}(t))^2) + \left( G(J_0, \phi_0^*) - \epsilon \Pi_M W(\phi_0^*) \right) - \left( G(J(t), \phi^*(t)) - \epsilon \Pi_M W(\phi^*(t)) \right) + R(J_0, \phi_0) - R(J(t), \phi(t)) \leq
\]

\[
\frac{1}{2} |\hat{J}_0^2 - (\hat{J}(t))^2| + 2\|G - \epsilon \Pi_M W\|_{D_\infty} + 2\|R\|_{D_\infty} \leq
\]

\[
\frac{3}{2} \|\hat{J}_0 + \hat{J}(t)\|_{D_\infty} \|\hat{J}_0 - \hat{J}(t)\|_{D_\infty} + 2\|G - \epsilon \Pi_M W\|_{D_\infty} + 2\|R\|_{D_\infty} \leq
\]

(5.49)

We now use (4.26) to estimate \( \|R\|_{D_\infty} \), (4.27) for \( \|G - \epsilon \Pi_M W\|_{D_\infty} \), the geometric proposition (Proposition 5.1) for \( \|\hat{J}_0 - \hat{J}(t)\|_{D_\infty} \), and finally the fact that \( J_0 \in S_\epsilon^{\sqrt{2}} + \frac{A}{4} \epsilon^c \) for \( \|\hat{J}_0\|_{\infty} \). (5.49) is therefore bounded by

\[
\frac{3}{2} \left( 2(\sqrt{2} + \frac{A}{4} \epsilon^c) + \frac{4E}{\sigma} t e e^{-\epsilon^{-\tau/4}} \right) \frac{4E}{\sigma} t e e^{-\epsilon^{-\tau/4}} + 2E\epsilon^{1+\tau} + 4E\epsilon e^{-\epsilon^{-\tau/4}}.
\]

(5.50)

Since \( 0 \leq t \leq \min\{T_0, T^*\} \), where \( T_0 \) is given by (5.40), this expression is less than or equal to

\[
\frac{3}{2} \left( 2(\sqrt{2} + \frac{A}{4} \epsilon^c) + \frac{E}{6} \epsilon^{1+\tau} \right) \frac{E}{6} \epsilon^{1+\tau} + 2E\epsilon^{1+\tau}.
\]

(5.51)

Using (5.36b), this simplifies to

\[
\frac{3}{2} \left( 2\sqrt{2} + 1 + \frac{1}{6} \right) \left( \frac{1}{6} \epsilon^{1+\tau} - \frac{8E}{\sigma} e e^{-\epsilon^{-\tau/4}} \right) + 2E\epsilon^{1+\tau},
\]

(5.52)

so finally, using (5.36b) \( \epsilon \leq (\delta/6)^{1/\tau} \) and the fact that \( 2\sqrt{2} + 1 + 1/6 < 4 \), we arrive to

\[
|E_\perp(J(t), \phi(t)) - E_\perp(J_0, \phi_0)| \leq 3E\epsilon^{1+\tau} \leq \epsilon_3^\delta.
\]

(5.53)
Now combining (5.47) and (5.53), we get

\[ E_{\perp}(J(t), \phi(t)) \leq E_{\perp}(I_0, \theta_0) + |E_{\perp}(J_0, \phi_0) - E_{\perp}(I_0, \theta_0)| + \]

\[ |E_{\perp}(J(t), \phi(t)) - E_{\perp}(J_0, \phi_0)| \leq \epsilon(Q + \frac{2}{3} \delta) \leq \epsilon Q^* \leq \epsilon \]  \hspace{1cm} (5.54)

for \( 0 \leq t \leq \min\{T^*, T_0\} \), from which we find

\[ \frac{1}{2} \|J^*(t)\|^2 \leq \frac{1}{2} (J^*(t))^2 \leq E_{\perp}(J(t), \phi(t)) \leq \epsilon \]

\[ \Rightarrow \|J^*(t)\|_\infty \leq \sqrt{2} \epsilon^{1/2} \]  \hspace{1cm} (5.55)

on the same interval.

We can now show that the time of escape \( T^* \geq T_0 \). By way of contradiction, suppose not; suppose \( T^* < T_0 \). Then \( J(T^*) \in \partial K_\infty \) and \( J_0 \in K_\infty - \frac{A}{4} \epsilon^a \) (see above) imply that \( \|J(T^*) - J_0\|_\infty \geq \frac{A}{4} \epsilon^a \). But

\[ \|J(T^*) - J_0\|_\infty \leq \|\hat{J}(T^*) - \hat{J}_0\|_\infty + \|J^*(T^*) - J_0^*\|_\infty \leq \]

\[ \|\hat{J}(T^*) - \hat{J}_0\|_\infty + \|J^*(T^*)\|_\infty + \|J_0^*\|_\infty \leq \]

\[ \frac{4E}{\sigma} T^* \epsilon e^{-\epsilon^{-1/4}} + 2\sqrt{2} \epsilon^{1/2}, \]  \hspace{1cm} (5.56)

by (5.13) and (5.55). Now from the assumption \( T^* < T_0 \) we find that this is

\[ \leq \frac{E}{6} \epsilon^{1+r} + 2\sqrt{2} \epsilon^{1/2} < 3\epsilon^{1/2} < \frac{A}{4} \epsilon^a \frac{12}{A} \epsilon^8 \tau < \frac{A}{4} \epsilon^a. \]  \hspace{1cm} (5.57)

The second to last inequality derives from the fact that \( a + 8\tau < 1/2 \) (use (4.18c) and \( p \geq 4 \)); the last inequality is a consequence of (5.36b) and \( E \geq 1 \). Thus \( \|J(T^*) - J_0\|_\infty < \frac{A}{4} \epsilon^a \) contradicts \( J(T^*) \in \partial K_\infty \), so \( T^* \geq T_0 \), and we see that the solution \((J, \phi)\) of (4.24) with initial condition \((J_0, \phi_0) = T^{-1}(I_0, \theta_0)\) remains in \( K_\infty \times T^3 \subset D_\infty \) on the time interval \([0, T_0]\). We thus have a well-defined solution \((I, \theta) = T(J, \phi)\) of (3.5) on the same interval. Furthermore, for \( 0 \leq t \leq T_0 \),

\[ \|\hat{I}(t) - \hat{I}_0\|_\infty \leq \|\hat{I}(t) - \hat{J}(t)\|_\infty + \|\hat{J}(t) - \hat{J}_0\|_\infty + \|\hat{J}_0 - \hat{I}_0\|_\infty \leq \]

\[ \|I(t) - J(t)\|_\infty + \|\hat{J}(t) - \hat{J}_0\|_\infty + \|J_0 - I_0\|_\infty \leq \]

\[ \frac{A}{4} \epsilon^c + \frac{4E}{\sigma} t \epsilon e^{-\epsilon^{-1/4}} + \frac{A}{4} \epsilon^c \leq \]

\[ \frac{A}{2} \epsilon^c + \frac{E}{6} \epsilon^{1+r} < \frac{A}{2} \epsilon^c + \frac{A}{2} \epsilon^c \left( \frac{E}{3A} \epsilon^8 \tau \right) < \frac{3}{4} A \epsilon^c. \]  \hspace{1cm} (5.58)
The next to last inequality holds because \( c + 8\tau < 1 + \tau \) by (4.18b); the last inequality follows from (5.36b) \( (\epsilon \leq (A/12E)^{1/4} < (3A/E)^{1/4}) \); and (5.41) is proved.

The following estimate for \( 0 \leq t \leq T_0 \), which parallels (5.47), is needed to conclude the proof:

\[
|E_\perp(I, \theta) - E_\perp(J, \phi)| \leq \frac{3}{2} \|I^* - J^*\|_\infty \left( \|I^* - J^*\|_\infty + 2\|J^*\|_\infty \right) + C_4 \epsilon \|\theta^* - \phi^*\|_\infty
\]

\[
\leq \frac{3}{2} \left( \frac{A}{4} \epsilon^c \right) \left( \frac{A}{4} \epsilon^c + 2\sqrt{2} \epsilon^{1/2} \right) + C_4 \sigma \frac{\epsilon^{1+d}}{4}
\]

\[
\leq \epsilon^{1+b} \left( \frac{3}{32} A^2 + \frac{3\sqrt{2}}{4} A + \frac{C_4 \sigma}{4} \right) \leq \frac{\epsilon \delta}{3}, \tag{5.59}
\]

as in (5.47). Combining (5.47), (5.53), and (5.59), we find

\[
|E_\perp(I(t), \theta(t)) - E_\perp(I_0, \theta_0)| \leq \epsilon \delta, \tag{5.60}
\]

which proves (5.42). It follows that for \( 0 \leq t \leq T_0 \),

\[
e\Pi_M W(\theta^*) \leq E_\perp(I(t), \theta(t)) \leq (Q + \delta)\epsilon
\]

\[\Rightarrow \theta \in \mathcal{A}(Q + \delta), \quad \text{and} \]

\[Q + \delta \leq Q^* \Rightarrow \mathcal{A}(Q + \delta) \cap \mathcal{C}(1) = \emptyset.
\]

The solution \((I, \theta)\) thus satisfies the channeling criterion as claimed for \( 0 \leq t \leq T_0 \), and the theorem is proved. \( \Box \)

### 6. CONCLUDING REMARKS

A few remarks should help clarify the meaning of the channeling theorem. First, the statement that the solutions \((I, \theta)\) discussed in the theorem “are approximated by solutions of a generalized continuum model” simply indicates that the difference between \((I, \theta)\) and \((J, \phi)\) is uniformly small on \([0, T_0]\), while \((J, \phi)\) is a solution of (4.24), which on the same interval may be written

\[
H'(J, \phi) = \frac{1}{2} J^2 + \epsilon \Pi_M W(\phi^*) + \mathcal{O}(\epsilon^{1+\tau}) \tag{6.1}
\]

by virtue of (4.26) and (4.27). Removing the \(\mathcal{O}(\epsilon^{1+\tau})\) term from (6.1) and renaming its phase variables \((p, q)\), we recover the ordinary continuum model

\[
H_c(p, q) = \frac{1}{2} p^2 + \epsilon \Pi_M W(q^*) \tag{6.2}
\]
which splits into the independent systems

\[ H_\perp(p^*, q^*) = \frac{1}{2}(p^*)^2 + \epsilon\Pi_\mathcal{M} W(q^*) \quad (6.3) \]

\[ H_\parallel(\hat{p}, \hat{q}) = \frac{1}{2}\hat{p}^2 \quad (6.4) \]

\((p^*, q^*)\) and \((\hat{p}, \hat{q})\) are the transverse and longitudinal phase variables, respectively; cf. \S 5.3A.) If \(\dim \mathcal{M} = 1\), \(H_\perp\) is a 1 degree of freedom planar continuum Hamiltonian, the scaled version of (2.4); if \(\dim \mathcal{M} = 2\) it is the scaled version of a 2 degree of freedom axial continuum Hamiltonian (2.3). This is the sense in which (4.24) is a generalized continuum model, and it is not surprising that the \(O(\epsilon^{1+\tau})\) remainder (comprising \([\epsilon^{-\tau/4}]\) terms) is needed to obtain the uniform approximation on exponentially long time intervals. In general, the continuum solution \((p, q)\) can uniformly approximate the exact solution \((I, \theta)\) to (3.5) on time intervals of length at most \(O(\epsilon^{-1/2})\) (see [20]), although it may be possible to construct a “phase-adjusted” solution based on the continuum solution which works on longer intervals, at least in the planar case.

A second point of interest is the set of admissible initial conditions

\[ \left( \hat{Z}^{\frac{1}{3}}_{\mathcal{M}} A^\alpha (N, C, 1/2) \times T^3 \right) \cap Q, \quad (6.5) \]

where

\[ Q = \{ (I_0, \theta_0) \mid E_\perp (I_0, \theta_0) \leq \epsilon Q \}. \quad (6.6) \]

In physical terms, the set (6.5) represents particles with suitable initial transverse energy (which excludes particles initially directed at nuclei), and with suitable initial directions. It is interesting to note that the \(\epsilon\)-dependence of the directions determined by (6.5) is in accordance with the incident energy dependence of channeling directions determined by Lindhard’s critical angle [29] (of course, this is just a consequence of the choice of scaling; cf. \S 3.3).

There is also the related question of what happens to trajectories with initial directions not governed by either Theorem 5.3 or Theorem 5.7. Because of the requirement \(c > \alpha\) in the analytic lemma (see (4.18b)), for sufficiently small \(\epsilon\) there is a subset of action space with relative volume \(O(\epsilon^{\alpha} - \epsilon^{c})\) which is excluded from the initial conditions of both theorems. This in turn may be traced to the proof of the channeling theorem, and ultimately to the need to avoid close encounters as demanded by the channeling criterion. It must be stressed that this gap in the set of initial directions for which particle motions are described is almost certainly not an artifact of the rough estimates carried out here (although the estimates are certainly crude); instead, it appears to be an unavoidable consequence of the channeling criterion. This points out the essential way that the channeling theorem differs from other Nekhoroshev-like results, which are not called upon to provide information about trajectories in
configuration space. It also leaves open the intriguing possibility that certain “dechanneling” phenomena may be represented rigorously in the perfect crystal model (some dechanneling is observed experimentally “on the shoulders” of axial directions; these shoulders appear to coincide roughly with the gaps in initial directions governed by the channeling theorem).

Much remains to be done in the mathematical theory of the motion of charged particles in crystals; some directions for further research are outlined in [19] and [20]. As it stands, the analysis of the perfect crystal model presented here establishes a general mathematical framework for particle channeling and should serve as the foundation for more comprehensive theories. Finally, it was a pleasure and a surprise for this author to uncover—with considerable assistance—the link between the vastly different but equally remarkable works of J. Lindhard and N. Nekhoroshev.

ACKNOWLEDGMENTS

The results presented here are a condensation of the PhD thesis [17]. Accordingly, I am indebted to more individuals than I can name; many of them were acknowledged in [17]. I thank P. Lochak, W.T. Kyner, and especially J.A. Ellison for their help; I also thank the American Fulbright Commission for the grant which allowed me to spend 1986–87 at the École Normale Supérieure in Paris, where much of this work was carried out.
REFERENCES


<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>774</td>
<td>L.A. Peletier &amp; W.C. Troy</td>
<td>Self-similar solutions for infiltration of dopant into semiconductors</td>
</tr>
<tr>
<td>775</td>
<td>H. Scott Dumas and James A. Ellison</td>
<td>Nekhoroshev's theorem, ergodicity, and the motion of energetic</td>
</tr>
<tr>
<td></td>
<td></td>
<td>charged particles in crystals</td>
</tr>
<tr>
<td>776</td>
<td>Stathis Filippas and Robert V. Kohn</td>
<td>Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.</td>
</tr>
<tr>
<td>777</td>
<td>Patricia Bauman, Nicholas C. Owen and Daniel</td>
<td>Maximum principles and a priori estimates for an incompressible material and non-linear elasticity</td>
</tr>
<tr>
<td></td>
<td>Phillips</td>
<td>Euler–Lagrange equations from non-linear elasticity</td>
</tr>
<tr>
<td>778</td>
<td>Jack Carr and Robert Pego</td>
<td>Self-similarity in a coarsening model in one dimension</td>
</tr>
<tr>
<td>779</td>
<td>J.M. Greenberg</td>
<td>The shock generation problem for a discrete gas with short range repulsive forces</td>
</tr>
<tr>
<td>780</td>
<td>George R. Sell and Mario Taboada</td>
<td>Local dissipativity and attractors for the Kuramoto–Sivashinsky</td>
</tr>
<tr>
<td></td>
<td></td>
<td>equation in thin 2D domains</td>
</tr>
<tr>
<td>781</td>
<td>T. Subba Rao</td>
<td>Analysis of nonlinear time series (and chaos) by bispectral methods</td>
</tr>
<tr>
<td>782</td>
<td>Nicholas Baumann, Daniel D. Joseph, Paul Mohr</td>
<td>Vortex rings of one fluid in another free fall</td>
</tr>
<tr>
<td></td>
<td>and Yusiko Renardy</td>
<td></td>
</tr>
<tr>
<td>783</td>
<td>Oscar Bruno, Avner Friedman and Fernando Reitich</td>
<td>Asymptotic behavior for a coalescence problem</td>
</tr>
<tr>
<td>784</td>
<td>Johannes C.C. Nitsche</td>
<td>Periodic surfaces which are extremal for energy functionals containing curvature functions</td>
</tr>
<tr>
<td>785</td>
<td></td>
<td></td>
</tr>
<tr>
<td>786</td>
<td>F. Abergel and J.L. Bona</td>
<td>A mathematical theory for viscous, free-surface flows over a perturbed plane</td>
</tr>
<tr>
<td>787</td>
<td>Gunduz Caginalp and Xinfu Chen</td>
<td>Phase field equations in the singular limit of sharp interface problems</td>
</tr>
<tr>
<td>788</td>
<td>Robert P. Gilbert and Yongzhi Xu</td>
<td>An inverse problem for harmonic acoustics in stratified oceans</td>
</tr>
<tr>
<td>789</td>
<td>Roger Fosdick and Eric Volkmann</td>
<td>Normality and convexity of the yield surface in nonlinear plasticity</td>
</tr>
<tr>
<td>790</td>
<td>H.S. Brown, I.G. Kevrekidis and M.S. Jolly</td>
<td>A minimal model for spatio-temporal patterns in thin film flow</td>
</tr>
<tr>
<td>791</td>
<td>Chao–Nien Chen</td>
<td>On the uniqueness of solutions of some second order differential equations</td>
</tr>
<tr>
<td>792</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>The thermistor problem for conductivity which vanishes at large temperature</td>
</tr>
<tr>
<td>793</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>The thermistor problem with one-zero conductivity</td>
</tr>
<tr>
<td>794</td>
<td>E.G. Kalnins and W. Miller, Jr.,</td>
<td>Separation of variables for the Dirac equation in Kerr Newman space time</td>
</tr>
<tr>
<td>795</td>
<td>E. Knobloch, M.R.E. Proctor and N.O. Weiss</td>
<td>Finite-dimensional description of doubly diffusive convection</td>
</tr>
<tr>
<td>796</td>
<td>V.V. Pukhnachov</td>
<td>Mathematical model of natural convection under low gravity</td>
</tr>
<tr>
<td>797</td>
<td>M.C. Knaap</td>
<td>Existence and non-existence for quasi-linear elliptic equations with the p-laplacian involving critical Sobolev exponents</td>
</tr>
<tr>
<td>798</td>
<td>Stathis Filippas and Wenxiong Liu</td>
<td>On the blowup of multidimensional semilinear heat equations</td>
</tr>
<tr>
<td>799</td>
<td>A.M. Meirmanov</td>
<td>The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution</td>
</tr>
<tr>
<td>800</td>
<td>Bo Guan and Joel Spruck</td>
<td>Interior gradient estimates for solutions of prescribed curvature equations of parabolic type</td>
</tr>
<tr>
<td>801</td>
<td>Hi Jun Choe</td>
<td>Regularity for solutions of nonlinear variational inequalities with gradient constraints</td>
</tr>
<tr>
<td>802</td>
<td>Peter Shi and Yongzhi Xu</td>
<td>Quasistatic linear thermoelasticity on the unit disk</td>
</tr>
<tr>
<td>803</td>
<td>Satyanad Kichenassamy and Peter J. Olver</td>
<td>Existence and non-existence of solitary wave solutions to higher order model evolution equations</td>
</tr>
<tr>
<td>804</td>
<td>Dening Li</td>
<td>Regularity of solutions for a two-phase degenerate Stefan Problem</td>
</tr>
<tr>
<td>805</td>
<td>Marek Fila, Bernhard Kawohl and Howard A. Levine</td>
<td>Quenching for quasilinear equations</td>
</tr>
<tr>
<td>806</td>
<td>Yoshikazu Giga, Shun’ichi Goto and Hitoshi Ishii</td>
<td>Global existence of weak solutions for interface equations coupled with diffusion equations</td>
</tr>
<tr>
<td>807</td>
<td>Mark J. Friedman and Eusebius J. Doedel</td>
<td>Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study</td>
</tr>
<tr>
<td>808</td>
<td>Mark J. Friedman</td>
<td>Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds</td>
</tr>
<tr>
<td>809</td>
<td>Peter W. Bates and Songmu Zheng</td>
<td>Inertial manifolds and inertial sets for the phase-field equations</td>
</tr>
<tr>
<td>810</td>
<td>J. López Gómez, V. Márquez and N. Wolanski</td>
<td>Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition</td>
</tr>
<tr>
<td>811</td>
<td>Xinfu Chen and Fahuai Yi</td>
<td>Regularity of the free boundary of a continuous casting problem</td>
</tr>
<tr>
<td>812</td>
<td>Eden, A., Foias, C., Nicolaenko, B. and Temam, R.,</td>
<td>Inertial sets for dissipative evolution equations Part I: Construction and applications</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>813</td>
<td>Jose–Francisco Rodrigues and Boris Zaltzman</td>
<td>On classical solutions of the two-phase steady-state Stefan problem in strips</td>
</tr>
<tr>
<td>814</td>
<td>Vioarel Barbu and Srdjan Stojanovic</td>
<td>Controlling the free boundary of elliptic variational inequalities on a variable domain</td>
</tr>
<tr>
<td>815</td>
<td>Vioarel Barbu and Srdjan Stojanovic</td>
<td>A variational approach to a free boundary problem arising in electro-photography</td>
</tr>
<tr>
<td>816</td>
<td>B.H. Gilding and R. Kersner</td>
<td>Diffusion-convection-reaction, free boundaries, and an integral equation</td>
</tr>
</tbody>
</table>
Shoshana Kamin, Lambertus A. Peletier and Juan Luis Vazquez, On the Barenblatt equation of elasto-plastic filtration

Avner Friedman and Bei Hu, The Stefan problem with kinetic condition at the free boundary

M.A. Grinfeld, The stress driven instabilities in crystals: mathematical models and physical manifestations

Bei Hu and Lihe Wang, A free boundary problem arising in electrophotography: solutions with connected toner region

Yongzhi Xu, T. Craig Poling, and Trent Brundage, Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean

Steven J. Altschuler, Singularities of the curve shrinking flow for space curves

Steven J. Altschuler and Matthew A. Grayson, Shortening space curves and flow through singularities

Tong Li, On the Riemann problem of a combustion model

L.A. Peletier & W.C. Troy, Self-similar solutions for diffusion in semiconductors


Minkyu Kwak, Finite dimensional description of convective reaction-diffusion equations

Minkyu Kwak, Finite dimensional inertial forms for the 2D Navier–Stokes equations

Victor A. Galaktionov and Sergey A. Posashkov, On some monotonicity in time properties for a quasilinear parabolic equation with source

Victor A. Galaktionov, Remark on the fast diffusion equation in a ball

Hi Jun Choe and Lihe Wang, A regularity theory for degenerate vector valued variational inequalities

Vladimir I. Oliker and Nina N. Uraltseva, Evolution of nonparametric surfaces with speed depending on curvature, II. The mean curvature case.

S. Kamin and W. Liu, Large time behavior of a nonlinear diffusion equation with a source

Shoshana Kamin and Juan Luis Vazquez, Singular solutions of some nonlinear parabolic equations

Bernhard Kawohl and Robert Kersner, On degenerate diffusion with very strong absorption

Avner Friedman and Fernando Reitich, Parameter identification in reaction-diffusion models

E.G. Kalnins, H.L. Manocha and Willard Miller, Jr., Models of q-algebra representations I. Tensor products of special unitary and oscillator algebras

Robert J. Sacker and George R. Sell, Dichotomies for linear evolutionary equations in Banach spaces

Oscar P. Bruno and Fernando Reitich, Numerical solution of diffraction problems: a method of variation of boundaries

Oscar P. Bruno and Fernando Reitich, Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain

Victor A. Galaktionov and Juan L. Vazquez, Asymptotic behaviour for an equation of superslow diffusion. The Cauchy problem

Josephus Hulshof and Juan Luis Vazquez, The Dipole solution for the porous medium equation in several space dimensions

Shoshana Kamin and Juan Luis Vazquez, The propagation of turbulent bursts

Miguel Escobedo, Juan Luis Vazquez and Enrike Zuazua, Source-type solutions and asymptotic behaviour for a diffusion-convection equation

Marco Biroli and Umberto Mosco, Discontinuous media and Dirichlet forms of diffusion type

Stathis Filippas and Jong-Sheng Guo, Quenching profiles for one-dimensional semilinear heat equations

H. Scott Dumas, A Nekhoroshev-like theory of classical particle channeling in perfect crystals

R. Natalini and A. Tesei, On a class of perturbed conservation laws

Paul K. Newton and Shinya Watanabe, The geometry of nonlinear Schrödinger standing waves

S.S. Sritharan, On the nonsmooth verification technique for the dynamic programming of viscous flow

Mario Taboada and Yuncheng You, Global attractor, inertial manifolds and stabilization of nonlinear damped beam equations

Shigeru Sakaguchi, Critical points of solutions to the obstacle problem in the plane

F. Abergel, D. Hilhorst and F. Issard-Roch, On a dissolution-growth problem with surface tension in the neighborhood of a stationary solution

Erasmus Langer, Numerical simulation of MOS transistors

Haïm Brezis and Shoshana Kamin, Sublinear elliptic equations in $\mathbb{R}^n$

Johannes C.C. Nitsche, Boundary value problems for variational integrals involving surface curvatures

Chao-Nien Chen, Multiple solutions for a semilinear elliptic equation on $\mathbb{R}^N$ with nonlinear dependence on the gradient

D. Brochet, X. Chen and D. Hilhorst, Finite dimensional exponential attractor for the phase field model

Joseph D. Fehrbach, Mullins-Sekerka stability analysis for melting-freezing waves in helium-4

Walter Schempp, Quantum holography and neurocomputer architectures

D.V. Anosov, An introduction to Hilbert's 21st problem

Herbert E Huppert and M Grae Worster, Vigorous motions in magma chambers and lava lakes