CHOOSING FROM A TOURNAMENT

By

H. MOULIN

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CHOOSING FROM A TOURNAMENT

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Abstract

A tournament is any complete asymmetric relation over a finite set $A$ of outcomes describing pairwise comparisons. A choice correspondence assigns to every tournament on $A$ a subset of winners. Miller's uncovered set is an example for which we propose an axiomatic characterization. The set of Copeland winners (outcomes with maximal scores) is another example; it is a subset of the uncovered set: we note that it can be a dominated subset. A third example is derived from the sophisticated agenda algorithm; we argue that it is a better choice correspondence than the Copeland set.
1. **Introduction**

Given a finite set $A$ of outcomes (candidates, decisions) a tournament $T$ on $A$ expresses decisive preference judgments for all pairs of outcomes: $aTb$ reads $a$ beats $b$ (no indifferences are allowed). When a tournament $T$ has no Condorcet winner (i.e., no outcome beats every other outcome) there is no straightforward notion of winner(s) for $T$. This paper explores some such notions and compares them normatively.

The above problem has long proved useful to many decision problems far more general than sport competitions; see, for example, the monograph by Moon (1968) and references therein to the psychometric and biometric literature. Here we concentrate on solution concepts that were originally inspired by the theory of collective choice, in particular the strategic analysis of voting rules based upon majority comparisons: hence our terminology of Condorcet consistency, sophisticated agenda and so on. Of course here we deal with abstract tournaments only, so that the potential applications are wider.

**Summary of the results**

A choice correspondence associates to any tournament $T$ a subset $S(T)$ of outcomes. The simplest example is the top cycle of $T$, made up of those outcomes that beat directly or indirectly every other outcome ($a$ beats $b$ indirectly if $a$ beats $c$ and $c$ beats $b$, or $a$ beats $c$, $c$ beats $d$, $d$ beats $b$, and so on). A minimal rationality requirement of any choice correspondence is that it produces a subset of the top cycle (Moon, 1968; Schwartz, 1972).

Miller's uncovered set is made up of those outcomes $a$ which, where opposed to any other $b$, either beat $b$ or beat some $c$ that beat $b$ (thus $a$ beats every other outcome in at most two steps). It was introduced independently by Miller (1977) and Fishburn (1977), and further explored by Miller (1980) and Shepsle and Weingast (1982) — see also McKelvey (1983) in the context of spatial voting. We propose our axiomatic characterization of the uncovered set (Theorem 1) based upon two familiar axioms from the theory of rationalizable choice functions (see Bordes (1983) and Moulin (1984)).
Next we turn (Section 5) to another choice correspondence, namely the Copeland set made up of those outcomes that beat the greatest number of opponents (Copeland, 1951). Miller observed that the Copeland set is contained in the uncovered set; however, it might be a dominated subset of the uncovered set, at least when there are 13 outcomes or more (Theorem 2).

In Section 6 we explore a family of deterministic choice correspondences using a fixed binary tree (as in a single elimination tournament). These solutions generalize to the context of tournaments the voting by binary choice methods introduced by Farquharson (1969) and further studied by Miller (1977, 1980), McKelvey and Niemi (1978), and Moulin (1979). We prove that they always choose one outcome from the top cycle, but may display perverse nonmonotonicity features. We also prove that no selection of the Copeland set can be obtained by such a binary tree (at least when there are 8 outcomes or more, Corollary to Lemma 10).

The last section (Section 7) is devoted to one choice correspondence derived from a specific binary tree, the multistage elimination tree (Miller, 1977; Moulin, 1979). Its solution is given by a remarkable algorithm called the sophisticated agenda algorithm (Shepsle and Weingast, 1982). This algorithm produces an undominated outcome of the uncovered set (Banks, 1984), thus is arguably superior to any selection of the Copeland set. When there are no more than four undominated outcomes in the uncovered set, all of them can be reached by an appropriate agenda (Banks, 1984). If there are at least five such outcomes, some might be out of reach (Lemma 14).

Open problems

Ranking the participants of a given tournament is another, related, problem. The Copeland scores (assigning to each outcome the number of outcomes it beats) induce such a preordering; Rubinstein (1980) and Henriet (1984) provide two distinct characterizations of it (the former based on a weak version of Arrow's Independence of Irrelevant Alternatives, the latter based on its invariance when the orientation of a cycle is reversed). Yet the arguments developed above raise
doubts about its reasonableness. Much more appealing is the idea (proposed independently by Kemeny (1959) and Slater (1961) to approximate a given tournament by its closest orderings (the distance being measured by the number of binary comparisons that must be reversed). Yet little is known about the top elements of the closest orderings: they are in the uncovered set and can be all Copeland nonwinners (Bermond, 1972); but are they undominated in the uncovered set? is there an easy algorithm to compute one of them?

Another widely open question is the generalization or choice correspondences to any complete relation on A (not necessarily asymmetric: indifferences are allowed). Already the definition of the top cycle and/or the uncovered set is far from simple if one wants to preserve some version of Condorcet consistency: on this see Miller (1982). Also the Copeland score can be defined in several non-equivalent ways (by counting one half point or zero point in case of a tie) among which the choice is not straightforward; see Henriot (1984).

2. **Choice Correspondence and Condorcet Consistency**

The finite set A contains all feasible outcomes. A tournament on A is a complete and asymmetric binary relation T: for each pair a, b of distinct outcomes, exactly one of aTb, bTa holds (and aTa for all a). We read aTb as: a beats b in the pairwise comparison a, b. Notice that tournaments allow cycles of arbitrary length (e.g., aTb, bTc, cTa is a cycle of length 3). A tournament with no cycles is an ordering of A (complete, transitive, asymmetric). We denote by ℭ the set of tournaments on A.

A choice correspondence (c.c.) is any multivalued mapping S from ℭ into A. To any tournament T, a choice correspondence associates a nonempty subset S(T) of "best" outcomes, called the choice set at T.

If an outcome beats every other outcome in pairwise comparisons we call it a Condorcet winner. Our first axiom states that a Condorcet winner should be uniquely chosen:
Condorcet consistency: for all $T \in \mathfrak{T}$, $a \in A \{aTb \text{ all } b \in A\} \Rightarrow \{S(T) = \{a\}\}$.

For instance, when tournament $T$ is an ordering (i.e., has no cycles) its top outcome should be uniquely chosen. A much stronger statement than Condorcet's consistency is:

Condorcet transitivity: for all $T \in \mathfrak{T}$, $a, b \in A \{[a \in S(T), bTa] \Rightarrow b \in S(T)\}$ \quad (1)

In words, any outcome that defeats some chosen outcome must be chosen as well. Observe that Condorcet transitivity is equivalently formulated as:

for all $T \in \mathfrak{T}$, $a, b \in A \ \{[a \in S(T), b \notin S(T)] \Rightarrow aTb\}$

In words, any chosen outcome beats any nonchosen outcome.

Lemma 1 (Schwartz, 1972). There is a unique smallest (w.r.t inclusion) choice correspondence satisfying Condorcet's transitivity. It is called the top cycle and defined as

$tc(T) = \{a \in A \mid \text{for all } b \in A, \text{ there is an integer } n \text{ and a sequence } a = a_0, a_1, \ldots, a_n = b \text{ s.t. } a_i Ta_{i+1}, \text{ all } 0, \ldots, (n-1)\}$ \quad (2)

Proof. Pick any c.c. $S$ satisfying (1) and let $tc$ be the c.c. defined by (2).

Choose $a \in tc(T)$ and $b \in S(T)$. Since there is a sequence $a = a_0, a_1, \ldots, a_n = b$ such as in (2) we have

$\{a_{n-1} Ta, b \in S(T)\} \Rightarrow a_{n-1} \in S(T)$

$\{a_{n-2} Ta_{n-1}, a_{n-1} \in S(T)\} \Rightarrow a_{n-2} \in S(T)$

\ldots

$\{aTa_1, a_1 \in S(T)\} \Rightarrow a \in S(T)$

This proves $tc \subseteq S$. It remains to check that $tc$ satisfies Condorcet transitivity.

This is obvious from (2).

Q. E. D.
An equivalent definition of \( tc \) goes by setting \( T^\ast \) to be the transitive closure of \( T \):

\[ a T^\ast b \text{ iff there is an integer } n \text{ and a sequence } \]
\[ a = a_0, a_1, \ldots, a_n = b \text{ s.t. } a_i T a_{i+1}, \text{ all } i = 0, \ldots, (n-1) \]

Then \( tc(T) \) is the set of maximal elements of \( T^\ast \).

The top cycle is too big a choice correspondence, as the following example shows: \( T \) coincides with the ordering \( 1 > 2 > 3 \ldots > n \) except for \( n T 1 \). Then \( tc(T) = \{1, \ldots, n\} \): the top cycle does not discriminate at all among outcomes, despite their asymmetry. All subsequent choice correspondences will actually choose \( \{1, 2, n\} \) thus eliminating \( 3, \ldots, n-1 \).

Another drawback of the top cycle arises when the tournament \( T \) is derived from binary majority comparisons (among finitely many agents, each endowed with a preference preordering over \( A \)). Then the top cycle typically contains Pareto dominated outcomes.

The following weakening of Condorcet's transitivity will prove compatible with all reasonable choice correspondences introduced below

\[ \text{Smith's Consistency: for all } T \in \mathcal{T}, \text{ all } B \subset A \{b T a \text{ for all } b \in B \}
\]

\[ \text{and all } a \in A \setminus B \Rightarrow \{S(T) \subset B\} \]  \hspace{1cm} (3)

If a subset \( B \) is ahead of \( A \setminus B \) (in the sense that any outcome in \( B \) beats any one in \( A \setminus B \)) then the choice set does not contain any outcome of \( A \setminus B \). This property is stronger than Condorcet's consistency, but weaker than Condorcet's transitivity.

Just like the c.c. satisfying Condorcet's transitivity are stable by intersection, those satisfying Smith's consistency are stable by union and thus have a largest element.

**Lemma 2.** There is a unique largest (w.r.t. inclusion) choice correspondence satisfying Smith's consistency. It is the top cycle.

The proof is straightforward.
We note finally

**Lemma 3.** The top cycle of a tournament \( T \) is a singleton only if it is the Condorcet winner of \( T \). Otherwise it is a cycle (with size at least 3); one can order it as \( \operatorname{tc}(T) = \{a_1, \ldots, a_p\} \) where

\[
a_iTa_{i+1}, \quad i = 1, \ldots, p-1 \quad \text{and} \quad a_pTa_1.
\]

**Proof.** If \( T \) has no Condorcet winner, every outcome in \( \operatorname{tc}(T) \) is beaten by at least one other outcome in \( \operatorname{tc}(T) \). Hence one can find a \( T \)-cycle within \( \operatorname{tc}(T) \). Pick one which is inclusion maximal, say \( C = \{a_1, \ldots, a_q\} \), and assume, per absurdum, that \( C \) is strictly smaller than \( \operatorname{tc}(T) \). Pick \( a \) in \( \operatorname{tc}(T) \setminus C \). If for some \( j \), \( 1 \leq j \leq q-1 \), we have \( a_jTa_{j+1} \), then \( \{a_1, \ldots, a_j, a, a_{j+1}, \ldots, a_q\} \) makes a bigger cycle, a contradiction. The same argument shows that \( a_qTa_1 \) is impossible.

Thus only two cases are possible: i) for all \( j = 1, \ldots, q \), \( a_jTa \) or ii) for all \( j = 1, \ldots, q \), \( a_jTa_j \). Then partition \( \operatorname{tc}(T) \setminus C \) as \( C_i \) (where i holds) and \( C_{ii} \) (where ii holds). If any one of \( C_i \) or \( C_{ii} \) is empty we have an easy contradiction by Smith's consistency (if \( \operatorname{tc}(T) = C \cup C_i \) then \( C \) is ahead of \( A \setminus C \); if \( \operatorname{tc}(T) = C \cup C_{ii} \), then \( C_{ii} \) is ahead of \( A \setminus C_{ii} \)). Now we can assume \( C_i \) contains some \( a \) and \( C_{ii} \) contains some \( b \). Consider a \( T \)-path from \( a \) to \( b \); such a path remains within \( \operatorname{tc}(T) \). Let \( b' \) be the first outcome on this path outside \( C_i \). Then \( b' \) must be in \( C_{ii} \) so that \( b'Ta_1Ta_2 \ldots Ta_qTb \). We have just constructed a bigger \( T \)-cycle than \( C \) which is the desired contradiction.

Q.E.D.

3. **The Uncovered Set**

For any tournament \( T \) on \( A \) define its covering relation as: \( b \) covers \( a \) iff

\[
a \neq b, \quad bTa \quad \text{and} \quad \text{for all } c \in A: aTc \Rightarrow bTc.
\]

(4)

The covering relation is transitive, yet not complete. Its maximal elements define the uncovered set:
Definition 1 (Miller, 1977). Given a tournament $T$, its uncovered set $uc(T)$ is defined by

$$a \in uc(t) \text{ iff for no } b \in A, \ b \text{ covers } a.$$ 

Lemma 4. The uncovered set $uc(T)$ contains outcome $a$ if and only if:

$$\text{for all } b \in A, \ b \neq a, \ aTb \ \text{and/or for some } c \in A, \ aTcTb.$$  \hspace{1cm} (5)

Proof. Suppose $a$ satisfies (5) and that $b$ covers $a$. Then $bTa$ so there must exist $c$ such that $aTc$ and $cTb$. By (4) $bTc$ so $b = c$ thus $aTb$, a contradiction (since $b \neq a$). Conversely, suppose $a$ does not satisfy (5). Then for some $b \neq a$, we have $bTa$ and, for all $c$, $aTc$ and $cTb$ cannot both be true. This is just statement (4).

Q. E. D.

From Lemma 4 and Definition 1 it follows that the uncovered set of a tournament is a subset of its top cycle:

$$uc(T) \subseteq tc(T) \quad \text{all tournament } T.$$ 

Furthermore, if subset $B$ is ahead of $A \setminus B$ ($bTa$ for all $b \in B$ and all $a \in A \setminus B$) then any $b \in B$ covers any $a \in A \setminus B$. Therefore $uc$ satisfies Smith's consistency (3).

As a first example, the tournament

![Diagram of a tournament]

has top cycle $\{a, b, c, d\}$ but uncovered set $\{a, c, d\}$ since $a$ covers $b$ (or equivalently, to go from $b$ to $a$ along a $T$-path, one needs 3 steps).

Consider next the example given above: $T$ is the ordering $1 > 2 > \ldots > n$ except for $nT1$. Here $2$ covers $3$, $3$ covers $4, \ldots, (n-2)$ covers $(n-1)$, therefore $uc(T) = \{1, 2, n\}$.

The uncovered set eliminates from the top cycle some bad outcomes: in particular if our tournament is deduced from a majority relation, its uncovered
set contains only Pareto optimal outcomes (the proof is obvious in view of Lemma 4). To compute it is not too difficult either: say that \( \Theta \) is the \( |A| \times |A| \) - matrix representing tournament \( T \).

\[
\theta_{ab} = 1 \quad \text{if } aTb \quad \text{(in particular } \theta_{aa} = 1) \\
0 \quad \text{otherwise}.
\]

Then the rows of \( \Theta^2 \) which contain no zero determine the outcomes in the uncovered set (this is an easy consequence of Lemma 4).

The uncovered set is not necessarily a \( T \)-cycle (contrary to the top cycle). However, if it is not a singleton, it cannot contain a winner, i.e., an outcome beating every other uncovered outcome.

**Lemma 5** (Miller, 1977). The uncovered set of a tournament \( T \) is a singleton if and only if \( T \) has a Condorcet winner \( a \) (in this case \( uc(T) = \{a\} \)). If \( uc(T) \) is not a singleton, then it contains at least three outcomes and the restriction of \( T \) to \( uc(T) \) has no Condorcet winner: there is no outcome \( a \) in \( uc(T) \) that beats every other \( b \) in \( uc(T) \).

**Proof.** Since the covering relation is transitive and \( A \) is finite, any outcome outside \( uc(T) \) must be covered by an outcome inside \( uc(T) \). Thus if \( uc(T) = \{a\} \), outcome \( a \) covers any other outcome, so it is a Condorcet winner. Suppose now that \( T \) has no Condorcet winner, yet \( a \in uc(T) \) beats every other \( b \in uc(T) \). Set \( B = \{b \in A \setminus a \mid bTa\} \subseteq A \setminus uc(T) \). We pick \( a^* \) in the uncovered set of the restriction of \( T \) to \( B \) and prove \( a^* \in uc(T) \) (hence a contradiction) by means of (5). For all \( b \in B \) there is a \( T \)-path within \( B \) from \( a^* \) to \( b \) of length at most 2. For all \( b \in A \setminus B \), either \( b = a \) or \( aTb \). Since \( a^*Ta \) there is a \( T \)-path of length at most 2 from \( a^* \) to \( b \).

This establishes that the restriction of \( T \) to \( uc(T) \) has no Condorcet winner, which implies that its size is at least 3.

Q.E.D.

The structure of \( uc(T) \) as described in Lemma 5 cannot be specified more generally; this is the meaning of our next result.
Lemma 6. Let $A_0$ contain at least three outcomes and let $T_0$ be any tournament on $A_0$ without a Condorcet winner. Then there exists a superset $A$ of $A_0$ and a tournament $T$ on $A$ such that

i) $uc(T) = A_0$

ii) the restriction of $T$ to $A_0$ is $T_0$.

Proof. Given $A_0', T_0$ as above, choose a set $A_1$ disjoint from $A_0$ and with the same size; also pick a bijection denoted $a \rightarrow a'$ from $A_0$ into $A_1$. Define $T$ on $A = A_0 \cup A_1$ as follows

\[
\begin{align*}
&\text{on } A_1: T \text{ copies } T_0: a'Tb' \text{ iff } aT_0b \\
&\text{on } A_0: T \text{ is } T_0 \\
&\text{if } a, b \in A_0 \text{ and } a \neq b: aTb' \\
&\text{if } a \in A_0: a'Ta
\end{align*}
\]

We claim that $uc(T) = A_0$. We use Lemma 4. Pick $a \in A_0$ and prove it satisfies (5): for all $b \in A_0 \setminus a$ we have $aTb'$, $b'Tb$ hence (5); for all $b' \in A_1$, $b' \neq a'$, we have $aTb'$; finally if $b$ is any outcome in $A_0 \setminus a$ s.t. $bT_0a$ (by assumption $a$ is no Condorcet winner) then $aTb'$, $b'Ta'$.

Next pick $a' \in A_1$ and prove it violates (5): by assumption there is $b \in A_0 \setminus a$ s.t. $bT_0a$. Careful inspection reveals that a $T$-path from $a'$ to $b$ has length at least 3.

Q.E.D.

4. The Uncovered Set for Variable Issues

This section is conceptually more elaborate than the rest of the paper. It can be skipped without affecting the understanding of any subsequent section.

We think of the subset $B$ of feasible outcomes (the issue) as varying within a given set $A$ and we seek to compare the uncovered set for the restrictions
of our tournament to each particular issue. Thus we consider the mapping

$$(T, B) \rightarrow uc(T, B) = \text{uncovered set of the restriction of } T \text{ to } B$$

defined for all tournament $T \in \mathcal{T}$ and all issue $B \subseteq A$.

Two essential invariance properties of $uc$ are easy to check:

**Neutrality** (nondiscrimination among outcomes)

If $\sigma$ is a permutation of $A$ and $T^\sigma$ is defined by $aT^\sigma b \iff \sigma^{-1}(a)T^{-1}(b)$, then:

for all $T$, all $B$

$$S(T^\sigma, B) = \sigma[S(T, B)].$$

**Arrow's Independence of Irrelevant Alternatives (AIIA)**

If $T$ and $T'$ have the same restriction on $B$, then

$$S(T, B) = S(T', B), \quad \text{ (all } T, T', B).$$

Now come the conditions relating the choice sets over some issue $B$ and over some other issue $B'$. From the theory of a single agent's choice function (see, e.g., Moulin (1984)) we borrow the following axiom.

**Expansion**  For all $T, B, B'$: $S(T, B) \cap S(T, B') \subseteq S(T, B \cup B').$

**Theorem 1.** The uncovered set (viewed as the mapping $(T, B) \rightarrow uc(T, B)$) satisfies Neutrality, Arrow's IIA and Expansion. Conversely any mapping $(T, B) \rightarrow S(T, B)$ satisfying Neutrality, Arrow's IIA, Expansion and Condorcet consistency (if $T$ has a Condorcet winner $a$ on $A$, then $S(T, A) = a$) must contain the uncovered set $(uc(T, B) \subseteq S(T, B)$ all $T, B)$.

**Proof.** The first statement is proved straightforwardly. For instance, Expansion follows from Lemma 4. Let us prove the converse property. First we observe that the Expansion axiom is equivalently written as

for all sequence $B_1, \ldots, B_K$ of subsets of $A$, and all tournament $T$:

$$\bigcap_{1 \leq k \leq K} S(T, B_k) \subseteq S\left(T, \bigcup_{1 \leq k \leq K} B_k\right).$$  \hspace{1cm} (6)
Next observe that Arrow’s IIA and Condorcet consistency together imply Condorcet consistency on every restricted issue, namely:

\[
\text{for all } T, B : \{ a \in T \text{ for all } b \in B \} \implies \{ S(T, B) = a \} \quad (7)
\]

Next we deduce from Neutrality and Arrow’s IIA that if \( B \) is a triple and the restriction of \( T \) to \( B \) is a cycle (say \( B = \{ abc \} \), \( a \in T \supseteq \{ abc \} \)) then \( S(T, B) = B \).

To see this, consider any permutation \( \sigma \) of \( A \) that permutes \( a, b, c \) then \( S(T^{\sigma}, B) = S(T, B) \) by Arrow’s IIA whence by Neutrality \( S(T, B) = \sigma(S(T, B)) \).

Finally we prove the inclusion \( UC \subseteq S \). Pick \( T, B \) and \( a \in UC(T, B) \). Partition \( B \setminus \{ a \} \) as \( B_+ = \{ b \in B_T \} \) and \( B_- = \{ b \in B_T \} \). For any \( b \in B_+ \) we have by (7)

\[
S(T, \{ ab \}) = \{ a \}.
\]

For any \( b \in B_- \) there exists (by Lemma 4) some \( c \in B_+ \) such that \( a \in T \supseteq \{ abc \} \). Thus the restriction of \( T \) to \( abc \) is a cycle so \( S(T, \{ abc \}) = \{ abc \} \). We invoke now (6):

\[
a \in \left( \bigcap_{b \in B_+} S(T, \{ ab \}) \right) \cap \left( \bigcap_{b \in B_-} S(T, \{ abc \}) \right) \Rightarrow a \in S \left( T, \left( \bigcup_{b \in B_+} \{ ab \} \right) \cup \left( \bigcup_{b \in B_-} \{ abc \} \right) \right) = S(T, B).
\]

This concludes the proof of Theorem 1.

In the study of a single agent’s choice functions, two more axioms, called Chernoff and Aizerman play, together with Expansion, a prominent role:

**Chernoff:** for all \( B, B' \), all \( T : \{ B \subseteq B' \} \implies \{ S(B') \cap B \subseteq S(B) \} \)

**Aizerman:** for all \( B, B' \), all \( T : \{ S(B') \subseteq B \subseteq B' \} \implies \{ S(B) \subseteq S(B') \} \)

We let the reader check that the uncovered set satisfies Aizerman but violates Chernoff. In fact we have the following negative result.

**Lemma 7.** If \( A \) contains at least 3 outcomes, no mapping \( (T, B) \rightarrow S(T, B) \) satisfies together Condorcet consistency, Arrow’s IIA and Chernoff.

**Proof.** Take \( B' = \{ a, b, c \} \) and \( T \) such that its restriction on \( B \) is a cycle: \( a \in T \supseteq \{ abc \} \). From Chernoff and \( a \in T \supseteq \{ abc \} \) follows \( S(B') \cap \{ ab \} \subseteq S(ab) = \{ a \} \Rightarrow b \notin S(B'). \)
Similar arguments from bTc and cTa show successively $c \notin S(B')$ and $a \notin S(B')$, a contradiction.

Q. E. D.

The above result is a mini-impossibility theorem: indeed, Chernoff's axiom is necessary to represent the choice function $B \rightarrow S(B)$ as the maximal element of some fixed acyclic relation on $A$ (in fact Chernoff and Expansion together are necessary and sufficient to such rationalization; see, e.g., Moulin (1984, Theorem 2). Thus Lemma 7 states that any choice correspondence for tournaments which selects a Condorcet winner when there is one, either violates Arrow's IIA axiom or is not rationalizable. In fact, the same proof yields a slightly more general statement, namely: if $S$ is Condorcet consistent then it violates Arrow's IIA or it contains no rationalizable choice correspondence (the latter property is called subrationalizability; see Moulin (1984, Section 4)).

Remark: The top cycle choice correspondence over variable issues (namely the mapping $(T, B) \rightarrow tc(T, B)$) is amenable to similar characterizations. Just add one more axiom, namely

$$\textbf{Sen} \quad \text{for all } T, B, B': \{B \subset B' \text{ and } S(B) \cap S(B') \neq \emptyset \} \Rightarrow \{S(B) \subset S(B')\}$$

Then the top cycle is the smallest mapping $(T, B) \rightarrow S(T, B)$ satisfying Neutrality, Arrow's IIA, Expansion, Sen and Condorcet consistency. The proof is similar to that of Theorem 1. Check first that the combination of Expansion axiom and Sen's axiom is equivalent to the following strong Expansion property:

$$\{S(B) \cap S(B') \neq \emptyset \} \Rightarrow \{S(B) \cup S(B') \subset S(B \cup B')\}.$$  Then deduce from strong Expansion that if the restriction of $T$ to $B$ is a cycle, namely $tc(T, B) = B$, then $S(T, B) = B$.

5. **Copeland Winners**

We are back to a fixed issue $A$ and look for the choice correspondences $T \rightarrow S(T)$ that behave "better" than the uncovered set. If by better we mean more deterministic and easier to compute, then a very simple choice correspondence emerges.
Given a tournament $T$ on $A$, the Copeland score (c-score) of outcome $a$ is the number of outcomes it beats:

$$c(a) = \left| \{ b \in A \mid a \succ T b \} \right|.$$ 

A Copeland winner is an outcome with maximal Copeland score. Their set is denoted $C(T)$,

$$C(T) = \{ a \in A \mid c(a) \geq c(b) \text{ all } b \in A \}.$$ 

If $a$ covers $b$ then the c-score of $a$ is strictly greater than the c-score of $b$. Thus

**Lemma 8.** A Copeland winner belongs to the uncovered set

$$C(T) \subseteq \text{uc}(T) \quad \text{all tournaments } T.$$ 

To compute the set $C(T)$ of a given tournament from its matrix $\Theta$, one needs only to compute $\Theta e$ (where $e$ is the vector with all components 1) which is just $c + e$ ($c$ is the vector of Copeland scores). This compares favorably to the computation of uc (which requires us to compute $\Theta^2$, see Section 2).

However, the Copeland winners may be very poor outcomes in the restriction of the tournament to the uncovered set itself. Consider the following example with 13 outcomes. Here we have four strong outcomes abcd and nine weak outcomes $\beta_i$, $\gamma_i$, $\delta_i$, $i = 1, 2, 3$. Tournament $T$ is depicted in Figure 1.

It is understood that any arrow which is not explicitly depicted goes down. Thus $a$ is beaten by $b, c, d$ and beats $\beta_i, \gamma_i, \delta_i$, $i = 1, 2, 3$. Also, the arrow from the circle around $\beta_1, \beta_2, \beta_3$ upward to $b$ means that each $\beta_i$ beats $b$. Similarly each $\beta_i$ beats each $\gamma_i$ and so on.

The Copeland scores are:

$$\begin{cases} 
  c(a) = 9 \quad &\text{(unique Copeland winner)} \\
  c(b) = c(c) = c(d) = 8 \\
  c(\beta_i) = c(\gamma_i) = c(\delta_i) = 5
\end{cases}$$

The uncovered set is $\{abcd\}$: namely, $\beta_i$ is covered by $d$, $\delta_i$ is covered by $c$ and
\( \gamma_1 \) is covered by \( b \). Thus in the uncovered set, \( a \) is a Condorcet loser. In particular, \( a \) is not in the top cycle of the restriction of our tournament to \( \{abcd\} \).

Let us define \( tc[uc(T)] \) to be the top cycle of the restriction of tournament \( T \) to its own uncovered set \( uc(T) \). Then we have just given an example where

\[
C(T) \cap tc[uc(T)] = \emptyset.
\]

**Theorem 2.** If \( A \) contains no more than 9 outcomes, then

\[
C(T) \subseteq tc[uc(T)] \quad \text{all tournament } T.
\]

If \( A \) contains no more than 12 outcomes, then

\[
C(T) \cap tc[uc(T)] \neq \emptyset.
\]
Proof. We prove the first claim. Suppose $A$ is such that $|A| \leq 9$ and some Copeland winner $a^*$ does not belong to $B = \text{tc}[\text{uc}(T)]$. Since the restriction of $T$ to $\text{uc}(T)$ has no Condorcet winner (Lemma 5), then $B$ has size at least 3. Also it contains a cycle of size 3, say $b_1, b_2, b_3$ ($b_1, b_2, b_3, b_4$). Let $C$ be the complement of $\{a^*\} \cup B$ in $A$, with size $p$, $p \leq 5$. Each $b_i$, $i = 1, 2, 3$ can beat at most $p - 2$ outcomes in $C$, otherwise its $c$-score would be at least $(p - 1) + 2$ (since $b_i$ beats $a$ and another $b_j$) whereas the score of $a$ is at most $p$ (every $b$ in $B$ beats $a$). Thus for all $i = 1, 2, 3$ there are at least two outcomes in $C$ that beat $b_i$. Since $p \leq 5$, this implies the existence of some $x$ in $C$ that beat two among $b_1, b_2, b_3$.

We distinguish now three cases: Case 1, $|B| = 3$ so $B = \{b_1, b_2, b_3\}$. Since $x \notin B$ it must be covered by some $y \in \text{uc}(T)$. This $y$ beats two among $b_1, b_2, b_3$, therefore it is in the top cycle of $\text{uc}(T)$, namely in $B$, a contradiction. Case 2, $|B| = 4$, so $B = \{b_1, b_2, b_3, d\}$ and $x$ beats $b_1, b_2$. Now $p = |C| = 4$ so set $C = \{x, e_1, e_2, e_3\}$. By the same argument as in Case 1 we know that $x$ must be covered by $d$: thus $d$ beats $x, b_1, b_2$ and $a$; since the $c$-score of $a$ is at most 4 we deduce that each $e_i$, $i = 1, 2, 3$ beats $d$. Next invoke Lemma 5. $d$ is not a Condorcet winner in $B$ so $b_3$ beats $d$. In fact, $b_3$ beats $x$ as well (otherwise $d$ would not cover $x$); it also beats $a$ and one of $b_1, b_2$. Its $c$-score cannot be greater than 4 so each $e_i$, $i = 1, 2, 3$ beats $b_3$. So the situation is that given in Figure 2.
Take in \( e_1, e_2, e_3 \) an uncovered outcome (w.r.t. \( \{e_1, e_2, e_3\} \)). That outcome beats \( d, b_3 \) so it cannot be covered by \( b_1 \) or \( b_2 \) \((dTb_i, i = 1, 2)\) nor \( x \) \((b_3Tx)\) nor \( a \) \((B \text{ beats } a)\). So it is uncovered, but then it must be in \( B \) (since it beats some outcome in \( tc \), see Lemma 1), a contradiction.

**Case 3.** \(|B| \geq 5\). Then the c-score of \( a \) is at most 3, so every outcome in \( B \) beats at most two other outcomes in \( B \). When \(|B| \geq 6\) this is impossible, when \(|B| = 5\) this implies that they all beat exactly two other outcomes in \( B \). So every outcome in \( C \) beats every one in \( B \), having then a c-score of at least 5, a contradiction.

The proof of the second claim in Lemma 9 exactly parallels the above argument and is therefore omitted.

Q.E.D.

The example in Figure 1 shows that the second statement in Theorem 2 cannot be improved. Here is an example to show that the first statement cannot be either. We have 10 outcomes, the four strong outcomes abcd together with six weak, \( \beta_i, \gamma_i, \delta_i \), \( i = 1, 2 \). The Copeland scores are 6 for each of \( a, b, c, d \); 4 for each \( \beta_i, \gamma_i, \delta_i \); and 3 for each \( \beta_2, \gamma_2, \delta_2 \). The uncovered set is \( \{abcd\} \) with top cycle \( \{bcd\} \).

![Diagram](image_url)

**Figure 2**
6. **Solving Tournaments by Binary Trees**

The familiar method for choosing a unique outcome from a tournament is by means of a (finite) binary tree where to each terminal node is attached one outcome:

![Binary Tree Diagram]

Figure 3. A binary tree on $A = \{a, b, c, d\}$

Given $A$, a **binary tree on** $A$ is a finite tree, where each node has zero or two successors, and to each terminal node is attached one outcome of $A$, each outcome of $A$ appearing at least once. To any binary tree $\Gamma$ on $A$ we associate the following single-valued choice correspondence that **solves** every tournament $T$, i.e., selects a unique outcome $\Gamma(T) \in A$ by the following algorithm:

i) Pick a nonterminal node $n$ whose two successors $m, m'$ are terminal nodes (such a node exists: consider a path with maximal length going upward from the origin and take the node next to the least).

ii) if $x, x'$ are the outcomes attached to $m, m'$ use $T$ to determine the winner outcome $y$ ($y = x$ if $xTx'$ or $x = x'$, otherwise $y = x'$).

iii) Chop the two branches $nm$ and $nm'$, thus making $n$ a terminal node with attached outcome $y$.

iv) Repeat this operation until a tree with one single node (the origin of $\Gamma$) is left: its attached outcome is $\Gamma(T)$.
Figure 4. The reduction algorithm

For instance, if $\Gamma$ is the tree of Figure 3 and $T$ is the tournament $(aTb, aTc, bTc, bTd, cTd, dTa)$ we get $\Gamma(T) = d$.

Figure 5

The class of choice functions $T \rightarrow \Gamma(T)$ derived from arbitrary binary trees deserves attention. Where tournaments are derived from a majority relation, these choice functions, and only these, can be achieved by sophisticated voting using binary majority votes (Farquharson, 1969).

Ideally one would like to find a few axiomatic properties characterizing this class. This goal, however, seems presently out of reach. More modestly we will exhibit two properties common to all choice functions in this class, allowing us in particular to prove that no one of them can select a Copeland winner for all tournaments (Corollary to Lemma 10). We shall emphasize also their potentially bad features, notably the lack of monotonicity.
Lemma 9. For any binary tree $\Gamma$ and any tournament $T$ the outcome $\Gamma(T)$ is in the top cycle $tc(T)$.

Proof. Paint all outcomes of $tc(T)$ in red and those in $A \setminus tc(T)$ in blue. By definition of the top cycle any red outcome beats any blue one. Since there is at least one red terminal node (each outcome appears at least once), the final winner must be red.

Q.E.D.

Another way of stating Lemma 10 is this: all choice functions $T \rightarrow \Gamma(T)$ satisfy Smith's consistency axiom.

For a fixed tournament $T$ the set of outcomes $\Gamma(T)$ that are selected along some binary tree $\Gamma$ is actually the whole top cycle $tc(T)$. To see this consider the successive elimination trees $\Gamma_\sigma$: let $\sigma = \{a_1, \ldots, a_p\}$ be an ordering of $A$, and define $\Gamma_\sigma$ as:

```
  a_2
   \downarrow
  a_3
   \downarrow
a_{p-1}
   \vdots
  a_p
```

Figure 6

Fix an outcome $a \in tc(T)$; there exists an ordering $\sigma$ of $A$ such that $a = \Gamma_\sigma(T)$ (Miller, 1977). Indeed, $tc(T)$ is a cycle (Lemma 3) so we can order it as $tc(T) = \{a_1, a_2, \ldots, a_p = a\}$ where $a_T a_{p-1} T a_2 T a_1 T a_p$. Any ordering $\sigma$ starting with $a_1 \ldots a_p$ proves the claim.

We noticed earlier that the top cycle is too big (see Section 2); this is clear as well on the tree of Figure 6: outcome $a_p$ can win by beating only one (well
chosen) outcome. This poor feature is common to all binary trees where each outcome appears in exactly one terminal node (or single elimination tournaments). In those there is always at least one outcome that can win by beating no more than \(\log_2 n\) other outcomes (since any binary tree with \(n\) terminal nodes has at least one node from which the path to the origin has length no more than \(\log_2 n\)). Thus we want to consider binary trees where each outcome is attached to many terminal nodes, so that it needs to beat on average "many" outcomes to be elected.

However, by multiplying the terminal nodes where the same outcome appears, another perverse feature might emerge. Consider the following game tree and tournaments:

![Binary Tree and Tournaments Diagram](image)

**Figure 7**

In \(T_1\), we have \(bT_1a\), so \(c\) is the winner in the left branch of \(T_0\) and the overall winner is \(a\). In \(T_2\), we have \(aT_1b\). Now \(d\) is the winner in the left branch of \(T_0\), and the overall winner as well. So switching the comparison of \(b\), \(a\) in favor of \(a\) is actually fatal to \(a\)!

Say that a choice correspondence \(T \rightarrow S(T)\) (single-valued or not) is **monotonic** if for all \(a \in A\), and \(T, T' \in \mathcal{T}\),

\[
\{ a \in S(T) \text{ and } [aTb \Rightarrow aT'b, \text{ all } b \in A \setminus a] \text{ and } [bTc \Leftrightarrow bT'c, \text{ all } b, c \in A \setminus a] \}
\]

\[
\Rightarrow \{ a \in S(T') \}
\]
In words, if the only change from $T$ to $T'$ is to switch some arrows in favor of $a$, then $a$ cannot be harmed. Notice that if each outcome of $A$ is attached to exactly one terminal node of $\Gamma$, the associated choice function is monotonic.

In Section 7 we study a specific binary tree (the multi-stage elimination process) which overcomes both difficulties mentioned above: its choice function $T \rightarrow \Gamma(T)$ is monotonic, and a selection of the uncovered set $\text{uc}(T)$. For the time being we state one more property of the choice functions generated by binary trees.

Given a tournament $T$ on $A$ and a proper subset $B \subset A$, $|B| \geq 2$, we say that $B$ is an adjacent set of $T$ if we have

$$\text{for all } b, b' \in B, \text{ for all } a \in A \setminus B: a \text{Ta} \iff a \text{Ta}' .$$

We denote by $\mathcal{T}_B$ the subset of tournaments which have $B$ as an adjacent set. If $T \in \mathcal{T}_B$ (and only in that case), we can merge all outcomes of $B$ into a single outcome and define unambiguously a restricted tournament, denoted $T_B'$, on $(A \setminus B) \cup \{B\}$ (the set with cardinality $|A| - |B| + 1$ where all of $B$ is a single outcome).

Lemma 10. Let $T \rightarrow \Gamma(T)$ the choice function derived from a binary tree $\Gamma$. Then for all proper subsets $B \subset A$ we have:

$$\text{for all } T, T' \in \mathcal{T}_B: \{T_B = T'_B \} \Rightarrow \{\Gamma(T) \in B \iff \Gamma(T') \in B\} \quad (8)$$

and also

$$\text{for all } T, T' \in \mathcal{T}_B: \{T_B = T'_B \text{ and } \Gamma(T) \in A \setminus B\} \Rightarrow \{\Gamma(T') = \Gamma(T)\} \quad (9)$$

Proof. Fix a binary tree $\Gamma$ on $A$. Consider the binary tree on $(A \setminus B) \cup \{B\}$ obtained by identifying all outcomes in $B$ to a single one, denoted $\{B\}$. Denote $\Gamma_B$ this binary tree (the tree is the same but there are fewer distinct nodes). Property (8) is implied by the following:

$$\text{for all } T \in \mathcal{T}_B: \Gamma(T) \in B \iff \Gamma_B(T_B) = \{B\} . \quad (10)$$
Now consider the reduction algorithm used to compute \( \Gamma(T) \) and let \( \Delta(\Gamma, T) \) be the mapping associating to each node of the tree \( \Gamma \) the corresponding provisional winner (Figure 5 describes such a mapping). A straightforward induction argument (omitted for the sake of brevity) reveals that \( \Delta(\Gamma, T) \) and \( \Delta(\Gamma_B, T_B) \) are consistent in the following sense: to any given node of the tree, \( \Delta(\Gamma, T) \) associates an outcome in \( B \) if and only if \( \Delta(\Gamma_B, T_B) \) associates (to that node) the outcome \( \{B\} \). Applying this to the origin node of \( \Gamma \) yields (10). Similarly, the same downward induction argument shows: to any given node of the tree, \( \Delta(\Gamma, T) \) associates \( a \in A \setminus B \) if and only if \( \Delta(\Gamma_B, T_B) \) associates to that node outcome \( a \) as well. Applying this to the origin yields

\[
\text{for all } T \in \mathcal{T}_B, \text{ all } a \in A \setminus B \quad \{\Gamma(T) = a\} \iff \{\Gamma_B(T_B) = a\}
\]

This in turn establishes (9).

Q. E. D.

**Corollary to Lemma 10.** Suppose the size of \( A \) is at least 8. Then there is no binary tree \( \Gamma \) of which the choice function always selects a Copeland winner:

\[
\text{for all } T \in \mathcal{T}: \quad \Gamma(T) \in C(T) .
\]

**Proof.** First we observe that when \( A \) contains four outcomes or less, there is a binary tree of which the choice function always picks a Copeland winner, for instance

```
     a
    / \
   b   c
   /    \
  d     
```

This in the statement of the Corollary, some lower bound on the size of \( A \) is needed. We do not know, however, the minimal lower bound.

For \(|A| \geq 8\) we prove that no selection of the Copeland winners correspondence satisfies (8). It is enough to prove the claim for \(|A| = 8\) (for \(|A| > 8\) consider
tournaments with a top cycle of size 3). Set \( A = \{a, b, c, d, e, x, y, z\} \) and \( B = \{a, b, c, d, e\} \). We construct two tournaments \( T, T' \) both in \( \mathcal{T}_B \) such that \( T_B = T'_B \) and yet \( C(T) = \{a\}, \ C(T') = \{z\} \). The tournament \( T_B = T'_B \) is like the ordering \( \{B\} > x > y > z \) except for \( zT[B] \). On \( B \), the restriction of \( T \) and \( T' \) are respectively

\[
T: \quad \text{ordering } a > b > c > d > e
\]

\[
T': \quad \text{tournament where each outcome has Copeland score 2, e.g.,}
\]

```
  a
 /\  \\
/  \ \\
 b - e - c
    \\
  d
```

The Copeland scores in \( T \) and \( T' \) are then

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>x</th>
<th>y</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>( T' )</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

\( \text{winner: } a \)

\( \text{winner: } z \)

Q.E.D.

7. **The Algorithm: Sophisticated Agenda**

We describe first the multi-stage elimination tree. There is (up to permuting the outcomes) exactly one such tree for each size of \( A \).

\[
A = \{1, 2\}
\]

\[
\Gamma_2(1, 2)
\]

\[
A = \{1, 2, 3\}
\]

\[
\Gamma_3(1, 2, 3)
\]

i.e.,

\[
\Gamma_2(1, 2) \quad \Gamma_2(1, 3)
\]

\[
\Gamma_3(1, 2, 3)
\]

23
and so on:

\[ A = \{1, 2, \ldots, n\} \]

\[ \Gamma_{n-1}(1, 2, \ldots, n-1) \]

\[ \Gamma_{n-1}(1, 2, \ldots, (n-2), n) \]

\[ \Gamma_n(1, 2, \ldots, n) \]

This tree was first proposed in the context of strategic voting (Moulin, 1979). Consider the induction formula (11) and suppose we start from the origin and move upward on the tree: the first decision is a choice between \( \Gamma_{n-1}(1, 2, \ldots, n-1) \) —thus eliminating \( n \) for good—and \( \Gamma_{n-1}(1, 2, \ldots, (n-2), n) \) —thus eliminating \( n-1 \) for good. Hence reading the tree upward (i.e., viewing it as an extensive game form, see Moulin (1979)) amounts to a successive elimination procedure: first \( n-1 \) or \( n \), then \( n-1 \) or whichever survived from \( n-1, n \), and so on. Of course, solving the tree downward (to compute its associated choice function \( T \rightarrow \Gamma(T) \)) is quite another story:

**Theorem 3** (Shepsle and Weingast, 1982). The choice function \( T \rightarrow \Gamma_n(T) \) achieved by the multi-stage elimination tree on \( A = \{1, 2, \ldots, n\} \) is given by the following algorithm called **sophisticated agenda**.

\[
\begin{align*}
\alpha_1 &= 1; \text{ for } i = 2, \ldots, n: \alpha_i = \begin{cases} 
1 & \text{if } iT\alpha_{i-1} \text{ and } iT\alpha_{i-2} \text{ and } \ldots \text{ and } iT\alpha_1 \\
\alpha_{i-1} & \text{if for some } j, 1 \leq j \leq i-1: \alpha_j T_i
\end{cases} \\
\alpha_n &= \Gamma_n(T) .
\end{align*}
\]
Proof. By induction on \( n \). For \( n = 1, 2 \) the claim is obvious. Suppose it is true up to \( n - 1 \) and consider a tournament \( T \) on \( \{1, 2, \ldots, n\} \). Denote \( T^{-n} \) (resp. \( T^{-,(n-1)} \)) the restriction of \( T \) to \( \{1, 2, \ldots, (n-1)\} \) (resp. to \( \{1, 2, \ldots, (n-2), n\} \)).

Set \( \beta_{n-1} = \Gamma_{n-1}(T^{-n}) \): by the induction assumption, it is computed by (12), which raises a sequence \( \beta_1, \ldots, \beta_{n-1} \). Similarly \( \gamma_{n-1} = \Gamma_{n-1}(T^{-,(n-1)}) \) is computed by (12) along a sequence \( \gamma_1, \ldots, \gamma_{n-1} \). Next by the inductive definition of \( \Gamma_{n'} \) (11), outcome \( a = \Gamma_n(T) \) is worth:

\[
a = \beta_{n-1} \text{ if } \beta_{n-1} T \gamma_{n-1} ; \quad a = \gamma_{n-1} \text{ if } \gamma_{n-1} T \beta_{n-1} .
\] (13)

Now consider the sequence \( \alpha_1, \ldots, \alpha_n \) defined by (12) from \( \{1, 2, \ldots, n\} \). By construction we have

\[
\{ \text{for } i = 1, \ldots, n-2: \alpha_i = \beta_i = \gamma_i \} \text{ and } \{ \alpha_{n-1} = \beta_{n-1} \} .
\]

In order to prove \( \alpha_n = a \) we distinguish three cases.

Case 1. \( \gamma_{n-1} = \gamma_{n-2} \).
Then for some \( i, 1 \leq i \leq n-2 \), \( \gamma_i T \alpha_n \). Since \( \alpha_i = \gamma_i \) this implies \( \alpha_n = \alpha_{n-1} \). On the other hand, \( \beta_{n-1} T \beta_{n-2} \) (by (12) again; remember the convention xTx), so from \( \gamma_{n-1} = \beta_{n-2} \) we get \( \beta_{n-1} T \gamma_{n-1} \). Therefore by (13) \( a = \beta_{n-1} \) so \( a = \alpha_{n-1} = \alpha_n \) and we are home.

Case 2. \( \gamma_{n-1} = n \) and \( \beta_{n-1} = \beta_{n-2} \).
In this case \( n \) beats any \( \alpha_i, 1 \leq i \leq n-2 \) (definition of \( \gamma_{n-1} \)) and \( \beta_{n-1} \) is one of them. So \( n \) beats \( \alpha_1 \ldots \alpha_{n-1} \) and \( \alpha_n = n \). On the other hand, \( \gamma_{n-1} = n \) beats \( \beta_{n-1} = \alpha_{n-2} \) so by (13) we get \( a = n \).

Case 3. \( \gamma_{n-1} = n \) and \( \beta_{n-1} = n-1 \).
Here both \( n \) and \( (n-1) \) beat \( \{ \alpha_1, \ldots, \alpha_{n-2} \} \). Thus \( \alpha_n = n \) if \( nT(n-1) \) and \( (n-1)T(n-1) \) if \( (n-1)Tn \). By (13), on the other hand, \( a \) is just the same.

Q. E. D.

Algorithm (12) was first introduced in the context of strategic voting, whence its name (Shepsle and Weingast, 1982). It describes the sophisticated equilibrium.
of the voting by successive elimination game form (described above) with agenda \( n, n-1, \ldots, 1 \).

Whenever \( A \) is ordered by an arbitrary permutation \( \sigma = \{a_1, \ldots, a_n\} \) we also denote by \( a_1 \ldots a_n \) the corresponding sequence in (12) (where \( \alpha = a_1, \alpha_i = a_i, \) or \( \alpha_{i-1} \ldots \) etc.) and we set \( \alpha_n = \Gamma_{\sigma}(T) \).

We study now the choice functions \( T \rightarrow \Gamma_{\sigma}(T) \) in the perspective suggested by Section 6. First a straightforward argument shows that \( \Gamma_{\sigma} \) is monotonic. Next we observe that the sophisticated agenda algorithm ends up within the uncovered set.

**Lemma 11.** Fix an outcome set \( A \) and a tournament \( T \). Let \( \sigma \) be any ordering of \( A \) and consider the set \( PW_{\sigma} \) of provisional winners \( \{a_1, \ldots, a_n\} \) in algorithm (12). Then \( a_n \) beats every other element in \( PW_{\sigma} \). Moreover, no outcome (outside \( PW_{\sigma} \)) can beat all elements of \( PW_{\sigma} \). In particular, \( \Gamma_{\sigma}(T) \in \text{uc}(T) \) all \( T \in \mathcal{T} \).

**Proof.** By the very definition of the algorithm, if \( \alpha_{i+1} \neq \alpha_i \) then \( \alpha_{i+1} \) must beat every outcome in the set \( \{a_1, \ldots, a_i\} \), hence our first statement. To prove the second statement suppose \( b \) outside \( PW_{\sigma} \) beats every element in \( PW_{\sigma} \). Say \( b = a_k \) for some \( i, 1 \leq k \leq n \). Then \( b \) beats \( \alpha_1, \ldots, \alpha_{k-1} \) and so equals \( \alpha_k \), a contradiction. The last statement is by definition of the covering relation.

Q.E.D.

Strikingly enough, we can say more.

**Theorem 4** (Banks, 1984). The sophisticated agenda algorithm ends up in the top cycle of the uncovered set:

\[ \Gamma_{\sigma}(T) \in \text{tc}[\text{uc}(T)] \text{ all ordering } \sigma \text{ of } A, \text{ all tournament } T. \]

**Proof.** We state first an auxiliary result. Let \( T \) be any tournament and \( a \) an outcome outside \( B = \text{tc}[\text{uc}(T)] \). Then

\[ \{a \text{ is covered by some } b \in B\} \text{ and/or } \{\text{for all } b \in B, \ bTa \} \quad (14) \]

To prove the claim, suppose \( a \) is not covered by any \( b \in B \). If \( a \) itself is uncovered, then \( bTa \) holds by definition of the top cycle. If \( a \) is covered, it is by some
Now take any \(b \in B\), then \(b T d\) (definition of \(tc\)) so \(b T a\) (by definition of the covering relation (4)).

Now to the proof of Theorem 2. Fix an ordering \(\sigma\) and a tournament \(T\). Suppose first that some \(\alpha_i \neq \alpha_n\) belongs to \(B\). Since \(\alpha_i T \sigma \alpha_n\), (12), we derive from (14) that \(\alpha_i\) is covered by some \(b \in B\), which contradicts Lemma 11. Thus all \(\alpha_i, i = 1, \ldots, n\), are outside \(B\). Suppose next no one of \(\alpha_1, \ldots, \alpha_n\) is covered by an outcome in \(B\). Then by (14) any \(b \in B\) beats all \(\alpha_1, \ldots, \alpha_n\), a contradiction of Lemma 11. So there is a largest integer \(i, i \leq n - 1\), such that \(\alpha_i\) is covered by some outcome \(b\) in \(B\). Then \(b\) beats \(\alpha_1 \ldots \alpha_i\) (since \(\alpha_i\) beats all its previous winners) as well as \(\alpha_{i+1}, \ldots, \alpha_n\) (by (14)), a contradiction of Lemma 11 again.

Q.E.D.

The algorithm of sophisticated voting requires at most to use the \(n(n-1)/2\) pairwise comparisons of tournament \(T\): this happens if, and only if, tournament \(T\) is an ordering and \(\sigma = \{a_1, a_2, \ldots, a_n\}\) is exactly the reverse ordering. Thus, in general, it is cheaper to compute the outcome of this algorithm than the Cope-land winners.

Viewed as a (single-valued) choice correspondence, the mapping \(T \rightarrow \Gamma_\sigma(T)\) lacks Neutrality: the particular choice of \(\sigma\) breaks the symmetry across outcomes. We can restore this by considering all orderings of \(T\) and their corresponding winners. Formally,

\[ S^\sigma(T) = \{a \mid \text{for some ordering } \sigma \text{ of } A: a = \Gamma_\sigma(T) \} . \]

**Lemma 12.** An outcome \(a\) belongs to \(S^\sigma(T)\) if and only if there exists a subset \(B \subseteq A\) such that

i) the restriction of \(T\) to \(B\) is an ordering, with \(a\) as its top outcome;

ii) no outcome outside \(B\) beats every outcome in \(B\).
**Proof.** If: Given B that satisfies i) and ii) order it according to T, say B = \{a_1, a_2, \ldots, a_p = a_j\} (where i < j ⇒ a_j T a_i). Then for any ordering \( T \) of A starting with \( \{a_1, \ldots, a_p\} \), the sophisticated agenda algorithm has \( a_p \) for final winner.

Only if: If \( a = \rho \) satisfies i) and ii) (Lemma 11).

Q. E. D.

The uncovered set uc(T) can be defined in a similar way as \( S^\ast(T) \): outcome a belongs to uc(T) if and only if there exists a subset B ⊆ A such that i) a ∈ B and a beats every outcome in B \( \Delta \) a, ii) no outcome outside B beats every outcome in B.

Reinforcing Lemma 5 we have now

**Lemma 13** (Banks, 1984). The set \( S^\ast(T) \) is a singleton if and only if T has a Condorcet winner. If \( S^\ast(T) \) is not a singleton then its size is at least three.

**Proof.** Suppose T has no Condorcet winner. For all a ∈ B there is b that beats a: for any ordering \( \sigma \) starting with \( a_1 = a, a_2 = b, \ldots \) the output \( PW_\sigma \) cannot be a. Hence \( S^\ast(T) \) contains at least two elements, say x, y. We prove now that it contains also a third one. Say that xTy. Since \( y \in S^\ast(T) \) is not covered by x, there is z such that yTz, zTx (Lemma 4). Consider an ordering \( \sigma \) starting with \( a_1 = x, a_2 = z, \ldots \). The algorithm of provisional winner, then, cannot produce x (by zTx we have \( \alpha_2 = z \)) nor y (since \( x = \alpha_1 \) beats y).

Q. E. D.

To emphasize the power of Theorem 4, one needs to be convinced that the inclusion \( S^\ast(T) \subseteq \text{tc(uc(T))} \) need not be an equality. Actually, we have

**Lemma 14.** If \( |\text{tc(uc(T))}| < 4 \) then \( S^\ast(T) = \text{tc(uc(T))} \). If \( |\text{tc(uc(T))}| \geq 5 \) then \( S^\ast(T) \not\subseteq \text{tc(uc(T))} \) is possible.

**Proof.** If T has no Condorcet winner, the size of \( \text{tc(uc(T))} \) is at least 3 (Lemma 5) and so is the size of \( S^\ast(T) \) (Lemma 13). Suppose now \( \text{tc(uc(T))} = \{a, b, c, d\} \). Up to relabeling the outcomes, the restriction of T to \( \text{tc(uc(T))} \) must be:
Suppose \( a \) is not in \( S^T(T) \). Since the restriction of \( T \) to \( \{a, b, c\} \) is transitive with top outcome \( a \), there exists (Lemma 12) an outcome \( x \) beating \( a, b \) and \( c \). We can even choose \( x \) uncovered (if it is not, take some uncovered outcome \( x' \) that covers \( x \)). Since \( x \) beats some outcome in the top cycle of \( uc(T) \), it is itself in \( tc(uc(T)) \) so \( x = d \), a contradiction.

Suppose next \( b \) is not in \( S^T(T) \): since \( b \) is not covered by \( a \) in \( T \), there is a \( z \) (outside \( tc(uc(T)) \)) such that \( bTz \), \( zTa \). Now \( \{b, c, z\} \) is transitive for \( T \), so (Lemma 12) there exists \( x \) beating \( b, c \), and \( z \). Again \( x \) can be chosen uncovered, hence in \( tc(uc(T)) \). But none of \( a, d \) can do the job. Similar arguments show that \( c \) and \( d \) both belong to \( S^T(T) \).

To prove the second statement of Lemma 14, we exhibit an example:

![Diagram](image)

**Figure 8**

Here \( uc(A) = \{a, b, c, d, e\} \) (\( x \) is covered by \( c \), \( y \) by \( b \))

\[
\text{tc}(uc(A)) = \{a, b, c, d, e\} \quad \text{(a cycle is } aTbTcTdTeTa)\]

yet \( S^T(T) = \{a, b, c, d\} \). Indeed \( e \notin S^T(T) \) since the subsets \( B \) over which \( T \) is transitive and with \( e \) on top are:

\[
\begin{align*}
e, x, a & : \text{beaten by } c \\
e, a, y & : \text{beaten by } d \\
e, y, x & : \text{beaten by } b .
\end{align*}
\]

Remark: The first statement in Lemma 14 and its proof are taken from Banks (1984).
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