TIME-DEPENDENT SOLUTIONS OF A NONLINEAR SYSTEM ARISING IN SEMICONDUCTOR THEORY, II: BOUNDEDNESS AND PERIODICITY

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TIME-DEPENDENT SOLUTIONS OF A NONLINEAR SYSTEM ARISING IN SEMICONDUCTOR THEORY, II: BOUNDEDNESS AND PERIODICITY

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Abstract

A reasonably standard model in semiconductor theory considers a system consisting of parabolic conservation laws for electron and hole densities with a Poisson equation (at each t) for the electrostatic potential; these may also be coupled with another parabolic equation for temperature. Existence of global solutions (on $\mathbb{R}^+ \times \Omega$) was shown in [6] based on a maximum principle estimate. A somewhat different estimate is obtained here from which one can obtain boundedness on $\mathbb{R}^+$ for solutions with bounded data. Another consequence is the existence of periodic solutions corresponding to time-periodic boundary data.
1. Introduction

This paper continues from [6] the investigation of a nonlinear system of partial differential equations arising in semiconductor theory\(^1\). The system under consideration is

\begin{align}
(1.1) \quad & \dot{u} - \nabla \cdot [D \nabla u - \mu u \nabla \psi] = -R \\
(1.2) \quad & \dot{v} - \nabla \cdot [E \nabla v + \nu v \nabla \psi] = -R \\
(1.3) \quad & -\Delta \psi = N - u + v \quad \text{(at each t)} \\
(1.4) \quad & \dot{\theta} - \nabla \cdot F \nabla \theta = \mu u + \nu v \|\nabla \psi\|^2 - (D \nabla u - E \nabla v) \cdot \nabla \psi \\
(1.5) \quad & R := (uv - n^2_1)/\tau
\end{align}

The spatial region under consideration is a bounded region \(\Omega\) in \(\mathbb{R}^m\) (clearly only \(m \leq 3\) can have physical significance) with sufficiently smooth boundary \(\partial \Omega\); We set \(Q := \mathbb{R}^+ \times \Omega\) and \(\Sigma = \mathbb{R}^+ \times \partial \Omega\) We assume Dirichlet conditions

\begin{align}
(1.6) \quad & \theta|_{\Sigma} = \tilde{\theta}, \quad \psi|_{\Sigma} = \tilde{\psi}
\end{align}

for (1.3), (1.4) but, as a matter of mathematical interest, use first order conditions of the form

\begin{align}
(1.7) \quad & u\cdot n_1 + \phi_1(\cdot, u, v) = 0 = v\cdot n_2 + \phi_2(\cdot, u, v) \quad \text{on } \Sigma
\end{align}

for (1.1), (1.2) where the conormal derivatives \(u\cdot n\) and \(v\cdot n\) are defined as

\begin{align}
& u\cdot n := D \nabla u \cdot n \\
& v\cdot n := E \nabla v \cdot n
\end{align}

\(^1\) Physically, \(u\) and \(v\) represent electron and hole densities, \(\psi\) is the electrostatic potential, \(\theta\) is the temperature (although we also introduce \(T := \mathcal{A}[\theta]\) where \(\mathcal{A}\) denotes a local averaging process), \(R\) is the recombination rate (given essentially as in the Shockley-Read-Hall "trapping" model; typically \(\tau\) is presented as \([\tau (u+n) + \tau (v+n)]\) but we will modify this below), \(N\) is the net intrinsic charge density, \(\mu\) and \(\nu\) are mobilities, \(D\), \(E\) and \(F\) are diffusion coefficients; physical units have been chosen to reduce various constants to unity. See, for example, [7], [8], etc.
(\vec{n} = \vec{n}(x) = \text{unit outward normal}), interpreted in the usual weak sense as appropriate.

We continue under essentially the same hypotheses as in [6] (see Section 2). The significant change is adjunction of a growth condition on \( \tau \) which permits a rather different maximum principle argument than was used there to obtain a new estimate. Whereas the previous estimate permitted exponential growth for \( u, v \) —even for bounded data—the new bound is constant. Thus it follows that with bounded data one has a bounded solution. It also follows that one can define a bounded invariant set for the Poincare period map and this is then used to demonstrate existence of a time-periodic solution corresponding to periodic boundary conditions in (1.6), (1.7).

2. Hypotheses and Some Results from [6]

We begin by recalling from [6] the set of hypotheses under which results were obtained there.

(H1) The diffusion coefficients \( D, E \) are positive continuous functions of \((t, x) \in Q, \nabla \psi \in \mathbb{R}^m, T \in \mathbb{R}^+ \). In the presence of an \textit{a priori} \( L^\infty \) bound on \( \nabla \psi \) one has a uniform lower bound \( \alpha > 0 \) and a uniform Lipschitz coefficient \( \beta \) for the dependences on \( \nabla \psi, T \). Further, \( D(\cdot, 0, 0), E(\cdot, 0, 0) \) are bounded \textit{a priori}. The diffusion coefficient \( F \) is in \( L^\infty(\Omega) \) with \( F \geq \alpha \).

(H2) \( \mu, \nu \) are positive \(^2\) constants.

(H3) \( N \) is a bounded, measurable function of \( x \in \Omega; \frac{n^2}{1} \) and \( \tau \) are positive functions of \((t, x) \in Q, \nabla \psi \in \mathbb{R}^m, T \in \mathbb{R}^+ \) and also, for \( \tau \), of \( u, v \in \mathbb{R}^+ \), measurable in \((t, x)\) and continuous in the other variables. One has a uniform upper bound \( M \) on \( \frac{n^2}{1} \) and, in the presence of \textit{a priori} bounds on \( \nabla \psi, u, v \), one has the uniform Lipschitz constant \( \beta \) for the dependences of

\(^2\) In [6] the condition (H2) required nonnegativity rather than positivity for \( \mu, \nu \) and the condition (2.1) in (H3) did not appear. Further, the coefficients \( D, E \) are now taken continuous in \( x \) as well as in \( t, T \).
and $\tau$ on $\nabla \psi, u, v$. Finally, one has the growth condition $^2$

$$1/M \leq \tau \leq \gamma_0 + \gamma r \quad \text{for} \quad 0 \leq u, v \leq r \in \mathbb{R}^+$$

with $\gamma < 1/\|N\|_\infty \max\{\mu, \nu\}$.

(H4) The boundary data $\tilde{\psi}, \tilde{\theta}$ in (1.6) are such that the solution $\psi_0$ of

$$-\Delta \psi_0 = 0 \quad \text{on} \ \Omega \ \text{for each} \ t, \ \psi_0|_\Sigma = \tilde{\psi}$$

is in $L^\infty (\mathbb{R}^+ \to C^1(\tilde{\Omega}))$ with $\|\nabla \psi_0\|_\infty \leq M$ and the solution $\theta_0$ of

$$\dot{\theta}_0 - \nabla \cdot F \nabla \theta_0 = 0 \quad \text{on} \ Q, \ \theta_0|_\Sigma = \tilde{\theta}$$

(with the given initial condition if one is specified, else any solution)

is in $W = C(\mathbb{R}^+ \to H^1(\Omega))$ with $\theta_0 \geq 0$.

(H5) The functions $\phi_1, \phi_2$ defining the boundary conditions (1.7) are functions of $(t, x) \in \Sigma$ and $u, v \in \mathbb{R}^+$, measurable in $t, x$ and continuous in $u, v$. One has $\phi_1$ increasing in $u$ and $\phi_2$ increasing in $v$ while for some positive constant $K_0$ one has

$$\phi_1(\cdots, 0, s) \leq 0 \leq \phi_1(\cdots, K_0, s), \ \phi_2(\cdots, s, 0) \leq 0 \leq \phi_2(\cdots, s, K_0)$$

for $s \in \mathbb{R}^+$. In the presence of a priori bounds on $u, v$ one has the uniform Lipschitz constant $\beta$ for the dependence of $\phi_1$ on $v$ and of $\phi_2$ on $u$.

(H6) The averaging map $\mathcal{U} : H^{1-\epsilon}(\Omega) \to C(\tilde{\Omega})$ has Lipschitz bound $\beta$. For $m = 1$ one can take $T = Q_\theta := \theta$. We assume the bound on $\tilde{\theta}$ in (H4) gives a uniform bound on $\mathcal{O}[\tilde{\theta}]|_\Sigma$.

Note that these are the same hypotheses as in [6] (somewhat rearranged) except as noted in footnote 2.

We now proceed to recall/extract from [6] some results about the system (A) consisting $^3$ of

3 As in [6] we have used (1.5) and have carried through the differentiation $\nabla \cdot \mu u \nabla \psi$ and $\nabla \cdot \nu \nabla \psi$ in (1.1), (1.2), using (1.3) and (H2), to obtain (2.2), (2.3).
\( (2.2) \quad \dot{u} - \nabla \cdot D\nabla u + \mu \nabla \psi \cdot \nabla u + \mu u(u - v) + \left[ -\mu N + v/\tau \right] u = \frac{n_1^2}{\tau} \)

\( (2.3) \quad \dot{v} - \nabla \cdot E\nabla v - \nu \nabla \psi \cdot \nabla u + \nu (v - u) + \left[ \nu N + u/\tau \right] v = \frac{n_1^2}{\tau} \)

\( (2.4) \quad -\Delta \psi = N - u + v \)

\( (2.5) \quad T := \alpha[\theta], \quad \dot{\theta} - \nabla \cdot F\nabla \theta := \left[ -\nabla \psi \cdot q \right] +, \quad q := \left[ -\mu u \nabla \psi + D\nabla u \right] + \left[ -\nu \nabla \psi - E\nabla v \right] \)

with the boundary conditions (1.6), (1.7) and subject to the hypotheses (H1) - (H6).

**THEOREM 1:** Consider the system \( \mathcal{S} = \{(2.2) - (2.5), (1.6), (1.7)\} \) subject to (H1) - (H6) and with nonnegative initial data \( u(s), v(s) \in L_{+}^\infty(\Omega), \quad \theta(s) \in L_{+}^\infty(\Omega) \cap H^{1-\epsilon}(\Omega) \).

Then there is a unique solution \( (u, v, \psi, \theta) \) on \( [s, \infty) \times \Omega \) with \( u, v, \theta \geq 0 \). Further, we have (in the presence of bounds on the initial data):

\[ [i] \quad u, v \text{ are in } L_{+}^\infty(\Omega) \text{ for each } t \text{ with, at worst, exponential growth in } t; \]

they are in \( L^2(\to H^{-1}(\Omega)) \) on bounded \( t \)-intervals. Also, \( \dot{u}, \dot{v} \) are in \( L^2(\to H^{-1}(\Omega)) \) on bounded \( t \)-intervals \( [s, t_{\text{\#}}] \).

\[ [ii] \quad \theta \text{ is continuous in } t \text{ to } H^{1-\epsilon}(\Omega) \text{ so } T := \alpha[\theta] \text{ is continuous on } Q. \]

\[ [iii] \quad \text{For } 0 \leq s \leq t^{\text{\#}}, \text{ the map: } [u(s), v(s), \theta(s)] \mapsto [u(t^{\#}), v(t^{\#})\theta(t^{\#})] \text{ is Lipschitzian on } L^2(\Omega) \times L^2(\Omega) \times H^{1-\epsilon}(\Omega) \text{ —subject to the nonnegativity and the bounds on the initial data assumed above—}

with constant \( \beta \) independent of \( s \in [0, t^{\text{\#}}] \) for fixed \( t^{\text{\#}}. \)

\[ [iv] \quad \text{The coefficients } D, E \text{ are continuous. For any bounded } t \text{-interval and with assumed bounds on the initial data, one has bounds on } D, E \text{ and on the}

\text{\( L^2([s, t_{\text{\#}}] \times \Omega) \)-norms of } D^{1/2}\nabla u \text{ and } E^{1/2}\nabla v. \]

**Proof:** The existence and uniqueness on \( [s, \infty) \times \Omega \) come from Theorem 4 and Remark 1 of [5] with a trivial shift of initial time. Property \([i]\) follows from Theorem 1 there and \([ii]\) follows from Lemma 2. The Gronwall estimates (5.6), (61.3), (6.16), etc., of [5] give \([iii]\) and \([iv]\) follows from (H1) here and the continuity of \( T := \alpha[\theta] \) and (6.3) in [5]. \( \square \)
3. A Maximum Principle Estimate

We consider, in this section, only the pair of equations (2.2), (2.3) — considering $D, E, \nabla \psi, n_i^2$ as given (specified on $Q \text{ a priori}$) and $\tau = \tau(\cdot, u, v)$ — with the boundary conditions (1.7) and initial data $u(0), v(0)$. The object of this section is to exhibit a constant $K$ — depending only on $\mu, \nu, \|N\|_\infty$, and the constants $M, \gamma_0, \gamma$ of (H3) — such that

\begin{equation}
0 \leq u, v \leq \max \{K, K_0, \|u(0)\|, \|v(0)\|\} =: \hat{K}
\end{equation}

with $K_0$ as in (H5). To obtain $K$ we look at the quadratic polynomials

\begin{align}
\xi^2 - \|N\|_\infty \xi - M^2 / \min\{\mu, \nu\}, \\
(1 - \mu \gamma \|N\|_\infty) \xi^2 - \mu \gamma_0 \|N\|_\infty \xi - M, \\
(1 - \nu \gamma \|N\|_\infty) \xi^2 - \nu \gamma_0 \|N\|_\infty \xi - M,
\end{align}

and let $K$ be the largest of the roots of these (so each of these polynomials is strictly positive for $\xi > K$).

Our strategy in proving (3.1) will initially use a pointwise analysis and then later use an approximation argument to remove the regularity assumption necessitated by that analysis.\footnote{A similar argument could have been used to prove the estimate of Theorem 1 of [6]. In contrast, we have not found a "weak" argument for the present estimate.}

LEMMA 2: Consider $(u, v)$ satisfying (2.2), (2.3), (1.7) on $Q_{p, q} := (0, t) \times \Omega$. Suppose one has: $D, E \geq \alpha > 0; \mu, \nu > 0; 0 < n_i^2 \leq M; \tau(\cdot, u, v)$ satisfying (2.1)

Suppose (H5) holds with strict inequalities and with $K_0$ taken so $0 < u(0), v(0) \leq K_0$ also. Let $K$ be the largest of the roots of the polynomials (3.2). Assume all the functions involved ($D, E, u, v, \nabla \psi$, etc.) are smooth. Then the solution $(u, v)$ satisfies $0 < u, v \leq \max\{K_0, K\}$ on $Q_{p, q}$. 

\[\]
Proof: We first show $u, v > 0$ by contradiction. Else, as $\tilde{Q}_\alpha$ is compact, there is a smallest $\tilde{t} \geq 0$ at which, for some $\bar{x} \in \tilde{Q}_\alpha$, one has, say, $u(\tilde{t}, \bar{x}) = 0$. Clearly one cannot have $\tilde{t} = 0$ as $u(0, \cdot) > 0$ so $\tilde{t} > 0$ and $u(\tilde{t}, \cdot)$ is minimized at $\bar{x}$. If $\bar{x} \in \partial \Omega$ one must have $\nabla u \cdot n \leq 0$ at $(\tilde{t}, \bar{x})$ so (1.7) would give $\phi_1(\tilde{t}, \bar{x}, 0, v) > 0$, contradicting the strict inequality $\phi_1(\cdot, \bar{x}, 0, v) < 0$ assumed in (H5). If $\bar{x} \in \Omega$ one has, at $(\tilde{t}, \bar{x})$, that $u = 0$, $\dot{u} \leq 0$, $\nabla \cdot D\nabla u \geq 0$, $\nabla u = 0$ so (2.2) gives $0 \geq n_1^2 / \tau$ which contradicts the positivity of $n_1^2$. Similarly, it is impossible for $v$ to vanish "first." Thus, $u, v > 0$ on $\tilde{Q}_\alpha$.

The argument that $u, v \leq K$ is similar but more complicated. Suppose the maximum for $u, v$ were attained—say, by $u$—at $(\tilde{t}, \bar{x})$. If $\tilde{t} = 0$ we are through; if $\tilde{t} > 0$ and $\bar{x} \in \partial \Omega$ one has $\nabla u \cdot n \geq 0$ so (1.7) would give $\phi_1(\tilde{t}, \bar{x}, U, v) \leq 0$ (with $U := \max \{u\} = u(\tilde{t}, \bar{x})$) contradicting (H5) if $U \geq K_0$. Suppose, then, one had $\bar{x} \in \Omega$ and, at $(\tilde{t}, \bar{x})$,

$$u = U, \quad 0 < v \leq U, \quad \dot{u} \geq 0, \quad \nabla \cdot D\nabla u \leq 0, \quad \nabla u = 0.$$ 

From (2.2) this gives

$$\mu U^2 - [\mu - 1/\tau] U v - \mu \|N\|_\infty U \leq n_1^2 / \tau$$

with $n_1^2 \leq M$, $n_1^2 / \tau \leq M^2$. We distinguish two cases:

Case 1 ($\mu \leq 1/\tau$): The second term $-[\mu - 1/\tau] U v$ is then nonnegative and may be omitted. Dividing by $\mu$ and estimating $n_1^2 / \tau \leq M^2$ one obtains $p_1(U) \leq 0$ where $p_1$ is the first polynomial of (3.2). By the definition of $K$, this gives $U \leq K$.

Case 2 ($\mu > 1/\tau$): Now we estimate $v$ by $U$ in the second term and estimate $n_1^2 \leq M$ to obtain

$$(1/\tau) U^2 - \mu \|N\|_\infty U \leq M / \tau.$$ 

Multiplying by $\tau$ and using (2.1) gives $p_2(U) \leq 0$ where $p_2$ is the second polynomial of (3.2). Again, this gives $U \leq K$.

If, instead, the maximum for $u, v$ were $V$—attained by $v$ rather than by $u$, with $0 < u \leq V = v(\tilde{t}, \bar{x})$, etc.—then one would have a similar analysis with (3.3)
replaced by
\[ \nu V^2 - [\nu - 1/\tau] u V - \nu \| N \|_\infty V \leq n_1^2 / \tau , \]

obtained from (2.3). In the second case \( \mu > 1/\tau \) at \( (\bar{t}, \bar{x}) \) one must now consider the third polynomial of (3.2).

Thus, in any of the various cases one has \( 0 < u, v \leq K_0 \) if the maximum occurs initially or at the boundary and \( 0 < u, v \leq K \) if it occurs for \( \bar{t} > 0, \bar{x} \in \Omega \). We have (3.1). \( \square \)

**Lemma 3**: Consider \((u, v)\) satisfying (2.2), (2.3), (1.7) weakly on \( Q_{\alpha*} \) with initial data \( 0 < u(0), v(0) \leq K_0^* \). Suppose one has: (H2); (H5); \( 0 < \alpha \leq D, E \in C(\bar{Q}_{\alpha*}) \); \( \nabla \psi \in L^2(\bar{Q}_{\alpha*}); N \in L^\infty(\Omega); \tau = \tau(t, x, u, v) \) measurable in \((t, x) \in Q_{\alpha*}\) and satisfying (2.1); \( 0 \leq n_1^2 \leq M \) measurable on \( Q_{\alpha*} \). Then (3.1) holds on \( Q_{\alpha*} \).

**Proof**: We proceed by an approximation argument using Lemma 2. Thus, we observe that the present hypotheses on \( D, E, \nabla \psi, N, \tau, n_1^2, \phi_1, \phi_2, u(0), v(0) \) permit us to find approximations (denoted by \( \hat{D}, \ldots, \hat{v}(0) \)) arbitrarily good in the senses of
\[
C(\bar{Q}_{\alpha*}), C(\bar{Q}_{\alpha*}), L^2(\bar{Q}_{\alpha*})^m, L^2(\Omega), \ldots, L^2(Q_{\alpha*}), \ldots, L^2(\Omega) \]

and satisfying the hypotheses Lemma 2. For \( \tau \) the nature of the approximation\(^5\) is such that
\[
\hat{u}, \hat{v} \rightarrow u, v \text{ in } L^2(Q_{\alpha*}) \text{ implies } \hat{\tau}(\cdot, \hat{u}, \hat{v}) \rightarrow \tau(\cdot, u, v) \text{ in } L^2(Q_{\alpha*})
\]

with a similar sense of approximation\(^5\) for \( \phi_1, \phi_2 \) replacing \( L^2(Q_{\alpha*}) \) by \( L^2(\Sigma_{\alpha*}) \) where \( \Sigma_{\alpha*} = (0, t_{\alpha*}) \times \partial \Omega \). We now let \((\hat{u}, \hat{v})\) be the corresponding solutions of (2.2), (2.3), (1.7), etc. Note that standard regularity results imply that \( \hat{u}, \hat{v} \) are smooth so all the hypotheses of Lemma 2 are satisfied and we may conclude that
\[ 0 < \hat{u}, \hat{v} \leq \max \{ K, K_0^* \} =: \hat{K}. \]

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\(^5\) One uses standard properties of Nemytsky operators to conclude that the nature of approximation for the functions required to obtain this is such that smooth approximants exist, subject to our hypotheses.
We also, of course, have uniform bounds on \(D, E, \nabla \psi, \ldots\) in the senses of approximation above. Multiplying (2.2) by \(\hat{u}\) and integrating over \((0, t) \times \Omega\) gives, with application of the Divergence Theorem,

\[
\frac{1}{2} \|\hat{u}(t)\|^2 + \alpha \int_0^t \|\nabla \hat{u}\|^2 \, dt \leq \frac{1}{2} \|\hat{u}(0)\|^2 + \mu \tilde{K} \int_0^t \|\nabla \hat{u}\| \|\nabla \hat{u}\| \, dt + \|Q_{\tilde{K}}\| \left[ \mu \tilde{K}^2 + \mu \|\tilde{K}\|_{\infty} \tilde{K} + M^2 \right].
\]

From this, application of the Cauchy and Gronwall inequalities gives a bound on \(\hat{u}\) in \(L^2((0, t_x) \to H^1(\Omega))\), uniform with respect to the family of approximations. Using this and (2.2), solved for \(\hat{u}^*\), one obtains a similarly uniform estimate for \(\hat{u}^*\) in \(L^2((0, t_x) \to H^{-1}(\Omega))\). We now use a compactness result of Lions \([3]\) and Aubin \([1]\) to conclude that all the approximations \(\hat{u}\) are in a fixed compact subset of \(L^2((0, t_x) \to H^{-1}(\Omega))\) where we take \(\epsilon < 1/2\) so we may also conclude \(L^2(\Sigma_{t_x})\) compactness for the traces. An essentially identical argument gives corresponding compactness for \(\hat{v}\).

Using this compactness we extract a sequence for which not only do we have \(\hat{D} \to D, \ldots, \hat{\psi}(0) \to \psi(0)\) in the senses noted earlier, but also suitable convergence \(\hat{u} \to \bar{u}, \hat{v} \to \bar{v}\). From the above we see that we may extract the sequence so as to have, in fact,

\[
\hat{u} \to \bar{u}, \hat{v} \to \bar{v} \quad \text{in } L^2(Q_{\tilde{K}}) \text{ and in } L^2(\Sigma_{t_x})
\]

\[
\nabla \hat{u} \to \nabla \bar{u}, \nabla \hat{v} \to \nabla \bar{v} \quad \text{weakly in } L^2(Q_{\tilde{K}})
\]

\[
\hat{u}^* \to \bar{u}^*, \hat{v}^* \to \bar{v}^* \quad \text{weakly in } L^2((0, t_x) \to H^{-1}(\Omega)).
\]

(We have used that fact that \(\nabla\) and \(\cdot\) are closed operators to conclude not only that \(\{\nabla \hat{u}\}\) is weakly convergent but that the limit is \(\nabla \bar{u}\), etc.) The nature of our assumptions on the approximations ensures \(\hat{\tau}(\cdot, \hat{u}, \hat{v}) \to \tau(\cdot, \bar{u}, \bar{v})\) in \(L^2(Q_{\tilde{K}})\), etc.; further, given the bounds \(0 < \tilde{K}, \hat{\omega} \leq \tilde{K}\) one has convergence for the quadratic terms. We also see that \(\hat{D} \nabla \hat{u} \to D \nabla \bar{u}\) weakly in \(L^2(Q_{\tilde{K}})\) so \(\nabla \cdot \hat{D} \nabla \hat{u} \to \nabla \cdot D \nabla \bar{u}\) weakly in \(L^2((0, t_x) \to H^{-1}(\Omega))\), etc.
We conclude that \((\tilde{u}, \tilde{v})\) is a weak solution of (2.2), (2.3), (1.7) with the initial data \((u(0), v(0))\). Since the solution is easily seen to be unique\(^6\), this shows \(\tilde{u} = u, \tilde{v} = v\) so \(\hat{u} \rightarrow u, \hat{v} \rightarrow v\). Since \(0 < \hat{u}, \hat{v} \leq \hat{K}\) one can now conclude that in the limit one has \(0 \leq u, v \leq \hat{K}\) as desired. \(\square\)

4. **Boundedness**

The results of the previous section give boundedness (in \(L^\infty\)) of \(u, v\) under the hypotheses imposed, but we wish also to treat the coupling with (2.5) and obtain a stability result for the temperature \(\theta\).

**Theorem 4:** Consider the system \((\mathcal{G}) := \{(2.2)-(2.5), (1.6), (1.7)\}\) subject to (H1) - (H6) and with nonnegative initial data \(u(0), v(0) \in L^\infty_+(\Omega), \theta(0) \in L^\infty_+(\Omega) \cap H^1(\Omega)\). Then the unique solution \((u, v, \psi, \theta)\) as given by Theorem 1 is bounded uniformly on \(\mathbb{R}^+\) in the sense of \([L^\infty_+(\Omega)]^2 \times C^1(\bar{\Omega}) \times H^1(\Omega)\).

**Proof:** From Lemma 3 we already know \(u, v\) are bounded by \(\tilde{K}\) on \(Q_T\) with \(t_T\) arbitrary so one has the desired bound for those components on \(\mathbb{R}^+\). As in Lemma 1 of [6], a uniform \(L^\infty(\Omega)\) bound on the right hand side of (2.4) permits application of Theorem 5.6.3 of [4] to get \(\psi\) bounded, uniformly on \(\mathbb{R}^+\), in \(W^{2,p}(\Omega)\) for arbitrary \(p\) \((1 \leq p < \infty)\) whence, by the Sobolev Embedding Theorem, \(\psi\) is bounded in \(C^1(\bar{\Omega})\). It remains only to bound \(\nabla \theta\).

Using (H4) we write \(\theta = \theta_0 + z\) where \(z\) is the solution of

\[
\dot{z} - \nabla \cdot F \nabla z = f := [-\nabla \psi \cdot \tilde{G}]_+ , \quad z(0) = 0, \quad z|_{\partial \Omega} = 0 .
\]

We begin by estimating \(f\) in \(L^2(\Omega)\); since we already have a uniform \(L^\infty(\Omega)\) bound for \(\nabla \psi\), this amounts to estimating

\[
\mathcal{G} := [D \nabla u - E \nabla v] - [\mu u + \nu v] \nabla \psi .
\]

We already have an \(L^\infty\) bound for the second part and need only estimate \(D \nabla u, E \nabla v\).

---

\(^6\) One uses a standard argument (compare Section 5 of [6]) applying the Gronwall inequality to estimate the difference of two supposed solutions.
The estimate of $D\nabla u$—and, similarly of $E\nabla v$—proceeds in two steps, writing
\[
\|D\nabla u\|_{L^2(\Omega)} \leq \|D\|_{\infty}^{1/2} \|D^{1/2} \nabla u\|_{L^2(\Omega)}.
\]
Given the bound on $\nabla \psi$, we have from (H1) that
\[
(4.2) \quad \alpha \leq D(\cdot, \nabla \psi, T) \leq \alpha + \beta T
\leq \beta \left(1 + \|\nabla z(t)\|_{L^2(\Omega)}\right)
\]
using (H6). Next, (2.2) gives
\[
\dot{u} - \nabla \cdot D\nabla u = B \cdot \nabla u + g
\]
where we already have $L^\infty$ bounds for $B, g$. Multiplying by $u$ and integrating over $(t, t+\delta) \times \Omega$ gives
\[
\frac{1}{2} \|u(t+\delta)\|^2 + \int_t^{t+\delta} \int_\Omega D|\nabla u|^2 + \int_t^{t+\delta} \int_{\partial \Omega} uD\phi_1(\cdot, u, v)
\leq \frac{1}{2} \|u(t)\|^2 + \tilde{K}\|B\|_{L^\infty} \int_t^{t+\delta} \|\nabla u\| + \tilde{K}|\Omega|\delta \|g\|_{L^\infty}.
\]
The bound on $(T = Q[\theta])_{|\Sigma}$ assumed in (H6) gives a bound on $D|_{\Sigma}$ so the third term on the left above is bounded below, using (H5). Thus, we obtain an estimate of the form\(^7\)
\[
(4.3) \quad \int_t^{t+\delta} \|D^{1/2} \nabla u\|_{L^2(\Omega)}^2 \leq C_\delta^2
\]
with $C_\delta$ a specifiable constant, depending only on estimates already obtained and
\(^7\) Compare Theorem 1 [iv].
on the choice of \( \delta > 0 \); for definiteness we choose, here, \( \delta = 1 \). Combining (4.2) and (4.3) — and the corresponding estimates for \( E \nabla v \) — gives

\[
(4.4) \quad \| f(t) \|_{L^2(\Omega)}^2 \leq (1 + \| \nabla z \|)^{1/2} \gamma(t)
\]

where \( \gamma: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) satisfies

\[
(4.5) \quad \int_t^{t+1} \gamma^2 \leq C^2
\]

corresponding to (4.3).

Now multiply (4.1) by \( 2e^{\epsilon t} z \), suitably choosing \( \epsilon > 0 \), and integrate over \((0, t) \times \Omega \) to obtain (with \( \delta > 0 \) arbitrary)

\[
\left[ e^{\epsilon t} \| z(t) \|^2 - 2\epsilon \int_0^t e^{\epsilon s} \| z \|^2 \right] + 2 \int_0^t e^{\epsilon s} \int_\Omega F \| \nabla z \|^2 \leq 2 \int_0^t e^{\epsilon s} \| f \| \| z \| .
\]

\[
\leq 2\delta \int_0^t e^{\epsilon s} \| z \|^2 + (1/2\delta) \int_0^t e^{\epsilon s} \| f \|^2 .
\]

By the Poincaré inequality one has \( \| z \| \leq C_{\Omega} \| \nabla z \| \) for \( z \) with \( z/\partial \Omega = 0 \) so, choosing \( \epsilon, \delta > 0 \) so that \( 2C_{\Omega}^2(\epsilon + \delta) < \alpha \), one has

\[
(4.6) \quad e^{\epsilon t} \| z(t) \|^2 + \int_0^t \int_\Omega F \| \nabla z \|^2 \leq C_{\alpha} \int_0^t e^{\epsilon s} \| f \|^2 .
\]

Next, multiply (4.1) by \( 2e^{\epsilon t} \dot{z} \) and integrate to get

\[
2 \int_0^t e^{\epsilon s} \| \dot{z} \|^2 + \left[ e^{\epsilon t} \alpha \| z(t) \|^2 - \epsilon \int_0^t e^{\epsilon s} \int_\Omega F \| \nabla z \|^2 \right]
\]

\[
\leq 2 \int_0^t e^{\epsilon s} \| f \| \| \dot{z} \| \leq \int_0^t e^{\epsilon s} \| \dot{z} \|^2 + \int_0^t e^{\epsilon s} \| f \|^2 .
\]
So, using (4.6),

\[
\left(4.7\right) \quad \int_{0}^{t} e^{\epsilon s}\|z\|^2 + \alpha e^{\epsilon t}\|\nabla z(t)\|^2 \leq \epsilon \int_{0}^{t} e^{\epsilon s} \int_{\Omega} F|\nabla z|^2 + \int_{0}^{t} e^{\epsilon s}\|f\|^2 \\
\leq (1 + \epsilon C_{\epsilon}) \int_{0}^{t} e^{\epsilon s}\|f\|^2
\]

and, by (4.4), this gives

\[
\left(4.8\right) \quad \xi_0^2(t) \leq \hat{C} \int_{0}^{t} e^{-\epsilon(t-s)} \left[1 + \xi_0^2(s)\right] \gamma(s) \, ds
\]

with (4.5) where we have set \(\xi_0^2(t) := \|\nabla z\|\). Now, setting

\[
\xi_1^2(t) := \sup_{[0, t]} \xi_0^2 := \sup_{\|\nabla z(s)\|_{L^2(\Omega)}} \{0 \leq s \leq t\}
\]

one obtains, for \(0 \leq t \leq \bar{t}\),

\[
\xi_0^2(t) \leq \hat{C} \int_{0}^{t} e^{-\epsilon(t-s)} \gamma(s) \, ds \left[1 + \xi_1^2(\bar{t})\right]
\]

\[
= \hat{C} \sum_{k=0}^{\infty} \int_{t-k-1}^{t-k} e^{-\epsilon(t-s)} \gamma(s) \, ds \left[1 + \xi_1^2(\bar{t})\right]
\]

taking \(\gamma(s) = 0\) for \(s < 0\)

\[
\leq \hat{C} \sum_{k=0}^{\infty} e^{-\epsilon k} \int_{t-k-1}^{t-k} \gamma(s) \, ds \left[1 + \xi_1^2(\bar{t})\right]
\]

\[
\leq \frac{\hat{C} C_{\epsilon}^2}{1 - e^{-\epsilon}} \left[1 + \xi_1^2(\bar{t})\right].
\]
Taking the $\sup_t$ over $[0, \bar{t}]$ on the left gives

$$\xi_1^2(\bar{t}) \leq [\tilde{C}C^2/(1-e^{-\varepsilon})] \left[ 1 + \xi_1(\bar{t}) \right]$$

for arbitrary $\bar{t} > 0$ which gives a uniform bound on $\xi_1(\bar{t})$ and so a bound on $\nabla z$ in $L^\infty(\mathbb{R}^+ \to L^2(\Omega))$. Combining this with the bound on $\nabla \theta_0$ implied by (H4), this gives the desired $L^\infty(\mathbb{R}^+ \to H^1(\Omega))$ bound on the temperature $\theta$. By (H6) this also gives an $L^\infty(\Omega)$ bound on the locally averaged temperature $\bar{T}$.

5. Periodicity

In this section we consider the system $(\mathcal{S}) = \{(2.2) - (2.5), (1.6), (1.7)\}$ subject to (H1) - (H6) without initial conditions but assuming all explicit dependence on $t$ is periodic with period $\bar{t}$ (with no loss of generality we may take $\bar{t} = 1$) and ask whether there must exist a correspondingly time-periodic solution. The basic approach is to show that the Poincaré map for (a suitable version of) the problem satisfies the hypotheses of the Schauder Fixed Point Theorem. The estimates of the preceding sections permit construction of a suitable invariant set for the map, continuity follows from Theorem 1, and compactness is obtained by a factorization of the Poincaré map enabling application of the Aubin–Lions compactness theorem for functions on the period interval.

**THEOREM 5:** Consider the system $(\mathcal{S}) = \{(2.2) - (2.5), (1.6), (1.7)\}$ subject to (H1) - (H6). Assume all explicit dependence on $t$ has period 1. Then there is a solution $(u, v, \psi, \theta)$ — in the spaces noted in Theorem 1 — which is periodic in $t$ with same period 1.

**Proof:** for $0 \leq s \leq 1$ we define the maps $P_s$ by taking initial data $[u(s), v(s), \theta(s)]$ at time $t = s$ as in Theorem 1, obtaining the corresponding solution on $[s, \omega) \times \Omega$, and evaluating this at $t = 1$. The Poincaré period map is thus $P_0$ and, as usual, if we can show this has a fixed point, then the assumed periodicity of the problem

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8 Mapping initial data for the problem at time $t = 0$ into the value of the solution evaluated at $t = \tau = 1$. 

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together with the uniqueness (given by Theorem 1) for the initial value problem ensure that the corresponding solution has period 1.

From Theorem 1 we have \( \sim_0 \) defined on the (nonnegative functions in)
\( L^2(\Omega) \times L^2(\Omega) \times H^1(\Omega) =: \mathcal{X} \) and we first seek a bounded convex invariant set.
Specifying \( K_0 \) from (H5) alone, since no initial condition is given, and then defining \( \tilde{K} \) as in Lemma 3, we see that if \( 0 \leq u(s), v(s) \leq \tilde{K} \) then one has \( 0 \leq u(1), v(1) \leq \tilde{K} \)
by Lemma 3 for
\[
[u(1), v(1), \cdot] := \sim_s [u(s), v(s), \cdot]
\]
(i.e., any admissible \( \theta(s) \)). We restrict \( \sim_s \) so the first components satisfy
\( 0 \leq u, v \leq \tilde{K} \) in \( L^2(\Omega) \). For the third component of \( \sim_0 \) we write \( \theta = \theta_* + \theta_0 + z \)
where \( \theta_0 \) is given by (H4) with 0 initial condition, \( z \) is given by (4.1) as in
Theorem 4, and \( \theta_* \) is given by the homogeneous linear initial value problem
\[
(5.1) \quad \dot{\theta}_* = \nabla \cdot \Gamma \nabla \theta_* \quad \text{,} \quad \theta_*(0) = \theta(0) \quad \text{,} \quad \theta_* \bigg|_{\partial \Omega} = 0 .
\]
It is standard (and not difficult to prove, either by spectral arguments or a Gronwall estimate) that
\[
\| \theta_*(t) \|_1 \leq e^{-\epsilon t} \| \theta(0) \|_1
\]
where \( \| \cdot \|_1 \) is the \( H^1(\Omega) \)-norm and \( \epsilon > 0 \) is taken sufficiently small\(^9\). Since we
have a fixed bound \( M_1 \) for \( \| \theta_0 + z \|_1 \) by (H4) and the argument of Theorem 4, a restriction of \( \sim_0 \) to \( \| \theta(0) \|_1 \leq M_0 \) would give
\[
\| \theta(1) \|_1 \leq M_0 e^{-\epsilon} + M_1 \leq M_0
\]
provided we choose \( M_0 \geq M_1/(1 - e^{-\epsilon}) \). This, then, gives the desired invariant set
for \( \sim_0 \):

\(^9\) Depending on the lower bound \( \alpha > 0 \) for \( F \) and the constant in the Poincaré inequality for \( \Omega \).
\[ \left\{ (u(0), v(0), \theta(0)) \in \mathfrak{X} : 0 \leq u(0), v(0) \leq \tilde{K}; 0 \leq \theta(0) ; \left\| \theta(0) \right\|_1 \leq M_0 \right\} =: \mathfrak{X}_\circ \]

and we take \( \mathcal{P}_0 \) restricted to this bounded convex \( \mathfrak{X}_\circ \subset \mathfrak{X} \) so \( \mathcal{P}_0 : \mathfrak{X}_\circ \to \mathfrak{X}_\circ \).

By Theorem 1 we know \( \mathcal{P}_0 \) is continuous—indeed, we have Lipschitz continuity of each \( \mathcal{P}_s \) with a constant \( L \) uniform in \( s \in [0, 1] \)—and for applicability of the Schauder Theorem we need only prove compactness of \( \mathcal{P}_0 \). To this end we employ a trick of factorization. Let \( \tilde{\mathfrak{X}} := L^2((0, 1) \to \mathfrak{X}) \) and define a map \( \tilde{A} : \tilde{\mathfrak{X}}_+ \to \mathfrak{X}_+ \) as follows:

(5.2) Given \( w \in \tilde{\mathfrak{X}}_+ \) (i.e. \{nonnegative triples in \( \tilde{\mathfrak{X}} \}), set \( \tilde{A}w := \int_0^1 \mathcal{P}_s w(s) ds \).

We also let \( \tilde{S} \) be the solution map

\[ \tilde{S} : \tilde{\mathfrak{X}}_+ \to \tilde{\mathfrak{X}}_+ : [u(0), v(0), \theta(0)] \mapsto [u(\cdot), v(\cdot), \theta(\cdot)] \mid_{[0, 1]} \]

given by Theorem 1. We will show that \( \tilde{S} \) is continuous and compact, that \( \tilde{A} \) is well-defined and Lipschitzian, and that \( \tilde{A} \tilde{S} = \mathcal{P}_0 \).

We begin by observing that \( w \in \tilde{\mathfrak{X}}_+ \) does not give \( w(s) \in \mathfrak{X}_+ \) for each \( s \) but, if \( w, \hat{w} \in \tilde{\mathfrak{X}}_+ \) one does have both \( w(s), \hat{w}(s) \in \mathfrak{X}_+ \) for almost all \( s \) and, by Theorem 1,

\[ \left\| \mathcal{P}_s w(s) - \mathcal{P}_s \hat{w}(s) \right\|_{\mathfrak{X}} \leq L \left\| w(s) - \hat{w}(s) \right\|_{\mathfrak{X}} \]

so, integrating,

\[ \| \tilde{A}w - \tilde{A}\hat{w} \|_{\mathfrak{X}} \leq \int_0^1 \| \mathcal{P}_s w(s) - \mathcal{P}_s \hat{w}(s) \|_{\mathfrak{X}} \leq L \int_0^1 \| w(s) - \hat{w}(s) \|_{\mathfrak{X}} \leq L \| w - \hat{w} \|_{\tilde{\mathfrak{X}}} \]

which shows \( \tilde{A} \) is well-defined and Lipschitzian \( ^10 \).

\(^10\) We have "cheated" slightly in that the constant \( L \) is only given by Theorem 1 as uniform in the presence of suitable \textit{a priori} bounds—but we only actually use \( \tilde{A} \) with such bounds, and could redefine elsewhere so this is global.
That $\mathcal{S}$ is well-defined and continuous (at least on $\mathcal{X}_+$) was given by Theorem 1. To see that $A\mathcal{S} = \mathcal{P}_0$ we note that if $w := [\dot{u}, v, \vec{v}] \in \mathcal{X}_+$ is the solution $w = \mathcal{S}w_0$, then the standard property of evolutionary systems (with uniqueness for the initial value problem) is that

$$P_{\mathcal{S}}w_0 = [\mathcal{S}w_0](1) = : w(1) = P_{\mathcal{S}}[\mathcal{S}w_0(s)] = P_{\mathcal{S}}w(s)$$

for any $s \in [0, 1]$. Thus, for $w(\cdot) = \mathcal{S}w_0$ one has

$$A\mathcal{S} = \int_0^1 P_{\mathcal{S}} w(s) = \int_0^1 P_{\mathcal{S}}w_0 = P_{\mathcal{S}}w_0$$

which gives $A\mathcal{S} = \mathcal{P}_0$.

Penultimately, we observe that (given the a priori bounds implicit in restriction to $X_+$) Theorem 1[1] indicates a bound in $L^2((0, 1) \to H^1(\Omega))$ for $u, v$ —indeed, this is essentially the estimate obtained from (3.4). Given this estimate together with the $L^\infty$ bounds for $D, E, \nabla \psi$ implied by our bounds on $u, v, T$ and using (H3), we have also the bound on $\dot{u}, \dot{v}$ in $L^2((0, 1) \to H^{-1}(\Omega))$ indicated in Theorem 1[1]. Thus, the Aubin-Lions compactness theorem [3] shows that $[u(\cdot), v(\cdot)]$ lie in a compact subset of $L^2((0, 1) \times \Omega)$.

Finally, we return to (4.1) for which multiplication by $2\nabla \cdot F\nabla z$ and integration give

$$\int_{\Omega} F|\nabla z(t)|^2 + 2 \int_0^t \|\nabla \cdot F\nabla z\|^2 = 2 \int_0^t <f, \nabla \cdot F\nabla z>$$

$$\leq \int_0^t \|f\|^2 + \int_0^t \|\nabla \cdot F\nabla z\|^2$$

noting that $\dot{z}|_{\partial \Omega} = (z|_{\partial \Omega})' = 0$. Using the bound on $f$ available from the analysis for Theorem 4, this gives a bound on $\nabla z$ in $L^2((0, 1) \to H^1(\Omega))$. Next, (4.7) gives a bound on $\dot{z}$ in $L^2$ and so a bound on $(\nabla z)'$ in $L^2((0, 1) \to H^{-1}(\Omega))$. Using the Aubin-Lions compactness theorem again, this gives compactness in $L^2$ for $\nabla z$
and so compactness for \( z \) in \( L^2((0,1) \to H^1_0(\Omega)) \). It is well known that (5.1) defines a compact holomorphic semigroup so one easily sees that \( \theta_{\star}(\cdot) \) lies in a compact subset in \( L^2_+((0,1) \to H^1_0(\Omega)) \) for \( \theta(0) \) bounded in \( H^1_0(\Omega) \). As \( \theta_0 \) is fixed, this gives a precompact range for \( \theta = \theta_{\star} + \theta_0 + z \), completing the argument for compactness of \( \tilde{S} : \mathbb{K} \to \mathbb{K} \).

As the composition of a continuous map with a compact one, this shows \( P_0 = \tilde{A} \tilde{S} \) is compact on \( \mathbb{K} \). Thus we may apply the Schauder Theorem. Existence of a fixed point of \( P_0 \) provides the desired periodic solution of (3).

6. Steady State

Existence of steady state solutions of (1.1) - (1.5) was considered in [5] but under somewhat different assumptions, especially concerning the convection term \( \nabla \cdot u(\mu \nabla \psi) - \nabla \cdot v(\nu \nabla \psi) \). Here, as in [5], \( \mu, \nu \) have been taken constant, whereas in [5] they were constant multiples of \( D, E \), respectively. Thus, it is not entirely redundant to consider this question in the present context although, as noted in remarks in both [5] and [6], neither of these treatments of the convection term is entirely faithful to the physics.

The question of existence of steady state solutions could be considered directly, employing a fixed-point argument resting on an a priori bound obtained along the lines of Lemmas 2 and 3. However, we will proceed to consider this as a corollary to Theorem 5.

THEOREM 6. Consider the system (3) := \( \{(2.2) - (2.5), (1.6), (1.7)\} \) subject to (H1) - (H6). Suppose, further, that the system is autonomous: no explicit time-

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11 This is known as the "Einstein relationship" for diffusion and convections.

12 A more correct treatment would replace \( \mu \nabla \psi \) by a velocity \( V = V(x, T, \nabla \psi) \) having the direction of \( \nabla \psi \) but bounded as \( |\nabla \psi| \) increases. Indeed, in some physical contexts \( |V| \) is not even monotone in \( |\nabla \psi| \) but rises to a maximum and then decreases slightly!
dependence in the coefficients or boundary conditions. Then there exists a steady state solution \((u, v, \psi, \theta)\) with \(\dot{u}, \dot{v}, \dot{\psi}, \dot{\theta}\) all 0.

**Proof:** Write \(w := (u, v, \theta)\) as in the proof of Theorem 5 and note that Theorem 5 gives existence of a periodic solution \(w_k\) with period \(2^{-k}\) for \(k = 0, 1, \ldots\). Clearly \(w_k\) also has period 1 for each \(k\); indeed, \(w_k(t + \tilde{t}) = w_k(t)\) for any multiple \(\tilde{t}\) of \(2^{-k}\).

Using the estimates obtained in the proof of Theorem 5 we have uniform bounds on

\[
\begin{align*}
&u_k, v_k \quad \text{in} \quad L^\infty \\
&\nabla u_k, \nabla v_k \quad \text{in} \quad L^2(\hat{Q}) \quad \hat{Q} := (0, 1) \times \Omega \\
&\dot{u}_k, \dot{v}_k \quad \text{in} \quad L^2((0, 1) \rightarrow H^{-1}(\Omega)) \\
&\nabla \psi_k \quad \text{in} \quad W^{1,p}(\Omega) \subset L^\infty \\
&\nabla \theta_k \quad \text{in} \quad L^\infty((0, 1) \rightarrow L^2(\Omega)) \\
&\dot{\theta}_k \quad \text{in} \quad L^2(\hat{Q}) \\
&T_k \quad \text{in} \quad C(\hat{Q})
\end{align*}
\]

and, extracting a subsequence if necessary, can assume corresponding weak convergences, etc. Let \(\tilde{w} := (\tilde{u}, \tilde{v}, \tilde{\psi}, \tilde{\theta})\) be the limit of such a subsequence.

Since for every binary rational \(\tilde{t}\) one has \(w_k(t + \tilde{t}) = w_k(t)\) for large enough \(k\), it follows that \(w\) is invariant under translation by all binary rationals. Since one always has continuity for the translation semigroup on \(L^p\) spaces, this shows \(w\) is constant in time. Since \(\dot{u}_k \rightharpoonup \tilde{u}\) in the sense of \(L^2((0, 1) \rightarrow H^{-1}(\Omega))\) and \(\tilde{u}\) is constant in, e.g., \(L^2((0, 1) \rightarrow H^1(\Omega))\), we conclude that \(\dot{u}_k \rightharpoonup 0 = \tilde{u}\) in the sense of \(L^2((0, 1) \rightarrow H^{-1}(\Omega))\). We similarly have suitable convergence of the other terms to conclude that \((\tilde{u}, \tilde{v}, \tilde{\psi}, \tilde{\theta})\) is a weak solution of the system \((\mathcal{J})\) with the time-derivative terms omitted. It is thus the desired steady state solution. \(\square\)
7. Remarks and Acknowledgments

We have shown, for the system (2.2) - (2.5), under the hypothesis (H1) - (H6), that (i) with bounded data (i.e., bounded uniformly in $t \in \mathbb{R}^+$) the solution will remain bounded, (ii) with periodic data (and correspondingly periodic explicit time-dependence for coefficients) there exists a periodic solution, and (iii) with autonomous coefficients and time-constant data there exists a steady state solutions.

There seems no point in repeating the remarks in [5], [6] regarding the physical relevance of the model which are equally applicable here. For comparison—not so much with the particular results of this paper as with the underlying results in [6] on the initial value problem and (note footnote 12) with the assumptions made—see [2].

The treatment here has clearly been based on the collaborative work [6] with G. M. Troianiello but, in addition, acknowledgment should be made of the value of conversations with Troianiello concerning some of the material presented here. In particular, it was an earlier version of the analysis of Lemma 2 which initiated the conversations leading to [6].

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