EXISTENCE OF A COMPETITIVE EQUILIBRIUM
IN $L_p$ AND SOBOLEV SPACES

BY
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EXISTENCE OF A COMPETITIVE EQUILIBRIUM
IN $L_p$ AND SOBOLEV SPACES

G. Chichilnisky and G.M. Heal*

Abstract

The paper proves existence of a competitive equilibrium in $L_p$ and $L_p$ spaces, $1 < p < \infty$, and Sobolev spaces of continuous and differentiable functions. This includes the Hilbert spaces $L_2$ and $L_2$. The conditions for existence are only on individual characteristics initial endowments and preferences. A competitive equilibrium is Pareto efficient, and is supported by positive prices in the same Hilbert space or in a dual $L_p$ space. The commodity spaces contain $L_\infty$, $L_\infty$ and $C(X)$ as dense subsets. Individual consumption sets may be the positive cone or the whole space of commodity bundles.

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1. Introduction
2. Definitions
3. One-consumer Results
4. Pareto Efficient Allocations
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1. Introduction

In this paper we prove the existence of a competitive equilibrium in an economy with infinitely many commodities and a finite number of agents. Our commodity spaces include Hilbert spaces; these are the closest analogs to Euclidean spaces amongst infinite dimensional spaces, and have already been found useful in applications to arbitrage in financial markets (Harrison and Kreps, 1979; Chamberlin and Rothschild, 1983), nonparametric econometrics (Bergstrom, 1983), demand behavior of one-agent economies (Chichilnisky, 1977, 1981a; Chichilnisky and Kalman, 1979), and utility theory (Chichilnisky, 1976). The results also apply to $L_p$ spaces, $1 < p < \infty$, and to Sobolev spaces of continuous and differentiable functions. All these spaces are reflexive. Hilbert spaces have inner products and a standard basis.

Our spaces contain strictly other infinite dimensional commodity spaces already used in general equilibrium theory, such as $\ell_\infty$ and $L_\infty$, and spaces of continuous bounded functions $C(X)$ (e.g., Debreu (1954) and Bewley (1972)), but present none of the duality problems of $\ell_\infty$, $L_\infty$, or $C(X)$. However, previous proofs of existence of an equilibrium in $L_\infty$ or $C(X)$ (e.g., Bewley (1972)) do not generalize to our spaces, because they rely on the positive cones having an interior, a condition which is satisfied neither in Hilbert nor in $L_p$ spaces.

The conditions we require to prove the existence of a competitive equilibrium are solely on individual preferences and initial endowments. Consumption sets are required to be either the positive cone or all of the commodity space. This latter case is of interest if one wishes to model the possibility of unlimited short sales in financial markets, a requirement which appears in Kreps (1981). As a by-product we prove the existence and continuity of feasible demand functions. Since our primitive concepts are individual preferences, the first welfare theorem can be tested, and competitive equilibria are indeed shown to be Pareto efficient. This is a difference from other recent results on existence with infinite-dimensional commodity spaces, e.g., Aliprantis and Brown (1983), who follow an interesting approach but assume demand functions as the basic characteristic of a household.
Our conclusion is that proving existence of a competitive equilibrium in Hilbert or \( L^p \) spaces is relatively straightforward once we solve the one-consumer problem, namely the existence for a single consumer of optimal individual choices and of associated supporting prices. This paper draws on earlier results on the one-consumer problem in Hilbert spaces in Chichilnisky (1977, 1981a,b) and Chichilnisky and Kalman (1979, 1980a) to provide the necessary one-person arguments. We then prove the existence of nontrivial Pareto efficient allocations for the many-person economy, using an argument which is also idiosyncratic to Hilbert (or \( L^p \), \( 1 < p < \infty \)) spaces. All these results rely only on the norm continuity of utilities, for which we exhibit necessary and sufficient conditions. The rest of the proof relies on a standard fixed-point argument in utility space, which with finitely many agents is finite-dimensional. This is a technique due to Negishi (1960), and used more recently by Magill (1981) and by Mas-Colell (1983) for \( L^\infty \) and Banach lattices respectively. This method of proof emphasizes that what sets the infinite-dimensional problem apart from the finite-dimensional one is not the difference in fixed-point theorems, but rather the difficulty in finding a resolution of the one-consumer problem. This point was made by Mas-Colell (1983) who also identified another problem for the infinite-dimensional case, namely establishing the existence of nontrivial efficient allocations. This problem can be solved readily in Hilbert and \( L^p \) spaces.

The paper studies pure exchange economies. However, consumption sets can be the whole commodity space as required for the analysis of arbitrage in Kreps (1981) and need not be equal to the positive orthant as required in Mas-Colell (1983), nor contained in the positive orthant as in Bewley (1972).

The proofs in the text deal with Hilbert spaces. The appendix shows that the proofs used here also apply to any \( L^p \) space with \( 1 < p < \infty \), and to Sobolev spaces consisting of continuous and differentiable functions. However, the approach is only valid for finitely many agents, thus contrasting with recent results for infinitely many consumers by Mas-Colell (1975), Brown and Lewis (1981), Balasko, Cass and Shell (1980), Jones (1982) and Ostroy (1982). The generalization to infinitely many agents would required fixed-point theorems in infinite-dimensional spaces, and would be an interesting extension.
2. **Definitions**

Commodities are indexed by the real numbers. Consumption bundles are therefore real-valued functions on \( R \) (the analysis can easily be extended to vector-valued functions on \( R^n \)). The space of commodity bundles, denoted \( H \), is an \( L^2 \) space of measurable functions \( x(t) \) with the inner product

\[
<x, y> = \int_R x(t) \cdot y(t) d\mu(t),
\]

where \( \mu(t) \) is any finite measure on \( R \) \((\int_R d\mu(t) < \infty)\) which is absolutely continuous with respect to the Lebesgue measure. The \( L^2 \) norm of a function \( x \) is as usual:

\[
\|x\| = (\langle x, x \rangle)^{1/2}.
\]

Commodity bundles in \( H \) need not be bounded functions (e.g., \( \lim_{t} x(t) = \infty \) is allowed), but the space \( L^\infty_\infty \) of all bounded measurable functions is contained as a dense subset of \( H \). This is because the measure \( \mu \) is finite, and is easy to see: approximate any \( x \) in \( H \) by the sequence of truncated functions \( \{x^n\} \), where \( x^n(t) = x(t) \) for \( t \leq n \), \( x^n(t) = 0 \) for \( t > n \). Clearly, \( x^n \) is in \( L^\infty_\infty \) for all \( n \), and

\[
\lim_{n} \|x^n - x\| = 0.
\]

A price \( p \) is a real-valued linear function on \( H \) giving positive value to positive commodity bundles. This latter condition implies that \( p \) is continuous on \( H \), that \( p \) is itself a (positive) function in \( H \), and that the value of a bundle \( x(t) \) in \( H \) at price \( p(t) \) is given by the inner product:

\[
\int_R p(t) \cdot x(t) d\mu(t)
\]

(see Chichilnisky (1977)).

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1 Chamberlin and Rothschild (1983) give a straightforward economic interpretation of \( L^2 \) norms in models with infinite-dimensional commodity spaces.
The price space is therefore $H^+$, the positive cone in $H$. It is clear that all the results obtained for $H = L_2$ follow for $H = l_2$, the Hilbert space of sequences with the inner product

$$<x, y> = \sum_{i=1}^{\infty} x(i) \cdot y(i) \mu(i),$$

where $\sum_{i=1}^{\infty} \mu(i) < \infty$. Moreover, the appendix shows that all the results apply, under somewhat modified conditions, to cases where the space of commodity bundles is either of the following two Sobolev spaces, with finite measure $\mu(t)$:

$$H^1 = \left\{ f : \int_{\mathbb{R}} (f(t)^2 + Df(t)^2) d\mu(t) < \infty \right\}$$

$$H^2 = \left\{ f : \int_{\mathbb{R}} (f(t)^2 + Df(t)^2 + D^2f(t)^2) d\mu(t) < \infty \right\}.$$

Both of these spaces are Hilbert spaces, with standard inner products and coordinate basis. Their interest resides in the fact that $H^1$ consists entirely of continuous functions, and $H^2$ of continuously differentiable functions. This is due to Sobolev's theorem (Chichilnisky, 1976). In these commodity spaces, therefore, prices are also continuous or differentiable functions, respectively. The appendix also shows that the results apply to $L_p$ spaces with $1 < p < \infty$ where

$$L_p = \left\{ f : \mathbb{R} \rightarrow \mathbb{R} : \int_{\mathbb{R}} |f|^p d\mu(t) < \infty \right\}.$$

These are Banach spaces, with the following nice duality property: the dual $(L_p)^*$ is $L_q$, for $1/p + 1/q = 1$. In particular, $(L_2)^* = L_1$, i.e., $L_1$ is a reflexive space.
To illustrate the results we shall consider examples of $L^2$ spaces where the finite measure $\mu(t)$ is $e^{-\lambda t}dt$, i.e., $e^{-\lambda t}$ is the density function of $\mu(t)$.

The order in $H$ denoted $\geq$ is given by: $x \geq y$ iff $x(t) \geq y(t)$ a.e.; $x \gg y$ indicates $x(t) > y(t)$ a.e. A function $u : R^2 \rightarrow R$ is said to satisfy the Caratheodory Condition if

\begin{itemize}
\item[(A)] $u(c,t)$ is continuous with respect to $c \in R$, for almost all $t$ in $R$, and measurable with respect to $t$ for all values of $c$.
\end{itemize}

In the following, a function $W : H \rightarrow R$ is said to be $H$-continuous or norm-continuous, when it is continuous with respect to the norm of $H$. The sequence $(x^n) \rightarrow x$ in the weak topology if and only if $\langle x^n, h \rangle \rightarrow \langle x, h \rangle$ for all $h$ in $H$. Weak continuity of a function $f : H \rightarrow R$ means continuity with respect to the weak topology in $H$.

At this point it may be useful to make a general observation about the implications of working with a Hilbert space as the commodity space, a choice which has been made and discussed several times before in the literature, e.g., Chichilnisky (1977, 1981a,b), Harrison and Kreps (1979) and Chamberlin and Rothschild (1983).

What is ultimately important about the choice of a commodity space is the economic results that it enables one to establish and the naturalness of the assumptions needed to prove the results. Working in a Hilbert space makes it possible to establish the existence of a competitive equilibrium under simple and natural conditions on the basic data of the problem, namely on preferences, endowments and consumption sets. The only one of these conditions that depends on the Hilbert space topology is the continuity condition on preferences, and in Lemmas 3 and 4 we characterize fully the class of additively separable real-valued continuous functions on $H$. This characterization is in terms of the properties of functions on finite-dimensional spaces. Boundedness of consumption sets is not needed to establish existence, although boundedness in $H$ certainly holds for a set which has the same bound in every finite-dimensional subspace, which is the criterion of boundedness in $L^\infty$. One version of our results uses boundedness in $H$. 

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3. **One-consumer Results**

Our first step is to establish how the one-consumer problem is solved in $H$. This is a crucial step in showing the existence of a competitive equilibrium, and one where the infinite dimensionality of the commodity space is felt most acutely. We shall establish a number of properties of the one-consumer problem in $H$, and give examples of how these properties do not hold in other spaces, such as $\ell_\infty$ or $L_\infty$.

We assume that each individual $i$ has an $H$-continuous utility function $W_i : H \rightarrow \mathbb{R}$ such that $W_i(0) = 0$ and $W_i(x) \geq 0$ when $x$ is in $H^+ = \{x \in H : x(t) \geq 0 \ \forall \ t\}$. $W_i$ is also increasing ($W_i(x) \geq W_i(y)$ if $x \succeq y$) and concave. $H$-continuous utilities were studied in Chichilnisky (1977, 1981a), where a full characterization was provided under the additional assumption of additive separability. Several examples are given at the end of this section, and in the next.

Each individual $i$ has an initial endowment vector $\omega_i \in H^+$, $\omega_i \neq 0$. There are $k$ individuals. Society's initial endowment is $\omega = \sum_{i=1}^{k} \omega_i \in H^+$. A feasible consumption bundle $x$ in $H$ is one that does not exceed society's endowment, i.e., $x(t) \leq \omega(t)$ a.e.

Our first lemma establishes the existence and continuity of feasible demand functions. These coincide with the usual demand functions on feasible consumption bundles. We work with feasible demand functions because even in Hilbert spaces the usual demand functions need not be well-defined: for strictly positive prices the budget set restricted to the positive orthant may be unbounded. We circumvent this problem by looking at utility maximization over the budget set restricted to the feasible set, i.e., utility maximization at feasible demands. As Example 1 below shows, in $\ell_\infty$, even this restriction is not enough to ensure the existence of a solution to the consumer maximization problem. Note that Lemma 1 is presented as a matter of interest, and that its results are not used in the proof of the main existence theorem. In particular, the concept of feasible demand is not used in the subsequent arguments.
Lemma 1: For any price \( p \in H^+ \), there exists a commodity bundle \( x_i(p) \in H^+ \), called a feasible demand vector, which maximizes \( W_i \) on the feasible budget set

\[
B^\omega_p = \{ x \in H^+ : \langle x, p \rangle \leq \langle \omega, p \rangle, \ x \leq \omega \}.
\]

If \( W_i \) is strictly concave, \( x_i(p) \) is unique and \( x_i(p) \) is a continuous function of \( p \) whenever \( p \) is strictly positive and \( x_i(p) \ll \omega \).

Proof: The set of positive feasible bundles is convex closed and bounded in \( H^+ \). For each price \( p \) in \( H^+ \), the budget set is convex and closed. We are looking for a maximum of an \( H \)-continuous concave function \( W_i \) on the intersection of these two sets. This intersection is a convex, closed and bounded subset of \( H \).

By Theorem 1 of Chichilnisky and Kalman (1979) such a maximum always exists, and is unique when \( W_i \) is strictly concave. Finally, strict concavity of \( W_i \) implies continuity of the interior solutions.

Remark 1. The proof of this result relies on the Banach-Saks theorem (Riesz-Nagy, 1955). This theorem establishes that in a reflexive Banach space \( B \), if a sequence \( (x_n) \rightarrow x \) weakly, then it has a subsequence \((x_{n_k})\) with

\[
\lim_{j} \left( \frac{x_{n_1} + \ldots + x_{n_j}}{j} \right) \rightarrow x
\]

in the norm. Recall that our Hilbert space \( H \) is a reflexive space (since it is self-dual), and that any convex, norm-bounded and closed subset of \( H \) is weakly compact by Alaoglu's theorem and the self-duality of \( H \) (Dunford and Schwartz, 1958).

Remark 2. The above result applies to any \( L^p \) space, \( 1 < p < \infty \), and to the Sobolev spaces \( H^1 \) and \( H^2 \) (see the appendix), since these spaces are respectively reflexive and self-dual. It should be noted that the result does not apply to \( L_\infty \), \( \ell_\infty \) or \( C(X) \), since none of these spaces is reflexive.

The following example demonstrates that the reflexivity of the space is necessary, by showing that Lemma 1 fails for \( \ell_\infty \). It also serves to provide a counterexample to the Banach-Saks theorem for \( \ell_\infty \). The example shows that in \( \ell_\infty \) the
feasible demand function may not be defined at some prices, even when the utility function is norm-continuous, and even though the feasible budget set is closed and bounded.

Example 1.

Let \( \mathbf{x}^k \) be a sequence in \( \ell^+_\infty \) with
\[
\mathbf{x}^k_i = \begin{cases} 
2 & \text{for } i \leq k \\
0 & \text{for } i > k 
\end{cases}
\]

Let \( (p) \in \ell^+_\infty \) be the price given by the positive linear function
\[
p(x) = \liminf_{i \to \infty} x_i; \quad p_i \text{ is continuous on } \ell^+_\infty \text{ with the sup. norm given by } \|x\|_\infty = \sup_i |x_i|.
\]
Finally, let society's endowment be \( \omega = (2, 2, \ldots, 2, \ldots) \), and the individual endowment be \( \omega_i = (1, 1, \ldots, 1, \ldots) \). Then for all \( k \), the value of \( (x^k) \) at price \( (p) \) is zero, and therefore each \( (x^k) \) is in the budget set \( B^\omega_p \). Consider now the utility function \( W : \ell^+_\infty \to \mathbb{R} \), defined by \( W(y) = \sum_{n=0}^{\infty} \frac{1}{2^n} y; \) \( W \) is concave, and continuous in the sup. norm. It is clear that \( \lim W(x^k) = 2 \). However, the sequence \( (x^k) \) has no limit in \( \ell^+_\infty \). Furthermore, the only vector \( v \) in \( \ell^+_\infty \) that is feasible \( (v \leq w) \) and satisfies \( W(v) = 2 \) is \( w \), and \( w \) is not in \( B^\omega_p \) (since \( p(\omega) = 2 > p(\omega_i) = 1 \)). Therefore we have constructed a feasible sequence \( (x^k) \) in the feasible budget set \( B^\omega_p \), whose utility values are increasing to 2, yet no vector in \( B^\omega_p \) actually takes the value 2. There is therefore no maximum for the function \( W \) on \( B^\omega_p \). The feasible demand is therefore not defined for these prices, even though the feasible budget set is bounded, and the utility function \( W \) is continuous and concave.

Remark 3. Note that the sum of the individual feasible demand vectors for a given price may exceed society's endowment. For any \( p \in H^+ \),
\[
\sum_{i=1}^{k} x_i(p) - \omega
\]
is called a feasible excess demand, and may be a strictly positive, or negative, vector. This concept is used only for the one-consumer case; in the existence theorem we deal with the usual demand behavior.
The second result addresses the problem of finding supporting prices for individually efficient commodity bundles:

**Lemma 2.** Let $x \in H^+$ be a commodity bundle, and let $W^x = \{ y \in H^+ : W(y) \geq W(x) \}$, where $W : H \to R$ is an $H$-continuous concave and increasing function, for which there exists $z$ with $W(z) > W(x)$. Then there exists a price $p \in H^+$ such that

(a) $\|p\| = 1$

(b) $\langle p, z \rangle > \langle p, x \rangle$ if $z > x$

and (c) $\langle p, y \rangle > \langle p, x \rangle$ for all $y$ in $W^x$.

((c) obviously implies (b) because $W$ is increasing.)

**Proof:** For a proof of this lemma, see Chichilnisky (1977, 1981a).

**Remark 4.** Lemma 2 is true for any Hilbert space $H$, including the Sobolev spaces $H^1_0$ and $H^2_0$ defined in the appendix. The crucial condition here is the $H$-continuity of $W$. Below we characterize certain $H$-continuous functions for $L^1_p$, $H^1_0$ and $H^2_0$. Lemma 2 is also true for $H = L^p$, $1 < p < \infty$, although in this case the prices are not in $L^p$ but in $L^q$ where $1/p + 1/q = 1$. $L^p$-continuity of $W$ is also studied in the appendix. It should be noted that Lemma 2 is not true for $H = L_\infty$, because $L_\infty$ is not a Hilbert nor a reflexive space. This means that in some cases one cannot support a convex set in $L_\infty$ by a positive function $p(t)$. It is easy to construct examples of strictly positive allocations and convex preferred sets of $L_\infty$-continuous functions such that if a function $p(t)$ supports the allocation, this function must be identically zero. The following example is related to one in Mas-Colell (1983).

**Example 2.**

Let

$$u_t(c) = \begin{cases} 2^t c & \text{for } |c| \leq 1/2^t \\ 1/2^t & \text{for } c > 1/2^t \\ -1/2^t & \text{for } c < -1/2^t \end{cases}$$

where $c \in (-\infty, \infty)$, see Figure 1.

---

2 It would be sufficient to require that the utility functions $W$ are defined and continuous on a neighborhood of the consummating sets. When consumption sets are the positive orthant $H^+$, this requires that $W$ admits a continuous extension to a neighborhood of $H^+$: this type of assumption was made in Smale (1974).
Figure 1: Illustration of Example 2. An $\ell_\infty$-continuous utility function giving rise to a preferred set supported only by zero prices. This example shows that Lemma 2 is not valid in $\ell_\infty$. 
For any sequence \( c \in \ell_\infty \), let

\[
W(c) = \sum_{t=1}^{\infty} u_t(c_t).
\]

Then if \( \sup_t |c_t| < K \),

\[
|W(c)| \leq K \left( \sum_{t=1}^{\infty} 1/2^t \right) < \infty.
\]

\( W \) is well defined, continuous concave on \( \ell_\infty^+ \) and increasing on \( \ell_\infty \). Let \( \omega \in \ell_\infty^+ \) be defined by \( \omega_t = 1/2^{2t+1} \), and let \( W(\omega) \) be the set \( \{ y \in \ell_\infty : W(y) \geq W(\omega) \} \).

Now assume that \( p \) is a supporting price for the set \( W(\omega) \) at \( \omega \), i.e., \( p \) is a continuous, positive linear function on \( \ell_\infty \) s.t. \( p(y) \geq p(\omega) \) whenever \( y \in W(\omega) \). Let \( p_t = p(e^t) \) where \( e^t_j = 1 \) if \( t = j \), and 0 otherwise. These \( p_t \)'s define the "sequence part" of the continuous linear functional \( p \).

By the usual marginal rate of substitution arguments, \( p_t = p_1 2^{t-1} \). We shall show that this leads to a contradiction when \( p_\perp \neq 0 \). Define \( z \in \ell_\infty^+ \) by \( z = 1/p_t \), and \( z^n \in \ell_\infty \) by \( z^n_t = z_t \) if \( t \leq n \) and 0 otherwise. Then \( z - z^n \) is positive for all \( n \), so that \( p(z) \geq p(z^n) \), since \( p \) is a positive function. But this implies \( p(z) \geq p(z^n) \). However, \( p(z^n) = \sum_{t=1}^{n} p z_t^n = n > p(z) \) for some \( n \) sufficiently large, which is a contradiction.

Therefore, \( p_\perp = 0 \) and \( p_t = 0 \) for all \( t \), i.e., the sequence part of any supporting price for \( W(\omega) \) at \( \omega \) is identically zero. The only possible prices for \( \omega \) are continuous linear functions on \( \ell_\infty \) whose sequence part is identically zero.

It is shown in Example 4 below that the function \( W \) is not norm-continuous in \( \ell_2 \). Obviously, in view of Lemma 2, if \( W \) were norm-continuous in \( \ell_2 \), there would be a strictly positive sequence of supporting prices \( (p_t) \) for \( W(\omega) \) at \( \omega \), \( (p_t) \in \ell_2^+ \).

The following results characterize additively separable \( H \)-continuous functions. Note that integral operators of the type characterized below are the
typical nonlinear operators defined in economics on infinite-dimensional spaces. We shall consider first the case where the measure $\mu(t)$ is $e^{-\lambda t}$, $0 < \lambda < 1$.

**Lemma 3.** Let

$$W(c) = \int_R e^{-\lambda t} u(c(t), t) dt,$$

where $u(c, t) : R^2 \to R$ satisfies the Caratheodory condition (A) above. Then $W$ defines a norm-continuous function from $H$ into $R$ if and only if

$$\left| u(c(t), t) \right| \leq a(t) + b(c)^2$$

where $b$ is a positive constant, $a(t) > 0$ and

$$\int_R a(t)e^{-\lambda t} < \infty.$$ 

If $u$ is concave and increasing, so is $W$.

**Proof:** See Chichlinsky (1977, proposition 1, p. 515).

The above lemma can be extended to spaces $H$ with any finite measure $\mu(t)$ which is absolutely continuous with respect to the Lebesgue measure on $R$:

**Lemma 4.** Let

$$W(c) = \int_R u(c(t), t) d\mu(t)$$

where $u : R^2 \to R$ satisfies (A). Then $W$ defines a norm-continuous function from $H$ into $R$ if and only if

$$\left| u(c(t), t) \right| \leq a(t) + b(c)^2$$

where $b$ is a positive constant, $a(t) > 0$ and

$$\int_R a(t) d\mu(t) < \infty.$$

The above two lemmas can also be extended to characterize continuous functions on the Sobolev spaces $H^1$ and $H^2$, and in $L_p$ spaces with $1 < p < \infty$. This is done in the appendix.

The following are examples of functions $W : H \to \mathbb{R}$ which are concave, increasing and norm-continuous.

Example 3.

Let

$$u_1(c) = \begin{cases} 1 - e^{-c} , & c > 0 \\ c , & c \leq 0 \end{cases} ;$$

$$u_2(c) = \alpha + \beta c , \quad \beta > 0 .$$

Clearly, for $i = 1$ or $2$,

$$|u_i(c)| \leq Bc^2 + D$$

for some positive $B$ and $D$, so that the conditions of Lemma 3 are satisfied (Figure 2). Hence the utility functions $W_1, W_2 : H \to \mathbb{R}$ defined by

$$W_1(c) = \int u_1(c)e^{-\lambda t} dt \quad \text{and} \quad W_2(c) = \int (\alpha + \beta c)e^{-\lambda t} dt ,$$

are both norm continuous, concave and increasing on the $L_2$ space $H$ with the finite measure $\mu(t) = e^{-\lambda t}$.

We now provide an example of a function which is not norm-continuous on $H$.

Example 4.

Consider the space of sequences $\ell_2$ with the finite measure $\mu(t) = \lambda^{-t}$, $\lambda > 1$, on the integers. For any sequence $(x_t)$ in $\ell_2$, define
Figure 2. Two examples of \( H \)-continuous functions. The graphs of \( u_1(c) \) and \( u_2(c) \), which give rise to norm-continuous functions on \( H \), as in Example 3, are shown.
\[ f(x_t) = \lim_{t} \inf |x_t| . \]

Obviously this function is only well-defined on bounded sequences in \( \ell_2 \), and in particular is not norm-continuous on \( \ell_2' \). We shall, moreover, show that \( f \) is not norm-continuous even when restricted to bounded sequences in \( \ell_2 \). Consider \((x^n)\), defined by

\[
x^n_i = \begin{cases} 
0 & \text{if } i \leq n \\
1 & \text{if } i > n 
\end{cases}.
\]

Then the limit of the sequence \((x^n)\) in the \( \ell_2 \) norm with finite measure \( \lambda^{-t} \) is the zero sequence \( x_i = 0 \) for all \( i \). Yet for all finite \( n \), \( f(x^n) = \lim_{t} \inf (x^n_1) = 1 \). Therefore, \( \lim_{t} f(x^n) = 1 \neq f \lim_{t} (x^n) = 0 \). It follows that \( f \) is not continuous in the \( \ell_2 \) norm even when restricted to bounded sequences of \( \ell_2' \).

The function \( W(x) \) defined in Example 2 above also fails to be continuous in the \( \ell_2 \) norm. To see this, note that if \( x_t = 1/2^{2t} \), then \( W(x) = \sum_{t=0}^{\infty} 2^t (1/2^{2t}) = 1 \).

Now consider the set of consumption vectors giving utility at least \( 1 - \epsilon \), for \( \epsilon > 0 \). If \( \bar{x} \) is such a vector, it must satisfy \( \bar{x} \geq 1/2^{2t} - \epsilon/2^t \) for all \( t \). But \( 1/2^{2t} - \epsilon/2^t > -\epsilon \). Hence the set of vectors giving welfare greater than \( 1 - \epsilon \), \( W^{-1}(1 - \epsilon, \infty) \) is contained in the set \( \ell_2^+ - (\epsilon) \) where \( \ell_2^+ \) is the positive cone and \( (\epsilon) \) is a sequence all of whose components equal \( \epsilon \). In \( \ell_2' \), \( \{ \ell_2^+ - (\epsilon) \} \) has no interior, as \( \ell_2^+ \) has no interior. Hence the set \( W^{-1}(1 - \epsilon, \infty) \) has no interior. However, this set is the pre-image of the open set \((1 - \epsilon, \infty)\) under \( W \). This means that \( W \) is not continuous in the \( \ell_2 \) norm because the inverse image of an open set under a continuous function is open.

We have now completed the results for the one-consumer case. The following section prepares the ground for the existence theorem by studying the structure of the set of Pareto efficient allocations.
4. Pareto Efficient Allocations

A consumption set for the $i$-th agent, denoted $X_i$, is either the positive cone of $H$, or all of $H$. There are $k$ agents.

An allocation $x$ is a vector $(x_1, \ldots, x_k) \in H^k$.

A feasible allocation is an allocation such that $\sum_{i=1}^{k} x_i \leq \omega$, and $x_i \in X_i$ for all $i$.

The set of feasible allocations is denoted $F$.

The utility value of an allocation $(x_1, \ldots, x_k) \in H^k$, denoted $W(x_1, \ldots, x_k)$, is the $k$-dimensional vector $(W_1(x_1), \ldots, W_k(x_k))$. The map $W : H^k \rightarrow R^k$ is called the utility map; it maps allocations in $H^k$ into the utility space of the $k$ agents, $R^k$.

A utility vector is a vector $(W_1(x_1), \ldots, W_k(x_k)) \in R^k$ where $(x_1, \ldots, x_k)$ is a feasible allocation. A utility vector $(W_1(x_1), \ldots, W_k(x_k))$ is weakly efficient if there is no other feasible allocation $(z_1, \ldots, z_k)$ such that $W_i(z_i) > W_i(x_i)$ for all $i$.

The Pareto frontier is the set of weakly efficient utility vectors in the positive cone $R^{k+}$.

Society's endowment $\omega$ is said to be desirable if $W_i(\alpha \omega) > W_i(0)$ for all $\alpha > 0$ and all $i$. This is always satisfied if $W_i$ is strictly increasing, since $W_i(0) = 0$ and the initial endowment is positive.

The next lemma extends the supporting price results of Lemma 2 to the many-agent case.

**Lemma 5.** Let $z = (x_1, \ldots, x_k)$ be a weakly efficient allocation and let $W_i$ be concave, increasing and $H$-continuous for all $i$, and take at least one value larger than $W_i(x_i)$. Then there exists a price $p \in H^+$, $\|p\| = 1$, such that $<p, y> \geq <p, x_i>$ for all $y$ satisfying $W_i(y) > W_i(x_i)$.

**Proof:** The proof extends Lemma 2 to the many-consumer case.
Consider the set $W_i = \{ x \in \mathcal{H} : W_i(x) \geq W_i(x_i) \}$. Let

$$V = \sum_{i=1}^{k} W_i \subset H,$$

and $\omega$ denote $x_1 + \ldots + x_k$; $\omega$ obviously belongs to $V$. Since each of the $W_i$'s is $H$-continuous and increasing, by Lemma 2 there exists a positive price $p_i \in H^+$ which supports $W_i$ at the point $x_i$, and we can take $\|p_i\| = 1$. In particular, $p_i$ will also support the convex set $W_i^{x_i} \cap X_i$.

Consider first the case where $X_i = H$. Then $W_i^{x_i} \cap X_i = W_i^{x_i}$, and efficiency of the allocation $(x_1, \ldots, x_k)$ implies that $\omega$ is in the boundary of $V$, for if $\omega \in \mathring{V}$, the interior of $V$, then there would exist $1 \in V$, with $1 = x_1^1 + \ldots + x_k^1$ (in $W_i^{x_i}$) such that $\sum_{i=1}^{k} x_i^1 < \omega$, and $W_i(x_1^1) > W_i(x_1)$, a contradiction. Now the set $V$ is convex and it has nonempty interior, because for each $i$, $W_i^{x_i}$ has a nonempty interior. It follows that there exists a nonzero supporting price $p$ for $V$ at $\omega$. Since $V$ contains a translate of the positive orthant, $p$ is positive, and we may take $\|p\| = 1$. It is now easy to check that $p$ supports each $W_i^{x_i}$ at $x_i$, for all $i$. This is because, by construction, $p$ is minimized over $V$ at $\omega$. The minima of a linear function on the sum of sets is equal to the sum of the minimum of the linear function on each set, i.e., $p(\omega) = \sum_{i=1}^{k} p(\bar{x}_i)$, where $\bar{x}_i \in W_i^{x_i}$ minimizes $p$ over the set $W_i^{x_i}$.

Now if $p(\bar{x}_i) > p(x_i)$ for some $i$, then for some $j$, $p(\bar{x}_i) < p(x_j)$, because $\sum_{i=1}^{k} p_i(x_i) = \omega = \sum_{i=1}^{k} p_i(\bar{x}_i)$, a contradiction.

Therefore $p_i(x_i) = \min p_i(x) \forall x \in W_i^{x_i}$, all $i$, so that $p$ supports $W_i^{x_i}$ at $x_i$, as we wished to prove.

Consider now the case where $X_i = H^+$ for all $i$. Note that if $p_i \in H^+$ supports $W_i^{x_i}$ at $x_i$, then $p_i$ also supports $W_i^{x_i} \cap X_i$ at $x_i$. Therefore the sets $W_i^{x_i} \cap X_i$ admit nonzero support prices in $H^+$ at $x_i$, for all $i$. This implies that there exists for each $i$ an open halfspace $H_i$ of $H$, such that $(W_i^{x_i} \cap X_i) \cap H_i^+ = \emptyset$.  

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Furthermore, since $W_i$ is increasing, the nonzero elements of the negative cone of $H$, $H^- = \{h : -h \in H^+\}$ are contained in $H_i$. Consider now $k \in H^-$, $\|k\| = 1$. Clearly $k$ is at a positive distance, say $\epsilon$, from $H - H_i$, since $H_i$ is open. This implies that there exists a vector at a positive distance from the Cone $C$ generated by the set $W_i^{x_i} \cap X_i$, with vertex $x_i$,

$$C = \{\lambda(y-x_i) : y \in W_i^{x_i} \cap X_i, \quad \lambda > 0\},$$

namely, the vector $k + X_i$. The existence of a vector at a positive distance from $C$ is condition (a) of Theorem 2.1 in Chichilnisky and Kalman (1980a, p.25). This theorem proved that the cone condition (a) is necessary and sufficient for the existence of a continuous nonzero supporting hyperplane for the set $W_i^{x_i} \cap X_i$ at $x_i$. We now show that this cone condition (a) is also equivalent to a condition used later in Mas-Colell (1983) which he calls "properness" at $x_i$. This latter condition requires the existence of a vector $h$ in $H^+$ and a number $\lambda$ such that if $x_i - \lambda h + y \in W_i^{x_i} \cap X_i$, then $\|y\| > \lambda \epsilon$. This "properness" condition simply means that there exists a vector $h$ at a positive distance $\epsilon$ from $W_i^{x_i} \cap X_i$ such that the cone with vertex $x_i$ generated by the open ball of radius $\epsilon$ around $h$, does not intersect the set $W_i^{x_i} \cap X_i$. From this latter formulation of the "properness" condition it is immediate that the cone condition (a) in Chichilnisky and Kalman (1980a) is identical to "properness" at the point $x_i$, for the set $W_i^{x_i} \cap X_i$.

Since the cone condition (a) is satisfied here at $x_i$, so is properness. Furthermore, a Hilbert space $H$ is a Banach lattice so that from Mas-Colell's Proposition VII.1, the weakly optimal allocation $\omega$ can be supported by a (nonzero) price $p \in H^+$. The rest of the proof is as for the case $X_i = H$. This completes the proof.

Remark 5. The above lemma shows \textit{inter-alia} that if $W_i$ is $H$-continuous, it satisfies properness at each given point $x_i \in H^+$: for any $x_i$ there will exist an $h \in H^+$ and an $\epsilon$ such that
implies \( \|v\| > \epsilon \lambda \). Note that \( \lambda \) and \( \epsilon \) may in general be different at different points \( x_i \). Mas-Colell (1983) requires the existence of an \( h \) and an \( \epsilon \) valid for all \( x_i \in H \), so that his condition is strictly stronger.

The next lemma studies the structure of the set of weakly efficient allocations.

**Lemma 6.** Assume that

(a) society's initial endowment \( \omega \) is desirable;

(b) each individual's consumption set \( X_i \) is convex, \( H \)-closed and bounded below in \( H \) (i.e., \( \exists h_i \in H : z \geq h_i \forall z \in X_i \)); and

(c) each individual's utility \( W_i : H \to \mathbb{R} \) is \( H \)-continuous, concave and increasing.

Then on each ray of the positive cone of utility space \( \mathbb{R}^{k+} \), there is a strictly positive weakly efficient allocation.

**Proof:** Since the initial endowment is desirable and each \( W_i \) is increasing there exists at least one strictly positive utility vector \( W_1(x_1), \ldots, W_k(x_k) \) on each ray of \( \mathbb{R}^{k+} \); we shall now show there exists a weakly efficient one as well.

Consider the map \( m : \mathbb{H}^k \to \mathbb{H} \), defined by \( m(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i \); this map is obviously continuous. Let

\[
H_\omega = \left\{ h \in \mathbb{H} : h \leq \omega \right\}
\]

\( H_\omega \) is clearly a closed subset of \( \mathbb{H} \). Since \( m \) is continuous, the set

\[
m^{-1}(H_\omega) = \left\{ (x_1, \ldots, x_k) \in \mathbb{H}^k : \sum_{i=1}^{k} x_i \leq \omega \right\}
\]

is closed in \( \mathbb{H}^k \). It is also clear that \( m^{-1}(H_\omega) \) is convex.

From this it follows that the set of feasible allocations \( F \) is convex and closed in \( \mathbb{H}^k \); \( F \) is the intersection of a convex set, the product of the \( X_i \)'s, with the convex closed set \( m^{-1}(H_\omega) \). Furthermore, \( F \) is bounded in \( \mathbb{H}^k \) because by assumption (b) for all \( (x_1, \ldots, x_k) \) in \( F \)
\[ h_i \leq x_i \leq \omega - \sum_{j \neq i} h_j, \]

where \( h_i \) is the lower bound for \( X_i \), and \( \omega \) is society's endowment. \( F \) is therefore closed, convex, and bounded in \( H^k \). In particular, \( F \) is weakly compact in \( H^k \), by Alaoglu's theorem (Dunford and Schwartz, 1958).

Consider now a sequence of utility vectors \((W^j)\) contained in the ray \( r \in \mathbb{R}^{k^+} \). Without loss of generality we may assume that \((W^j)\) is increasing in \( r \). By definition, \( W^j = (W_1(x_1^j), \ldots, W_k(x_k^j)) \) for some sequence \( (z^j) = (x_1^j, \ldots, x_k^j) \in F \). Let \( S = \sup_j(W^j) \) in \( r \). We shall prove that \( S \) is a utility vector corresponding to some feasible allocation in \( F \), and therefore \( S \) will be weakly efficient in \( r \).

Let \( z \) be the weak limit of the \((z^j)\) in \( F \), which exists because \( F \) is weakly compact. We may apply the Banach–Saks theorem, because \( H^k \) is reflexive, and deduce that there is a subsequence \((z^m)\) of \((z^j)\) such that

\[
\lim_{m \to \infty} \left( \frac{z^1 + \ldots + z^m}{m} \right) = z
\]

in the norm. Since \( F \) is convex, \( \frac{z^1 + \ldots + z^m}{m} \) is in \( F \) for all \( m \). Furthermore, by concavity of the \( W_i \),

\[
W_i \left( \frac{z^1 + \ldots + z^m}{m} \right) \geq \frac{W_i(z^1) + \ldots + W_i(z^m)}{m}
\]

We can choose the subsequence \( z^m \) so that

\[
\lim_{m} \frac{W_i(z^1) + \ldots + W_i(z^m)}{m} = \lim_{m} W_i(z^m) = \lim_{j} W_i(z^j) = S_i
\]

the \( i \)-th component of \( S = \sup_j(W) \); this is just the Banach–Saks theorem in the line. Note, however, that

\[
W_i \left( \frac{z^1 + \ldots + z^m}{m} \right) \leq W_i(z^m)
\]

because by assumption \((W^j)\) is increasing in the line \( r \). Now (1), (2) and (3) imply
that the utility vectors associated with the sequence \((z^1 + \ldots + z^m)/m\), i.e.,
\(W_i(z^1 + \ldots + z^m)/m\), converge to \(S\). This means that \(S = \sup_j W_j^j\) is reached within
\(F; W(z) = (W_1(z_1), \ldots, W_k(z_k))\) is a greatest utility vector on the ray \(r\), as we
wished to prove.

From Lemma 6, one obtains immediately:

**Corollary 7.** Under the assumptions of Lemma 6, the Pareto frontier is closed
in \(\mathbb{R}^{k+}\).

**Remark 6.** The results of Lemma 6 and Corollary 7 cannot be obtained in \(\ell_\infty\).
This is because the Banach-Saks theorem does not hold for \(\ell_\infty\) with the sup. norm,
since \(\ell_\infty\) is not reflexive. The following example is related to one in Mas-Colell
(1983):

**Example 5.**

Consider two agents in an economy where the space of commodity bundles
is \(\ell_\infty\). The consumption set of each agent is the positive orthant, \(\ell_\infty^+\), which is
convex, closed and bounded below. Society's initial endowment is the positive
sequence \(\omega = (1, 1, \ldots, 1, \ldots)\). Individual utilities are

\[
W_1(x) = \liminf_{t} \frac{x_{t}/2}{t} + \sum_{t=0}^{\infty} \frac{1}{2^t} (x_{t}/2);
\]

\[
W_2(y) = \sum_{t=0}^{\infty} \frac{1}{2^t} (y_{t}).
\]

Both \(W_1\) and \(W_2\) are norm continuous on \(\ell_\infty\) with the usual sup. norm.
They are also concave and monotone, yet the Pareto frontier is not closed. It is
easy to check that the Pareto frontier of this two-agent economy in \(\mathbb{R}^{2+}\) consists
of the line segment \([(1/2, 1), (1, 0)]\) and the point \((0, 1)\). The point \((1/2, 1)\) is not
in the Pareto frontier, and there are no weakly efficient allocations on any ray \(r\)
to the left of the point \((1/2, 1)\) except the point \((0, 1)\). To check this, consider
giving initially all the endowment to agent 1: we obtain utility vector \((1, 0)\). Then
subtract from agent 1 vectors of the form
\[(x^m) : x^m_i = \begin{cases} 
1 & \text{if } i \leq n \\
0 & \text{if } i > n 
\end{cases} \]

Agent 1's utility drops from 1 to 1/2, but never falls below 1/2. At the same time, agent 2's utility increases to 1, but never quite reaches 1. If all the endowment goes to agent 2, we obtain \((0, 1)\). This is illustrated in Figure 3.

The next lemma extends Lemmas 6 and 7 to cases where the individual consumption sets are not bounded in any way. The lemma requires, instead, that the family of individual preferences satisfy the following curvature condition (C):

(C) For any allocation \(z = (x^1, \ldots, x^k)\) there exists an allocation \(\tilde{z} = (\tilde{x}^1, \ldots, \tilde{x}^k)\) with \(W_i(x^i) = W_i(\tilde{x}^i)\) for all \(i\) and a price \(q(z)\) with \(<q, y> \geq <q, \tilde{x}^i>_i\) for \(y\) satisfying \(W_i(y) > W_i(\tilde{x}^i)\) such that \(\inf_{t} q(t) > \epsilon > 0\). In addition, each \(W_i\) is strictly concave.

Condition (C) requires that the preferred sets of the \(k\) individuals at any given allocation admit one price bounded above zero which simultaneously supports all of these sets. Potentially, each set will be supported at different points (see Figure 4). This assumption allows us to drop the requirement that the consumption sets be bounded below. Allowing unbounded consumption sets is of interest in modelling equilibria in financial markets in which unlimited short sales are allowed, as in Kreps (1981).

**Lemma 8.** Assume that:

(a) society's initial endowment \(\omega\) is desirable;

(b) each consumption set is the whole commodity space \(X^i = H\);

(c) each utility function \(W_i\) is concave, increasing and \(H\)-continuous, and the family of \(W_i\) satisfy the curvature condition (C).

Then on any ray of the positive cone in \(R^{k^+}\) there is a strictly positive weakly efficient utility vector, and the Pareto frontier is closed.

**Proof:** Consider as in Lemma 6 a sequence \((z^i)\) in \(F\,\!, \) with \(W_i(z^{i+1}) > W_i(z^i)\) for all \(i\) and \(j\). We cannot now apply the Banach-Saks theorem directly, because we have not assumed the \(X^i\) to be bounded below. However, by condition (C), the set
Figure 3: This illustrates Example 5. Lemma 6 does not hold in $\ell_p$; this example shows that the Pareto frontier in utility space ceases to be closed. Lemma 6 shows that in Hilbert spaces this frontier is always closed.
Figure 4: An illustration of the curvature condition to prove existence with unbounded consumption sets. The preferred-or-indifferent sets of four individuals are drawn. All have a common support, but at different points.
A of allocations \( z \in F \subset H^k \) with
\[
W_1(z_1), \ldots, W_k(z_k) \geq W_1^{1}(z_1), \ldots, W_k^{1}(z_k)
\]
can be shown to be a bounded set in \( H^k \). This is proved in Lemma 11 of the appendix. By the \( H \)-continuity and concavity of the utility functions, \( A \) is also closed and convex. We can therefore apply the proof of Lemma 6 to \( A \) rather than to \( F \). Since the sequence \( (z^j) \) is in \( A \) we obtain a norm limit \( z = (z_1, \ldots, z_k) \) in \( A \), with
\[
(W_1(z_1), \ldots, W_k(z_k)) = \sup \{ W_1(z_1^j), \ldots, W_k(z_k^j) \}.
\]
Since \( (W_1(z_1^j), \ldots, W_k(z_k^j)) \) is in the ray \( r \) for all \( j \), so is \( (W_1(z_1), \ldots, W_k(z_k)) \).

The following is an example of a function defined and continuous in \( H \), strictly concave, monotone, giving zero value to the zero consumption bundle, and satisfying the curvature condition (C):

**Example 6**

Let \( W(x) = \int_0^1 (1 - e^{-x(t)}) e^{-\lambda t} dt \), and let the commodity space be the space of \( L_2 \) measurable functions on the unit interval with the inner product
\[
\langle x, y \rangle = \int_0^1 x(t) \cdot y(t) dt.
\]
\( H \)-continuity on a neighborhood of \( R^+ \) follows from Lemma 3.

It is easily verified that for each \( t \), the instantaneous function \( 1 - e^{-x(t)} \) is strictly concave, increasing, and maps zero to zero. For any consumption path \( \bar{x}(t) \), \( 0 \leq t \leq 1 \), define the level set of \( \bar{x} \), \( L(\bar{x}) \), as
\[
L(\bar{x}) = \{ x \in H : W(x) = W(\bar{x}) \}.
\]
All individuals have identical preferences. We shall demonstrate that the curvature condition (C) is satisfied by showing that for any consumption path \( x(t) \) there exists in \( L(x) \) a path supported by the price \( p(t) = 1 \) a.e. The first step is to characterize paths supported by \( p(t) = 1 \) a.e., and the second is to show that there exists such a path in \( L(x) \) for any \( x(t) \).
A path supported by \( p(t) = 1 \) a.e. solves the problem:

\[
\text{minimize} \quad \int_0^1 p(t)x(t)\,dt \\
\text{subject to} \quad \int_0^1 (1 - e^{-x(t)})e^{-\lambda t}\,dt \geq K
\]

where \( K \) is the utility value of the level set of the path. This is an isoperimetric variational problem (see Dasgupta and Heal, 1974) whose solution satisfies

\[(E) \quad x(t) = x(0) - \lambda t\]

where \( x(0) \) is a constant whose value will depend on \( K \). So we now have to verify that for any path \( \tilde{x}(t) \), there exists in \( L(\tilde{x}) \) a path satisfying \( (E) \). Along a path satisfying \( (E) \) we have

\[
\int_0^1 (1 - e^{-x(t)})e^{-\lambda t}\,dt = \int_0^1 (e^{-\lambda t} - e^{-x(0)})\,dt \\
= \frac{1}{\lambda} \left( 1 - \frac{1}{e^\lambda} \right) - e^{-x(0)}
\]

Now the maximum possible value of the function \( \int_0^1 (1 - e^{-x(t)})e^{-\lambda t}\,dt \) is

\[\frac{1}{\lambda} \left( 1 - \frac{1}{e^\lambda} \right)\], as \( (1 - e^{-x(t)}) \) is bounded above by one. Hence by picking \( x(0) \) appropriately, the integral can be given any value between its maximum and minus infinity, along a path satisfying \( (E) \). Hence on every level set of \( \int_0^1 (1 - e^{-x(t)})e^{-\lambda t}\,dt \) there is a path whose support is orthogonal to the constant unit sequence or function, and in particular bounded above zero.

The following are examples of \( H \)-continuous functions which do not satisfy the curvature condition.
Example 7

Let \( k = 2 \), and

\[
W_i = \int u_i(c(t))e^{-\lambda t} \, dt, \quad i = 1, 2.
\]

Then \( u_i(c(t)) = \alpha_i + \beta_i c(t) \), \( \alpha_i, \beta_i \) in \( \mathbb{R}^+ \) and \( \beta_1 \neq \beta_2 \), does not satisfy (C).

In the following, a convex preference \( W : H \to \mathbb{R} \) is said to be weakly regular if:

- \( \text{(R)} \) The set of supporting hyperplanes to the preferred sets \( W^X \cap X, \ x \in H \), is weakly bounded away from zero, where \( X \) is the consumption set.

Example 8

- \( \text{(A)} \) Condition (R) is rather weak when \( X = H \). In this case, examples of preferences satisfying (R) are those smooth preferences whose gradient vectors are constrained to lie in a compact set disjoint from the origin (compactness could be either weak or strong). Another example in \( l^+_2 \) is a preference with supporting hyperplanes \( p \in l^+_2 \) such that \( \min_{t \in T} p(t) > \epsilon > 0 \) for all but finitely many \( t \). This means that marginal utilities are bounded away from zero except on a finite set of commodities.

- \( \text{(B)} \) When the consumption set \( X \) is either \( H \) or \( H^+ \), condition (R) is satisfied when there exists a finite set of vectors \( v_1, \ldots, v_{l} \in H \) such that all supporting hyperplanes for \( W^X \cap X \) have normals \( p \), with \( \langle p, v_j \rangle > \epsilon > 0 \) for at least one \( v_j \) within a given set \( \{v_1, \ldots, v_{l}\} \). This latter example shows that the condition (R) is weaker than the properness condition of Mas-Colell (1983), who requires that \( \langle p, v \rangle = 1 \) for a single vector \( v \).

We have now established all the preliminary results needed for a proof of the existence of a competitive equilibrium.
5. **Existence and Optimality of Competitive Equilibrium**

Let \( \Delta \) denote the unit simplex in \( \mathbb{R}^k \),

\[
\Delta = \left\{ y \in \mathbb{R}^k : \sum_{i=1}^{k} y_i = 1 \right\}
\]

A feasible allocation \( (x_1, \ldots, x_k) \) in \( H^k \) is a quasi-equilibrium when there is a price \( p \neq 0 \), with \( \langle p, \omega \rangle = \langle p, x \rangle \) and \( \langle p, z \rangle \geq \langle p, x \rangle \) for any \( z \) with \( W_i(z) \geq W_i(x) \), all \( i \). A feasible allocation is an equilibrium when \( W_i(z) > W_i(x) \Rightarrow \langle p, z \rangle > \langle p, x \rangle \), all \( i \). The latter holds at a quasi-equilibrium such that \( \langle p, \omega \rangle > 0 \) for any \( i \).

**Theorem:** Consider an economy with a desirable initial endowment\(^3\) \( \omega = (\omega_1, \ldots, \omega_k) \) in \( H^k \), and such that the individual utilities \( W_i : H \to \mathbb{R} \) are \( H \)-continuous, increasing and concave, and are not satiated on the consumption sets \( X_i \). Assume either that consumption sets \( X_i \) are translations of the positive cone \( H^+ \) and one agent's utility satisfies (R), or else that consumption sets are the whole space and preferences satisfy the curvature condition (C). Then there exists a competitive quasi-equilibrium allocation \( (x_1^*, \ldots, x_k^*) \in H^k \), with a supporting price \( p \) in \( H^+ \), \( \|p\| = 1 \). The allocation \( (x_1^*, \ldots, x_k^*) \) is a competitive equilibrium when all initial endowments \( \omega_i \) are strictly positive. The competitive equilibrium is Pareto efficient.

**Proof:** In view of the results of Sections 3 and 4, we may now use a version of Negishi's (1960) argument, following an adaptation by Mas-Colell (1983). We define a correspondence \( \phi : \Delta \to T \), where \( T \) is the set \( \left\{ y \in \mathbb{R}^k : \sum_{i=1}^{k} y_i = 0 \right\} \), with the property that any of its zeros is a quasi-equilibrium.

---

\(^3\) The condition that the initial endowment be desirable could be substituted by a weaker condition which appears in McKenzie (1981).
With any \( S \in \Delta \), associate the feasible allocation \( v(S) = x_1(S), \ldots, x_k(S) \) which gives the greatest utility vector colinear with \( S \). Such an allocation exists and gives a nonzero utility vector by Lemmas 6 and 8.

Without loss of generality, assume \( \sum_{i \in K} x_i(S) = \omega \). Now let \( P(S) = \{ p \in H^+: \| p \| = 1, \text{ } p \text{ supports the weakly efficient allocation } v(S) \} \). \( P(S) \) is convex, and is nonempty by Lemma 5. \( \phi(S) \) is now defined as

\[
\phi(S) = \left\{ \langle p, \omega - x_1(S) \rangle, \ldots, \langle p, \omega - x_k(S) \rangle : p \in P(S) \right\}.
\]

\( \phi(S) \) is nonempty and convex-valued; \( \sum_{i = 1}^k z_i = 0 \) for \( z \in \phi(S) \); and \( 0 \notin \phi(S) \) if and only if \( v(S) \) is a quasi-equilibrium. The next step is to show that \( \phi \) is upper hemi-continuous, i.e., if \( S^n \to S \), \( z^n \in \phi(S^n) \), \( z^n \to z \) then \( z \in \phi(S) \).

Consider now the feasible allocation \( v(S) \) in \( H^k \), where \( S = \lim_{n \to \infty} S^n \). Let \( u \) be any other allocation with \( W_i(u) > W_i(x_i(S)) \), all \( i \). Since \( S^n \to S \), eventually one has \( W_i(u) > W_i(x_i(S^n)) \), which implies \( \langle p^n, u \rangle > \langle p^n, x_i(S^n) \rangle = \langle p^n, \omega_i \rangle - z^n_i \), where \( z^n_i \) is the \( i \)-th component of \( z^n \in \phi(S^n) \), and \( p^n \in P(S^n) \): this follows from the definitions of \( z^n \) and \( p^n \).

Let \( (p^n) \) be any such sequence of price vectors in \( P(S^n) \). Since \( \| p^n \| = 1 \) for all \( n \), and closed bounded sets in \( H \) are weakly compact by Banach-Alaoglu's theorem (Dunford and Schwartz, 1958), there exists a \( p \) with \( \| p \| < 1 \) and a subsequence \( (p^m) \) of \( (p^n) \) such that \( \langle p^m, f \rangle \to \langle p, f \rangle \) for all \( f \) in \( H \). Note that each \( p^m \) supports the preferred sets of all agents, so that by condition (R), the weak limit of \( p^m = p \neq 0 \).

In particular, such a \( p \) exists for \( f = u \), i.e., \( \langle p^m, u \rangle \to \langle p, u \rangle \).

Therefore in the limit \( \langle p, u \rangle > \langle p, \omega_i \rangle - z_i \). Since this is true for all \( u \) with \( W_i(u) > W_i(x_i(S)) \), it is also true for \( u \) with \( W_i(u) > W_i(x_i(S)) \) and in particular for \( u = x_i \), i.e.,

\[
\langle p, x_i(S) \rangle > \langle p, \omega_i \rangle - z_i \quad \text{for all } i.
\]

Since \( \sum_{i} x_i(S) = \sum_{i} w_i \), we have \( \langle p, x_i(S) \rangle = \langle p, w_i \rangle - z_i \) for all \( i \), implying that \( z \in \phi(S) \), as we wanted to prove.
The proof is completed by showing that $\phi$ has a zero. This is a standard application of Kakutani's fixed point theorem. Consider the map $\psi : \Delta \to \Delta$ defined by $\psi(S) = S + \phi(S)$. It is upper hemi-continuous, nonempty and convex valued. In the boundary of $\Delta$, $\psi$ points inwards: if $S_i = 0$ for some $i$, then $x_i(S)$ is indifferent to $0$ for $i$, so that $0 \geq p \cdot x_i(S) \geq 0$. This implies that $z_i = p \cdot (\omega_i - x_i) \geq 0$ so that $\psi$ points inwards at the boundary of $\Delta$. Since $\psi$ satisfies appropriate boundary conditions, is upper hemi-continuous and convex valued, $\psi$ has therefore a fixed point, which is a zero of $\phi$.

Finally, we show that there exists a competitive equilibrium when all initial endowments $w_i$ are strictly positive.

When $w_i$ is strictly positive, the value of $\omega_i$ at the equilibrium prices is also strictly positive, i.e.,

$$\int_{\mathbb{R}} p(t) \cdot \omega_i(t) d\mu(t) > 0 .$$

More generally, in a Hilbert space, $p \gg 0$, $\omega \gg 0$ and $\langle p, \omega \rangle = 0$ always imply $p = 0$.

It follows that when all initial endowments are strictly positive, then $\langle p, \omega_i \rangle > 0$ for all $i$ and therefore the quasi-equilibrium is an equilibrium. In this case, in particular, the value of the society's initial endowment $\omega$ is always positive, i.e.,

$$\langle p, \omega \rangle > 0 .$$

That the competitive equilibrium is Pareto efficient follows from standard arguments, see, for example, Debreu (1954).

Remark 7. In other infinite-dimensional spaces such as $\ell_\infty$, $L_\infty$ or $C(X)$, the fact that all initial endowments are strictly positive does not imply the existence of a competitive equilibrium. This is because in contrast to Hilbert spaces, in such space $p \gg 0$, $\omega \gg 0$ and $\langle p, x \rangle = 0$ does not imply $p = 0$. The following is an example.
Example 9.

Consider the strictly positive vector $\omega \in \ell_\infty$, defined by $\omega_i = 1/2^i$, and the positive continuous linear function $p(y) = \liminf_i y_i$. Then $\langle p, \omega \rangle = 0$, even though $p \succ 0$ and $\omega \gg 0$.

Therefore $p \succ 0$ and $\omega \gg 0$ and $\langle p, x \rangle = 0$ does not imply $p = 0$ in $\ell_\infty$. 
6. Appendix

A. Extension of the existence results to Sobolev spaces $H^1$ and $H^2$ and to $L^p$ spaces with $\infty > p > 1$

All the results stated in the paper apply to $H^1$, $H^2$ and to $L^p$ spaces ($1 < p < \infty$), provided the assumptions are made in the respective norms of these spaces. We made two types of assumptions. One is on $H$-continuity of the utility functions $W_i$: this appears in all the lemmas, from 1 to 8, and in the existence theorem. The second is an assumption on boundedness of sets in $H$. This appears in Lemmas 6, 7 and the existence theorem.

The boundedness condition is standard and requires no further explanation: it means the same in all spaces, namely that there is an upper bound to the norm of all elements in the (bounded) set. Of course, since the norms of $L^p$, $H^1$ and $H^2$ are different from the $L^2$ norm, this must be interpreted accordingly, e.g., boundedness in $H^1$ requires that the functions and their derivatives have uniformly bounded $L^2$ norms; in $H^2$, instead, the functions and their first two derivatives must have uniformly bounded $L^2$ norms.

The assumption of $H$-continuity, however, requires a different treatment. It is not the same to be norm-continuous in $L^2$, as in $L^p$, or $H^1$. The following results characterize continuous functions in $H^1$, $H^2$ and $L^p$ ($\infty > p > 1$). The analogy with the results in Section 3, Lemmas 4 and 5, is immediate. As before, all measures $\mu(t)$ are finite.

Lemma 9. Let $W = \int_R u(c(t), t) d\mu(t)$, with $u$ satisfying the Caratheodory condition (A). Then $W$ defines a norm-continuous function from $L^p$ to $R$, ($1 < p < \infty$), if and only if

$$|u(c(t), t)| \leq a(t) + bc(t)^p$$

where $a(t) \geq 0$, $\int_R a(t)^p d\mu(t) < \infty$ and $b > 0$.

The proof is the same as given for the case of $L^2$ in Chichilnisky (1977).
Lemma 10. A function $W = \int_{\mathbb{R}} u(c(t), t) d\mu(t)$ is continuous from $H^1$ to $\mathbb{R}$ if and only if the conditions in Lemma 9 are satisfied for $u$ and for $Du$. $W$ is continuous from $H^2$ to $\mathbb{R}$ if the conditions in Lemma 9 are satisfied for $u$, $Du$ and $D^2u$.

The proof is as in Chichilnisky (1977).

B. Results for unbounded consumptions sets

Recall the curvature condition (C):

(C) For any allocation $z = (x_1, \ldots, x_k)$ there exists an allocation $\tilde{z} = (\tilde{x}_1, \ldots, \tilde{x}_k)$ with $W_i(x_i) = W_i(\tilde{x}_i)$ for all $i$ and a price $q(z)$ with $\langle q, y \rangle > \langle q, \tilde{x}_i \rangle$ for all $y$ satisfying $W_i(y) > W_i(\tilde{x}_i)$, and such that $\inf_t q(t) > \epsilon > 0$. In addition, the $W_i$ are strictly concave.

Recall also that the set of feasible allocations $F$ is

$$F = \left\{ (x_1, \ldots, x_k) \in H^k : \sum_{i=1}^{k} x_i \leq \omega \right\}$$

where $\omega$ is society's initial endowment, $w \in H^+$. Let $W_{i}^{\tilde{z}} = \{ z \in H : W_i(z) \geq W_i(\tilde{z}_i) \}$.

Lemma 11. Assume that the utility functions are $H$-continuous, and satisfy the curvature condition (C). Then the set $A$ of allocations $z = (z_1, \ldots, z_k)$ in $F$ with $(W_1(z_1), \ldots, W_k(z_k)) \geq (W_{i}^{\tilde{z}_i}(z'_1), \ldots, W_{i}^{\tilde{z}_i}(z'_k))$ is a bounded set in $H^k$, for any $(z'_1, \ldots, z'_k)$ in $H^k$.

Proof: Define the set $A \subseteq H$:

$$A = \{ z \in H : W_i(z) \geq W_i(z'_i) \text{ and } \langle z, q \rangle \leq \langle \omega, q \rangle \}$$

where $q \in H^+$ is the vector given by the curvature condition (C).

The first step of the proof is to show that $A$ is bounded in $H$. By (C), there exists $\tilde{z}$ with $W_i(\tilde{z}) = W_i(z'_i)$ and such that the set $W_{i}^{\tilde{z}}$ has a supporting hyperplane
SH(\tilde{z}) at \tilde{z} which is orthogonal to q. Write this hyperplane as \( \{ z \in H : \langle z, q \rangle = K \} \).

Since for all \( z \in W_i \) we have \( \langle z, q \rangle \geq K \), A is contained in the following set:

\[
A \subseteq \{ z : K \leq \langle z, q \rangle \leq \langle \omega, q \rangle \},
\]

i.e., A is "sandwiched" between two hyperplanes, see Figure 5.

Now suppose A is not bounded. Then we can pick an increasing sequence of real numbers \( N_j \), \( \lim_{j \to \infty} N_j = \infty \), and a sequence of points \( y_j \) in A, such that

\[
\| y_j - \tilde{z} \| > N_j \quad \text{for each } j.
\]

As \( y_j \in A \),

\[
k_i \leq \langle y_j, q \rangle \leq \langle \omega, q \rangle \quad \text{for each } j,
\]

and since A is convex, \( \lambda y_j + (1 - \lambda)\tilde{z} \) is in A for \( 0 \leq \lambda \leq 1 \). With each point \( y_j \) we shall associate three further points. \( x_j \) is the point on the line segment \( \lambda y_j + (1 - \lambda)\tilde{z}, \ 0 \leq \lambda \leq 1 \), at distance one from \( \tilde{z} \). Such a point always exists for \( j \) sufficiently large. \( n_j \) is the point on the line through \( \tilde{z} \) and \( y_j \) satisfying \( \langle n_j, q \rangle = \langle \omega, q \rangle \). Such an \( n_j \) must exist for \( j \) sufficiently large, because if \( n_j \) does not exist, then \( y_j \) must belong to \( SH \cap A \), and since \( SH \) is a hyperplane and \( y_j \neq \tilde{z} \) this would contradict strict concavity of \( W_i \). Finally, \( m_j \) is the point in the hyperplane \( SH(\tilde{z}) \) nearest to \( y_j \). Figure 5 illustrates these points.

Now clearly,

\[
\| n_j - \tilde{z} \| > \| y_j - \tilde{z} \| > N_j,
\]

and

\[
(n_j - \tilde{z}) = (m_j - \tilde{z}) + (n_j - m_j).
\]

Taking inner products with q

\[
\langle (n_j - \tilde{z}), q \rangle = \langle (m_j - \tilde{z}), q \rangle + \langle (n_j - m_j), q \rangle.
\]

Since by construction \( (m_j - \tilde{z}) \) is orthogonal to q, for all \( j \)

\[
\langle (n_j - \tilde{z}), q \rangle = \langle (n_j - m_j), q \rangle.
\]

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Figure 5. The set A in Lemma 11.
But \((n_j - m_j) = \lambda q\) for some scalar \(\lambda\), so that

\[
\langle n_j - z, q \rangle = \lambda \| q \|
\]

Now

\[
\langle x_j - z, q \rangle = \langle n_j - z, q \rangle \frac{\| x_j - z \|}{\| n_j - z \|}
\]

But \(\| x_j - z \| = 1\) by construction, and also by construction \(\lim_{j \to \infty} \| n_j - z \| = \infty\).

Hence \(\lim_{j \to \infty} \langle x_j - z, q \rangle = 0\).

Now, \(\{x_j\}\) is a sequence lying in a closed and bounded subset of \(H\), namely the unit sphere centered at \(\tilde{z}\). It therefore has a subsequence which converges weakly to a point \(m\) in \(\text{SH}(\tilde{z})\). By the Banach-Saks theorem, there exists a subsequence denoted also by \((x_j)\), such that the sequence of means

\[
m_n = \left\{ \frac{x_1 + x_2 + \ldots + x_n}{n} \right\}
\]

converges in the norm of \(H\) to the same limit \(m\). By convexity the \(m_n\) are all contained in \(A\). Their limit \(m\) is in \(\text{SH}(\tilde{z})\) and is distinct from \(\tilde{z}\) because \(\| x_j - z \| = 1\) for all \(j\). \(A\) is norm-closed because \(W_i\) is norm-continuous. The limit \(m\) is also in \(A\). Hence there are two points \(m\) and \(\tilde{z}\) both in \(W_i z_i^1\) and both lying in \(\text{SH}(\tilde{z})\), contradicting the strict concavity of the \(W_i\). Hence \(A\) must be bounded, as asserted.

Next consider the set

\[
B = \left\{ (z_1', \ldots, z_k') \in H^k \mid z_i \in W_i z_i^1 \text{ and } \sum_{i=1}^k z_i \leq \omega \right\}
\]

By the curvature condition \((C)\), for each \(z_i^1\) there exists \(\tilde{z}_i\) such that \(W_i(\tilde{z}_i) = W_i(z_i^1)\) and \(W_i\) has a supporting hyperplane \(\text{SH}(\tilde{z}_i)\) orthogonal to \(q\) at \(\tilde{z}_i\). Let the equation of this hyperplane be \(\langle z, q \rangle = K_i\), and let \(M = \min_i \{K_i\}\). So for any \(i = 1, \ldots, k\),
\[ z_i' \in W_1 \Rightarrow <z, q> \geq M. \]

Now for any \( z_i \),
\[
\sum_i z_i < \omega \Rightarrow \sum_i <z_i, q> \leq <\omega, q>
\]
so that for any \( z_j \),
\[
<z_j, q> \leq <\omega, q> - \sum_{i \neq j} <z_i, q>.
\]

But
\[
z_i' \in W_1 \Rightarrow <z, q> \geq M
\]

so
\[
<z_j, q> \leq <\omega, q> - (k-1)M
\]
and
\[
M \leq <z_j, q> \leq <\omega, q> - (k-1)M.
\]

Hence, if \((z_1', \ldots, z_k') \in B\), then each \( z_i' \), \( i = 1, \ldots, k \), satisfies

(i)
\[
z_i' \in W_1
\]

(ii)
\[
M \leq <z_j, q> \leq <\omega, q> - (k-1)M.
\]

Since (i) and (ii) define a set of the same form as the set \( A \) shown in the first part of the lemma to be bounded, \( B \) is bounded in \( H^k \), as required. \( \blacksquare \)
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