MULTISUMMABILITY AND STOKES MULTIPLIERS OF LINEAR MEROMORPHIC DIFFERENTIAL EQUATIONS

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B.L.J. Braaksma

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Dedicated to Henry Antosiewicz on the occasion of his 65th birthday.

0. Introduction

In this paper we consider linear meromorphic differential equations

\[ D[y] = 0, \quad \text{where} \quad D[y] = x \frac{dy}{dx} - A(x)y \]

with \( A(x) \) an analytic \( n \times n \)-matrix in a reduced neighborhood of \( 0 \in C \) having a pole at \( x = 0 \) and \( y(x) \in C^n \). It is well-known that there exists a formal fundamental matrix

\[ \hat{Y}(x) = \hat{F}(t)t^\Lambda \exp Q(t^{-1}), \quad t = x^{1/p}, \]

where \( p \in N \), \( Q(t) \) a diagonal matrix whose entries are polynomials in \( t \), \( \Lambda \) a constant matrix such that \( \Lambda Q(t) = Q(t)\Lambda \) and \( \hat{F} \) an \( n \times n \)-matrix with entries in \( C[t] \).

Here we show that the formal series \( \hat{F}(t) \) are multisummable in the sense of Ecalle [8], [9] and Martinet and Ramis [14]. The multisums \( F \) of \( \hat{F} \) exist on certain sectors \( S \) with vertex at 0 such \( F(t) \sim \hat{F}(t) \) as \( t \to 0 \) on \( S \) and

\[ Y(x) = F(t)t^\Lambda \exp Q(t^{-1}), \quad t = x^{1/p}, \]

is a fundamental matrix of (0.1) on \( S_1 \), where \( x \in S_1 \), iff \( t \in S \). The sectors \( S \) are such that every pair of distinct diagonal elements of \( Q(t^{-1}) \) has at most one dominance change on \( S \), in accordance with Jurkat [13], sect.7f.

In two consecutive sectors \( S \) and \( \tilde{S} \) of this type we get two multisums \( F \) and \( \tilde{F} \) and two fundamental matrices \( Y \) and \( \tilde{Y} \). Then \( \tilde{Y}(x) = Y(x)C \) where \( C \) is a nonsingular matrix, the Stokes matrix. For the elements of \( C \), the Stokes multipliers, we give relations in terms
of Borel transforms and accelerates of the formal matrix \( \hat{F} \) in case an extra condition is fulfilled. The Stokes multiplier are determined by the behavior of those accelerates at their singularities, so by resurgence in the sense of Ecalle. Another approach to the study of Stokes multipliers has recently been given by Immink [12].

The multissummability result is derived from a general multissummability property of equations \( D[y] = g \) where \( D \) is as in (0.1) and the coefficient matrix \( A(x) \) and \( g(x) \) are multisums of formal series. If this equation has a formal series solution then it is multissummable and its sum is an analytic solution.

This paper may be seen as an example of Ecalle's general theory of resurgent functions [7] and in particular of his later theory of accelero summability [8], [9]. Very general results concerning \( D[y] = g(x, y) \) with \( g \) nonlinear in \( y \) have been announced by Ecalle during the "Journées de la résurgence" in June 1989 in Paris. There he gave a rough idea of a proof.

We have worked out the linear case in his spirit using a special case of his concept of accelero summability viz. multissummability. Here we follow the treatment of multissummability given recently by Martinet and Ramis [14]. We will return to the nonlinear case in another paper, using the same methods.

The multissummability property of formal fundamental matrices of (0.1) may also be derived from the decomposition of \( \hat{F}(t) \) as a product of summable factors given by Ramis [15]. Another proof will be given in a forthcoming paper of Balser et al. [1] using a cohomological method (cf. [17]).

The organization of this paper is as follows. In section 1 we collect the information on acceleration and multissummability needed in the sequel. The multissummability of formal solutions of \( D[y] = g \) is studied in section 2. Here we need some properties of solutions of associated convolution equations which are proved in section 3. In section 4 we consider the multissummability property of fundamental matrices of (0.1). Then we give a decomposition of multissummable functions by means of finite Laplace transforms in section 5. This we use in section 6 to obtain relations for Stokes multipliers. We end with an example for the determination of these multipliers for a differential equation with two levels.
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1. Acceleration and multizummability. We give definitions and properties of these notions as given by Martinet and Ramis in a forthcoming paper [14]. If \( \theta_- < \theta_+ \) we define

\[
S(\theta_-, \theta_+) := \{ x \in C_{\infty} | \theta_- < \arg x < \theta_+ \},
\]

where \( C_{\infty} \) denotes the Riemann surface of \( \log x \). Let \( k > 0, S := S(\theta_-, \theta_+), \theta_- < \theta_+ \). If \( f \) is defined in a neighborhood of \( \infty \) in \( S \), then we say \( f \) is of exponential growth of order \( \leq k \) if there exists a positive constant \( C \) such that

\[
f(x) = O(1) \exp(C|x|^k) \text{ as } x \to \infty \text{ in } S.
\]

For short we then write \( f \in E(k, S) \).

Assume \( S \) is as above, \( f \) is analytic in \( S, k > 0, (1.2) \) holds and there exists \( \epsilon > 0 \) such that

\[
f(x) = O(x^{\epsilon-k}) \text{ as } x \to 0 \text{ in } S.
\]

Then we define the Laplace transform of order \( k \) of \( f \) in \( S \) by

\[
(L_{k,S}f)(x) := \int_0^{\infty} \exp(i\theta) \{ \exp - (\xi/x)^k \} f(\xi)d(\xi^k), \ x \in S(C)
\]

where

\[
S(C) := \{ x \in S \left( \theta_- - \frac{\pi}{2k}, \theta_+ + \frac{\pi}{2k} \right) | \text{Re} \left( e^{i\theta} x^{-k} \right) > C \text{ for some } \theta \in (\theta_-, \theta_+) \}
\]
Here $\theta \in (\theta_-, \theta_+)$ is chosen so that $\text{Re} \left( e^{ik\theta} x^{-k} \right) > C$. If there is no fear of confusion we write $L_k$ instead of $L_{k,S}$.

If $z \in S$ and $f$ is as above except that we do not assume (1.2) we define

\[
(L_k^z f)(x) := \int_0^z \{\exp - (\xi/x)^k\} f(\xi) d(\xi^k), \quad x \neq 0,
\]

where the path of integration is in $S$. Then $(L_k^z f)(x)$ is an entire function of $x^{-k}$.

If $g$ is an analytic function in a region $S(C)$ as in (1.5) with $C \geq 0$, $\theta_- < \theta_+$, then we define the Borel transform of order $k$ of $g$ in $S(C)$ by

\[
(B_{k,S(C)} g)(\xi) := \frac{1}{2\pi i} \int_\gamma \{\exp (\xi/x)^k\} g(x) d(x^{-k}),
\]

where $\gamma$ is a contour in $S(C)$ from $0 \exp \{i(\theta + \arg \xi)\}$ to $0 \exp \{i(-\theta + \arg \xi)\}$ for some $\theta \geq \frac{\pi}{2k}$ provided the integral converges, and $\xi \in S(\theta_-, \theta_+)$.  

If $\theta_- < \theta_+$ and $f$ is analytic in $S := S(\theta_-, \theta_+)$ and if (1.2) and (1.3) hold with $k > 0$, $\varepsilon > 0$ then

\[
B_{k,S(C)} L_{k,S} f = f.
\]

If $k > 0$ we define $\rho_k f$ by $(\rho_k f)(x) := f(x^{1/k})$ and

\[
\begin{align*}
\left\{ \begin{array}{l}
(f \ast_k g) := (\rho_k)^{-1}((\rho_k f) \ast (\rho_k g)), \text{ i.e.} \\
(f \ast_k g)(x) = \int_0^x f((x^k - t^k)^{1/k}) g(t) d(t^k),
\end{array} \right.
\]

provided the integral exists. In particular,

\[
\frac{x^{\lambda-k}}{\Gamma(\frac{\lambda}{k})} \ast \frac{x^{\mu-k}}{\Gamma(\frac{\mu}{k})} = \frac{x^{\lambda+\mu-k}}{\Gamma(\frac{\lambda+\mu}{k})}, \quad \text{if } \text{Re} \lambda > 0, \text{ Re } \mu > 0.
\]

If $f$ and $g$ are analytic in $S = S(\theta_-, \theta_+)$, $\theta_- < \theta_+$ and (1.2) and (1.3) are satisfied by $f$ and $g$ with some $k > 0$, $\varepsilon > 0$, $C > 0$, then

\[
L_{k,S} (f \ast_k g) = (L_{k,S} f)(L_{k,S} g).
\]
If $\varphi$ and $\psi$ are analytic in $S(C)$ and $\varphi(x)$, $\psi(x)$ are $O(x^\varepsilon)$ as $x \to 0$ in $S(C)$ where $\varepsilon > 0$ then

\begin{equation}
(B_k, S(C)\varphi)^*_k (B_k, S(C)\psi) = B_k, S(C)(\varphi \psi).
\end{equation}

Next, we consider the acceleration operator $A_{k',k}$ where $k' > k > 0$. Let $f$ be analytic in $S := S(\theta_-, \theta_+)$ where $\theta_- < \theta_+$ and assume (1.2) and (1.3) with $C > 0$, $\varepsilon > 0$. Then we define

\begin{equation}
(A_{k',k}, Sf)(\xi) := (B_{k',k}, S(C)\mathcal{L}_{k, S} f)(\xi),
\end{equation}

if $\xi \in S' := S(\theta_- - \frac{\pi}{2k}, \theta_+ + \frac{\pi}{2k})$ where

\begin{equation}
\frac{1}{\kappa} = \frac{1}{k} - \frac{1}{k'}, \quad \kappa = \frac{k'k}{k' - k}.
\end{equation}

Thus

\begin{equation}
\mathcal{L}_{k', S'} A_{k',k}, Sf = \mathcal{L}_{k, S} f.
\end{equation}

If we substitute for $\mathcal{L}$ and $B$ their integral expressions and then change the order of integration we get

\begin{equation}
(A_{k',k}, Sf)(\xi) = \xi^{-k'} \int_0^\infty \exp(i\theta) C_{k'/k}((t/\xi)^k) f(t) dt^k,
\end{equation}

where $\xi \in S(\theta_- - \frac{\pi}{2k}, \theta_+ + \frac{\pi}{2k})$, $\theta \in (\theta_-, \theta_+)$ such that $\Re(e^{i\theta}/\xi)^\kappa > 0$ and

\begin{equation}
\xi^{-k'} C_{k'/k}((t/\xi)^k) := (B_{k'}, \exp(-(t/x)^k))(\xi).
\end{equation}

The last relation defines Ecalle's function $C_\alpha(t)$:

\begin{equation}
C_\alpha(t) := \sum_{n=1}^\infty \frac{t^n}{n! \Gamma(-n/\alpha)} \quad \text{if } \alpha > 1.
\end{equation}

The asymptotic behavior of this function is given by

\begin{equation}
C_\alpha(t) \sim \sqrt{\frac{\alpha - 1}{2\pi}} \left(\frac{t}{\alpha}\right)^{\frac{1}{2} \beta} \exp\left\{-\left(\alpha - 1\right) \left(\frac{t}{\alpha}\right)^\beta\right\}, \quad t \to \infty
\end{equation}
on \(|\arg t| \leq \frac{\pi}{2\beta} + \delta\) where \(\beta := \frac{\alpha}{\alpha - 1}\) and \(\delta > 0\) is some constant.

From this it follows that the definition of \(A_{k',k,S}\) may be extended to functions \(f\) analytic in \(S := S(\theta_-, \theta_+)\) with \(f \in E(\kappa, S)\) and (1.3) with some \(\varepsilon > 0\) by means of (1.16) again for \(\xi \in S(\theta_--\frac{\pi}{2\kappa}, \theta_+ + \frac{\pi}{2\kappa})\), \(\text{Re}(e^{i\theta}/\xi)^\kappa > C, C\) as in (1.2).

We also will use a truncated operator \(A_{k',k}^t\). Let \(f\) be analytic in \(S = S(\theta_-, \theta_+)\) and satisfy (1.3) in \(S\) with some \(\varepsilon > 0\). Let \(z \in S\). Then we define \(A_{k',k}^t f\) by

\[
(A_{k',k}^t f)(\xi) = \xi^{-k'} \int_0^z C_{k'/k}(t/\xi)^k f(t) d(t^k), \quad \xi \neq 0,
\]

where the path of integration is in \(S\). This is \(\xi^{-k'}\) times an entire function of \(\xi^{-k}\). Moreover, we have

\[
(1.21) \quad \mathcal{L}_{k'} A_{k',k}^t f = \mathcal{L}_{k'}^t f, \quad A_{k',k}^t f = B_{k'} \mathcal{L}_{k'} f,
\]

since (1.17) implies \(\mathcal{L}_{k'}((\xi^{-k'} C_{k'/k}(t/\xi)^k))(x) = \exp[-(t/x)^k]\). Using (1.11) and (1.12) one may show that if \(f\) and \(g\) are analytic in \(S\) and are in \(E(\kappa, S)\) whereas moreover both are \(O(x^{\varepsilon - k})\) as \(x \to 0\) in \(S\) with some \(\varepsilon > 0\) then

\[
(1.22) \quad A_{k',k,S}(f^* g) = (A_{k',k,S} f)^* (A_{k',k,S} g).
\]

For a positive constant \(k\), if \(f\) is analytic on \(S\) and

\[
(1.23) \quad f(x) \sim \hat{f}(x) := \sum_{j=0}^{\infty} c_j x^{\lambda_j} \quad x \to 0 \text{ in } S, \quad \lambda_0 < \lambda_1 < \ldots
\]

with \(\lambda_0 > -k\), then we have in case \(f \in E(k, S)\)

\[
(1.24) \quad \mathcal{L}_{k,S} f(x) \sim \sum_{j=0}^{\infty} c_j \Gamma \left(1 + \frac{\lambda_j}{k} \right) x^{\lambda_j + k} =: \hat{L}_k \hat{f}(x),
\]

as \(x \to 0\) on \(\bar{S}\) where \(\bar{S} := S(\theta_--\frac{\pi}{2k}, \theta_+ + \frac{\pi}{2k})\), whereas in case \(f \in E(\kappa, S)\) we have

\[
(1.25) \quad A_{k',k,S} f(x) \sim \sum_{j=0}^{\infty} c_j \left\{ \Gamma(1 + \frac{\lambda_j}{k}) / \Gamma(\frac{\lambda_j + k}{k'}) \right\} x^{\lambda_j + k-k'} =: \hat{A}_{k',k} \hat{f}(x)
\]

as \(x \to 0\) on \(\bar{S} := S(\theta_--\frac{\pi}{2\kappa}, \theta_+ + \frac{\pi}{2\kappa})\).
If \( f \) is analytic on \( \tilde{S} \) and (1.23) holds as \( x \to 0 \) on \( \tilde{S} \), then

\[
(1.26) \quad B_{k,\tilde{S}} f(x) \sim \sum_{0}^{\infty} c_j x^{k_j} / \Gamma(\lambda_j / k) =: \hat{B}_k \hat{f}(x) \text{ as } x \to 0 \text{ on } S.
\]

In (1.24) and (1.26) we have equality if \( f = \hat{f} \) is a finite series.

Next assume \( f \) is analytic in \( S \), \( f \in \mathcal{E}(\kappa, S) \), (1.23) holds with \( \lambda_j = j \) and \( f \) is of Gevrey level \( p > 0 \) on \( S \). The last assumption means that if \( \delta > 0 \), \( \rho > 0 \) then there exist positive constants \( C \) and \( R \) such that

\[
|f(x) - \sum_{0}^{N-1} c_n x^n| \leq C \Gamma(1 + \frac{N}{p}) |R x|^N
\]

for all \( N \in \mathbb{N}, x \in S(\theta_- + \delta, \theta_+ - \delta) \triangleq \Delta(0; \rho) \). Then \( A_{k', k, S} f \) is of Gevrey level \((p^{-1} + \kappa)^{-1}\) on \( S(\theta_- - \frac{\pi}{2 \kappa}, \theta_+ + \frac{\pi}{2 \kappa}) \). In particular, if \( p = \infty \), i.e. if \( \hat{f} \) is a convergent series, then \( \hat{A}_{k', k} \hat{f} \) (cf. (1.25)) is \( \kappa \)-summable in the directions \( \alpha \) with \( \alpha \in (\theta_-, \theta_+) \) with sum \( A_{k_1, k, S} f \). These properties may be shown by estimating \( A_{k_1, k, S} \varphi \) for an analytic function \( \varphi \) on \( S \) such that \( |\varphi(t)| \leq M|t|^N \exp(c|t|^k) \) on \( S \) with \( M, N \) and \( c \) positive, by means of the saddle point method using (1.18) and (1.19).

Next we define multisummability. Let \( r \in \mathbb{N} \),

\[
(1.27) \quad \left\{ \begin{array}{l} 0 < k_1 < \ldots < k_r, \\
\kappa_j := \frac{k_{j+1} - k_j}{k_{j+1} - k_j} = \left( \frac{1}{k_j} - \frac{1}{k_j + 1} \right)^{-1} \text{ if } j = 1, \ldots, r - 1 \text{ and } \kappa_r := k_r.
\end{array} \right.
\]

Let \( \theta^-_j < \theta^+_j, S_j := S(\theta^-_j, \theta^+_j), S'_j := S \left( \theta^-_j - \frac{\pi}{2 \kappa_j}, \theta^+_j + \frac{\pi}{2 \kappa_j} \right) \) and assume \( S_{j+1} \subset S'_j \). Let

\[
\hat{f}(x) = \sum_{n=1}^{\infty} c_n x^n \in \mathbb{C}[x]_{1/k_1}
\]

which means that \( \hat{B}_{k_1} \hat{f} \) is convergent, i.e.

\[
(1.28) \quad \varphi_1(x) := B_{k_1} \hat{f}(x) = \sum_{n=1}^{\infty} c_n x^{n-k_1} / \Gamma(n/k_1)
\]

is convergent for \( 0 < |x| < \rho \) with some \( \rho > 0 \).
Assume that \( \varphi_1 \) can be continued analytically on \( S_1 \) so that \( \varphi_1 \in E(\kappa_1, S_1) \). Then

\[
\varphi_2 = A_{k_2, k_1, S_1} \varphi_1
\]

exists for \( \xi \in S'_1 \) such that \( \Re(e^{i\theta}/\xi)^{\kappa_1} > C \) for some \( C \) as in (1.2) and some \( \theta \in (\theta_1^-, \theta_1^+) \).

Assume that \( \varphi_2 \) can be continued analytically on \( S_2 \) and \( \varphi_2 \in E(\kappa_2, S_2) \) etc. Recursively define

(1.29)

\[
\varphi_j := A_{k_j, k_{j-1}, S_{j-1}} \varphi_{j-1} \quad \text{if} \quad j = 2, \ldots, r
\]

and assume \( \varphi_j \) can be continued analytically on \( S_j \) such that \( \varphi_j \in E(\kappa_j, S_j) \). Then \( \hat{f} \) is said to be \((k_r, \ldots, k_1)\)-summable on \( S'_r \) and its \((k_r, \ldots, k_1)\)-sum \( f \) is defined by

(1.30)

\[
f := \mathcal{L}_{k_r, S_r} \varphi_r.
\]

Then we have \( f(x) \sim \hat{f}(x) \) as \( x \to 0 \) on \( S'_r \). In case \( r = 1 \) we obtain ordinary \( k_1 \)-summability of Ramis [15].

Using the remark after (1.26) concerning functions which are of some Gevrey level we see that \( \hat{\varphi}_2 \) is \( \kappa_1 \)-summable in the direction \( \alpha \) with \( \theta^- < \alpha < \theta^+ \), since \( \varphi_1 \) is of Gevrey level \( \infty \). Moreover, \( \varphi_j \) is of Gevrey level \((k_1^{-1} - k_j^{-1})^{-1}\) on \( S'_{j-1} \).

If \( c \) is a constant we also say \( c + \hat{f}(x) \) is \((k_r, \ldots, k_1)\)-summable if \( \hat{f} \) has this property and its multisum is \( c + f(x) \) where \( f \) is the multisum of \( \hat{f} \).

If \( 0 < m_1 < \ldots < m_s \) and \( \{k_1, \ldots, k_r\} \subset \{m_1, \ldots, m_s\} \) then \((k_r, \ldots, k_1)\)-summability implies \((m_s, \ldots, m_1)\)-summability. If with the notation above we have \( h \leq j, k_j < m \leq k_{j+1} \), then we may extend the previous definitions as follows:

(1.31)

\[
\begin{align*}
A_{m, k_h, S_h} \varphi_h & := A_{m, k_j, S_j} \varphi_j =: Bmf, \\
A_{m, k_h} z_j \varphi_h & := A_{m, k_j} z_j \varphi_j, \quad \text{where} \ z_j \in S_j.
\end{align*}
\]

If \( k_j < m < k_{j+1} \), then \( A_{m, k_h, S_h} \varphi_h = Bmf \) is analytic on \( S(\theta_j^- - \frac{1}{2} \pi/\kappa, \theta_j^+ + \frac{1}{2} \pi/\kappa) \) where \( \kappa^{-1} = k_j^{-1} - m^{-1} \).
2. A multsummability theorem. Here we consider the differential equation

\[(2.1) \quad D[y] = g\]

with the following assumptions:

i) \(D\) is defined by (0.1) where now the \(n \times n\)-matrix \(A(x)\) satisfies

\[(2.2) \quad A(x) = \bigoplus_{h=0}^{r} x^{-k_h} A_h + \big( \bigoplus_{h=0}^{r} x^{1-k_h} I_h \big) A_+(x)\]

where \(r \in \mathbb{N}_+, \ k_0 = 0 < k_1 < \cdots < k_r, \ k_h \in \mathbb{N}, \ n = n_0 + \cdots + n_r, \ n_h \in \mathbb{N}, \ A_h\) a constant \(n_h \times n_h\) matrix, \(A_h\) is invertible if \(h > 0, \ I_h\) is the \(n_h \times n_h\)-identity matrix, \(A_+\) and \(g\) are \((k_r, ..., k_1)\)-sums on a sector \(S\) of formal series

\[(2.3) \quad \hat{A}_+(x) = \sum_{j=0}^{\infty} A_+^j x^j, \quad \hat{g}(x) = \sum_{j=0}^{\infty} g_j x^j,\]

ii) (2.1) possesses a formal solution \(\hat{y}(x) = \sum_{j=0}^{\infty} c_j x^j\).

Then we show that \(\hat{y}\) is \((k_r, ..., k_1)\)-summable on subsectors of \(S\) and its sum satisfies (2.1).

To specify those subsectors we use the following definition in connection with (2.2):

**Definition.** Let \(j \in \{1, ..., r\}\). Then \(\xi\) is a **singular value** of level \(k_j\) for (2.2) if \(k_j \xi^{k_j}\) is an eigenvalue of \(A_j\). The set of singular values of level \(k_j\) will be denoted by \(V_j\). A **singular direction** of level \(k_j\) is an argument of an element of \(V_j\). The set of singular directions of level \(k_j\) will be denoted by \(W_j\). If \(\sigma \in W_j\), then \(\sigma \pm \frac{\pi}{2k_j}\) are Stokes directions of (2.2).

Then we have

**Theorem 1.** With the assumptions made above the formal solution \(\hat{y}(x)\) of (2.1) is \((k_r, ..., k_1)\)-summable on any subsector \(\tilde{S}\) of \(S\) such that for all \(j \in \{1, ..., r\}\) the sector \(\tilde{S}\)
does not contain any pair of Stokes directions $\sigma - \frac{\pi}{2k_j}, \sigma + \frac{\pi}{2k_j}$ with $\sigma \in W_j$. The multisum $y(x)$ of $\hat{y}(x)$ depends on $\hat{S}$ and satisfies (2.1).

We give a proof of this theorem by means of convolution equations as in Ecalle's theory of resurgent functions. We only need to prove the theorem with $c_0 = \hat{y}(0) = 0$, since the general case may be reduced to this case. We use the notation of (1.27) - (1.30) with $\hat{f} = x \hat{A}_+$ and $\hat{g}, \varphi_j = \alpha_j$ and $\gamma_j$ respectively.

Next define for $j = 1, \ldots, r$:

\begin{equation}
M_j := x^{k_j} I_0 \oplus \cdots \oplus x^{k_j} I_j \oplus x^{k_j+1} I_{j+1} \oplus \cdots \oplus x^{k_r} I_r, \quad g_j := M_j g.
\end{equation}

Let $d$ be a halfline $\arg x = \theta$ in $C$ with $\theta^-_j < \theta < \theta^+_j$ and let $C^\infty_0(d)$ be the space of $C^\infty$-functions $\psi : d \to C^n$ with compact support. On $C^\infty_0(d)$ we define the operator $Q_j$ by

\begin{equation}
Q_j := B_{k_j} M_j D_L k_j.
\end{equation}

Here we use obvious extensions of the definitions for $L_k$ and $B_{k_j}$ in section 1.

We may elaborate this definition as follows: If $w \in C^n$ we denote by $w^{(h)}$ the projection of $w$ on the space spanned by the unit vectors with indices $n_0 + \cdots + n_{h-1} + 1, \ldots, n_0 + \cdots + n_h$ if $h \in \{0, 1, \ldots, r\}$. Utilizing (1.12), (1.26) and

\begin{equation}
\left( B_{k_j} x^{1+k_j} \frac{d}{dx} L_k \psi \right) (\xi) = k_j \xi^{k_j} \psi(\xi)
\end{equation}

we obtain for $\psi \in C^\infty_0(d)$:

\begin{equation}
\begin{aligned}
(Q_j \psi)^{(h)} &= k_j \xi^{k_j} \psi^{(h)} - \frac{1}{(1-(k_h/k_j))} \left\{ (\xi^{-k_h} A_h)^* \psi^{(h)} +
\right.
\left. + \left( \xi^{-k_h} \alpha_j^* k_j \psi^{(h)} \right) \right\}, \quad \text{if } 0 \leq h \leq j - 1, \\
(Q_j \psi)^{(j)} &= (k_j \xi^{k_j} I_j - A_j) \psi^{(j)} - (\alpha_j^* k_j \psi)^{(j)}, \\
(Q_j \psi)^{(h)} &= -A_h \psi_h + \frac{\xi^{k_h-2k_j}}{(1+(k_h/k_j))} k_j (k_j \xi^{k_j} \psi^{(h)}) - (\alpha_j^* k_j \psi)^{(h)}, \\
&\quad \text{if } j + 1 \leq h \leq r.
\end{aligned}
\end{equation}
We extend the definition of $Q_j$ to analytic functions $\psi : S_0 \to \mathbb{C}^n$ where $S_0$ is a subsector of $S_j$ with vertex 0 such that $\psi(x) = O(x^{1-k_j})$ as $x \to 0$ on $S_0$. To $D[y] = g$ we now associate the convolution equations

\begin{equation}
Q_j \psi_j = \tilde{\gamma}_j \text{ where } \tilde{\gamma}_j := B_{kj} g_j = B_{kj} M_j g,
\end{equation}

where $j = 1, \ldots, r$. We solve these equations successively for $j = 1, j = 2, \ldots$ and $j = r$. This involves 3 steps which we formulate as lemmas. Let

\begin{equation}
\hat{\psi}_j(\xi) := \sum_{h=1}^{\infty} c_h \xi^{\delta - k_j} / \Gamma\left(\frac{h}{k_j}\right) = \hat{B}_{kj}\hat{\psi}(\xi), \quad j = 1, \ldots, r.
\end{equation}

**Lemma 1.** The equation $Q_1 \psi_1 = \tilde{\gamma}_1$ has a unique solution $\psi_1(\xi)$ which is the sum of the convergent series $\hat{\psi}_1(\xi)$ in a neighborhood of 0. This solution $\psi_1$ can be analytically continued on $S_1 \setminus V_1$.

**Lemma 2.** If $j \in \{1, \ldots, r\}$ and $\psi_j$ is an analytic solution (possibly multivalued) of (2.8) in $S_j \setminus V_j$, then $\psi_j \in E(\kappa_j, S_j)$. 

**Lemma 3.** Suppose $j \in \{1, \ldots, r - 1\}$ and $\psi_j$ is an analytic solution of (2.8) on some subsector $\tilde{S}_j := S(\varphi_j^-, \varphi_j^+)$ of $S_j \setminus V_j$ such that

\begin{equation}
\psi_j(\xi) \sim \hat{\psi}_j(\xi), \quad \xi \to 0
\end{equation}

on $\tilde{S}_j$. Then

\begin{equation}
\psi_{j+1} := A_{kj+1, k_j, \tilde{S}_j} \psi_j
\end{equation}

is an analytic solution of $Q_{j+1} \psi = \tilde{\gamma}_{j+1}$ in a neighborhood of 0 in $\tilde{S}_j(\delta) := S(\varphi_j^-, \frac{\pi}{2\alpha_j} + \delta, \varphi_j^+ + \frac{\pi}{2\alpha_j} - \delta)$ for any $\delta > 0$, and (2.10) with $j$ replaced by $j + 1$ holds on $\tilde{S}_j(0)$.

We postpone the proof of these lemmas to section 3. From lemma 1 and lemma 3 with $j = 1$ we see that (2.8) with $j = 2$ holds in a neighborhood of 0 in $\tilde{S}_1(\delta), \delta > 0$, and
(2.10) with \( j = 2 \) holds on \( \hat{S}_1(0) \) provided \( \hat{S}_1 \subset (S_1 \setminus V_1) \). From the convolution equation (2.8) with \( j = 2 \) we see that \( \psi_2 \) can be continued analytically on \( S_2 \setminus V_2 \).

Now we apply lemma 3 with \( j = 2 \) and \( \hat{S}_2 \subset (S_2 \setminus V_2) \cap \hat{S}_1(0) \). From this and (2.8) with \( j = 3 \) we get a solution \( \psi_3 \) of (2.8) with \( j = 3 \) on \( S_3 \setminus V_3 \) (possibly multivalued) such that (2.10) with \( j = 3 \) holds on \( \hat{S}_2(0) \). We may repeat this procedure for \( j = 3, \ldots, r - 1 \) with \( \hat{S}_j \subset (S_j \setminus V_j) \cap \hat{S}_{j-1}(0) \). Thus we obtain \( \psi_r \) as solution of (2.8) with \( j = r \) on \( S_r \setminus V_r \), which satisfies (2.10) with \( j = r \) on \( \hat{S}_{r-1}(0) \) and \( \psi_r \in E(k_r, S_r) \). So \( y = \mathcal{L}_{k_r, \hat{S}_r} \psi_r \) exists on a neighborhood of 0 in \( \hat{S}_r(\delta) \) if \( \delta > 0 \) and \( y(x) \sim \hat{y}(x) \) as \( x \to 0 \) on \( \hat{S}_r(0) \). From (2.5) and (2.8) it follows that

\[
B_{k_r, M_r} D[y] = B_{k_r, M_r} g
\]

and so \( D[y] = g \) on a neighborhood of 0 in \( \hat{S}_r(\delta) \) if \( \delta > 0 \). More precisely, using \( \psi_r \in E(k_r, \hat{S}_r) \) we see that \( y \) exists and is a solution on

\[
\{ x \in \mathbb{C} \mid \varphi_r^+ - \frac{\pi}{2k_r} < \arg x < \varphi_r^- + \frac{\pi}{2k_r}, \quad \text{Re}(e^{i\theta}/x)^{k_r} > C \text{ for some } \theta \in (\varphi_r^-, \varphi_r^+) \}. \tag{2.12}
\]

Finally we show that if \( \hat{S} := S(\tau^-, \tau^+) \) is chosen as in the theorem, then we may choose \( \varphi_j^+, j = 1, \ldots, r \), such that \( \hat{S} \subset \hat{S}_r(0) \), \( 0 \neq \hat{S}_j \subset (S_j \setminus V_j) \cap \hat{S}_{j-1}(0), j = 1, \ldots, r \), or equivalently

\[
\theta_j^- \leq \varphi_j^- < \varphi_j^+ \leq \theta_j^+, \varphi_j^- - \pi/(2k_j) \leq \varphi_{j+1}^- - \pi/(2k_{j+1}) \leq \varpi \leq \tau^+ - \pi/(2k_{j+1}) \leq \varphi_{j+1}^+ + \pi/(2k_{j+1}) \leq \varphi_j^+ + \pi/(2k_j) \tag{2.13}
\]

and \( (\varphi_j^-, \varphi_j^+) \cap W_j = \emptyset, j = 1, \ldots, r - 1 \).

We know from the definition of multisummability (cf. (1.27)-(1.30)) that

\[
\theta_j^- - \pi/(2k_j) \leq \theta_{j+1}^- - \pi/(2k_{j+1}) < \theta_{j+1}^+ + \pi/(2k_{j+1}) \leq \theta_j^+ + \pi/(2k_j).
\]

Let \( h \in \{1, \ldots, r\} \) be such that \( \pi/k_{h+1} < \tau^+ - \tau^- \leq \pi/k_h \). Then we choose \( \varphi_j^- := \tau^- + \pi/(2k_j) \) and \( \varphi_j^+ := \tau^+ - \pi/(2k_j) \) if \( j \geq h + 1 \). Hence, if \( j \geq h + 1 \), then (2.13) holds. If \( \sigma \in (\varphi_j^-, \varphi_j^+) \cap W_j, j \geq h + 1 \), then \((\tau^-, \tau^+)\) would contain the Stokes rays \( \sigma - \pi/(2k_j) \) and \( \sigma + \pi/(2k_j) \) contradicting the assumption on \( \hat{S} \). Hence \((\varphi_j^-, \varphi_j^+) \cap W_j = \emptyset \) if \( j \geq h + 1 \). If
\( j \leq h \), we use \( \tau^+ - \tau^- \leq \pi/k_h \). Then we may choose \( \sigma_j^- - \pi/(2k_j) \leq \tau^- \), \( \tau^+ \leq \varphi_j^+ + \pi/(2k_j) \) such that (2.13) holds and \( (\varphi_j^-, \varphi_j^+) \cap W_j = \emptyset \) if \( j \leq h \).

3. Proof of lemmas 1, 2, and 3. First we rewrite the equations \( Q_j \psi = \tilde{\gamma}_j \) by means of an operator \( T_j \) defined on the space of analytic functions \( f \) on a neighborhood of 0 in \( S(\theta_j^-, \theta_j^+) \) such that \( f(\xi) = O(\xi^{1-k_j}) \) as \( \xi \to 0 \) on this sector. For such functions \( f \) we define \( T_j f \) by:

\[
(T_j f)^{(h)} := \{\Gamma(1 - k_h k_j^{-1})k_j \xi^{k_j} \}^{-1}((\xi^{-k_h} A_h) k_j \star f^{(h)}) + (\xi^{-k_h} k_j \alpha_j k_j \star f^{(h)}),
\]

if \( 0 \leq h \leq j - 1 \),

\[
(T_j f)^{(j)} := (k_j \xi \cdot I_j - A_j)^{-1}(\alpha_j k_j \star f)^{(j)},
\]

\[
(T_j f)^{(h)} := A_h^{-1}\{\frac{\Gamma(-1+(k_h/k_j))}{k_j} k_j \xi^{k_j} f - \alpha_j k_j \star f\}^{(h)},
\]

if \( j + 1 \leq h \leq r \).

If \( j = 1 \) we may replace \( S(\theta_1^-, \theta_1^+) \) by a reduced neighborhood of 0.

Correspondingly we define \( \gamma_j^* \) by

\[
(\gamma_j^*)^{(h)} := (k_j \xi^{k_j})^{-1}\gamma_j^{(h)}, \quad \text{if} \quad 0 \leq h \leq j - 1,
\]

\[
(\gamma_j^*)^{(j)} := (k_j \xi \cdot I_j - A_j)^{-1}\gamma_j^{(j)},
\]

\[
(\gamma_j^*)^{(k)} := -A_h^{-1}\gamma_j^{(h)}, \quad \text{if} \quad j + 1 \leq h \leq r.
\]

Hence \( Q_j \psi_j = \tilde{\gamma}_j \) is equivalent to \( \psi_j = T_j \psi_j + \gamma_j^* \).

Proof of lemma 1. We use a method which in case \( r = 1 \) occurs in Horn [11].

Horn’s method is modified here in a manner which has similarities with the method used by Harris, Sibuya and Weinberg [10] (cf. also [6], [4], [5]).

If \( N \in \mathbb{N} \), \( N > k_1 \), we define

\[
f_N(\xi) := \sum_{m=1}^{N-1} c_m \frac{\xi^{m-k_1}}{\Gamma(m/k_1)}, \quad r_N = \psi_1 - f_N.
\]
Then $\psi_1 = T_1 \psi_1 + \gamma_1^*$ is equivalent to $r_N = T_1 r_N + \delta_N$ where $\delta_N := \gamma_1^* + T_1 f_N - f_N$. Since $D\dot{y} = \dot{g}$ we see that $\dot{\psi}_1 := \dot{B}_k, \dot{y}$ is a formal solution of $\dot{B}_k, M_1 D\dot{\psi}_1, \psi = \dot{B}_k, M_1 \dot{y}$, so $Q_1 \psi = \dot{\gamma}_1$ and $\psi = T_1 \psi + \gamma_1^*$. From this it is easily verified that $\delta_N(\xi) = O(\xi^{N-k_1})$ as $\xi \to 0$.

Let $V := \{ \xi \in \mathbb{C} \mid |\xi| \leq \rho \}$ with $\rho > 0$ so small that $k_1 \xi^{k_1} I_1 - A_1$ is invertible for $\xi \in V$. Let $W$ be the Banach space of analytic functions $f$ on $V$ such that
\[
\|f\| := \sup_{\xi \in V} |\xi^{k_1-N} f(\xi)| < \infty.
\]

From $\alpha_1 = \dot{B}_k, (x A_+ (x))$ and (1.26) we deduce $\alpha_1(\xi) = O(\xi^{1-k_1}), \xi \to 0$. If $f \in W$ and $\xi \in V$ we obtain with the help of (1.9) and (1.10):
\[
|f(1 \k_1 \psi_1)| \leq \|f\| |k_1 \xi^{N-k_1}| = N^{-1} k_1 |\xi|^N \|f\|,
\]
\[
|\alpha_1(1 \k_1 \psi_1)| \leq C \|f\| |\xi^{1-k_1}| k_1 \xi^{N-k_1} = CB(k_1^{-1}, N k_1^{-1}) |\xi|^{N+1-k_1} \|f\|,
\]
\[
|f(1 \k_1 \alpha_1 \k_1 \psi_1)| \leq CB(k_1^{-1}, N k_1^{-1}) |\xi|^{N+1} \|f\|,
\]
\[
|f(1 \k_1 \alpha_1 \k_1 \psi_1)| \leq B(k_1 k_1^{-1} - 1, N k_1^{-1} + 1) |\xi|^{N+k_1-k_1} \|f\|,
\]

if $h > 1$. Here $C$ is a constant independent of $N$ and $B$ is the Beta-function. Moreover, $B(k_1^{-1}, N k_1^{-1})$ and $B(k_1 k_1^{-1} - 1, N k_1^{-1} + 1) \to 0$ as $N \to \infty$ if $h > 1$.

Hence $T_1$ is a contraction on $W$ if $N$ is sufficiently large. Thus we have a unique solution of $r_N = T_1 r_N + \delta_N$ in $W$. So $\psi_1 = f_N + \delta_N$ is an analytic solution of $\psi = T_1 \psi + \gamma_1^*$ in $V$ and $\dot{\psi}_1$ is convergent with sum $\psi_1$ on $V$.

The solution $\psi_1$ may be continued analytically on $S(\theta_1^-, \theta_1^+)$ except for singular (branch-) points $\xi$ where $k_1 \xi^{k_1} I_1 - A_1$ is singular, because these are the only singular points of the analytic convolution equation $\psi_1 = T_1 \psi_1 + \gamma_1^*$.

Proof of lemma 2. We fix $j$ in lemma 2 and omit the index $j$ in the proof. Let $\psi$ be an analytic solution of $Q \psi = \gamma$ in $\bar{S} = \{ \xi \in S(\theta_1^- \theta_1^+), |\xi| \geq R \}, R > 0$. So $\psi = T \psi + \gamma^*$ in $\bar{S}$.
Let $\xi_0$ be a point in $\tilde{S}$ with $|\xi_0| = R$ and $t_0 = \xi_0^{1/k}$. We modify the operator $T$ by $\tilde{T}$ on $\tilde{S}$ by replacing in the right-hand side of (3.1) the convolutions $g * f$ by $g * f$, where

$$
(g * f)(\xi) := \int_{t_0}^{t} g((t - \tau)^{1/k}) f(\tau^{1/k}) d\tau |_{t = \xi^k}, \quad \xi \in \tilde{S}.
$$

However, we do not change terms of the form $\xi^a * \alpha$ in (3.1). Then we get for $\psi$ in $\tilde{S}$: $\psi = \tilde{T}\psi + \tilde{\gamma}^*$, where the function $\tilde{\gamma}^*$ depends on $\gamma^*$ and the restriction of $f$ to $S(\theta^-, \theta^+) \setminus \tilde{S}$.

Let $S^* = \{\xi \in C| \theta^- + \delta \leq \arg \xi \leq \theta^+ - \delta, |\xi| \geq R\}$, where $\delta > 0$ is sufficiently small. We consider $\tilde{T}$ in the space $W_c$ of analytic functions $f$ on $S^*$ such that

$$
\|f\|_c := \sup_{\xi \in S^*} |f(\xi)| \exp(-c|\xi|^\kappa) < \infty,
$$

where $c > 0$ will be chosen later on. From the construction of $\tilde{\gamma}^*$ it follows that $\tilde{\gamma}^*|_{S^*} \in W_c$ if $c \geq c_0$, where $c_0$ is some suitable positive constant. Similarly $\alpha \in W_c$ if $c \geq c_0$.

We estimate $\tilde{T}$ on $W_c$ using the following estimates which we prove later on:

If $a, b, c$ and $\mu$ are positive, then

(I) $\int_0^t (t - \tau)^{a-1} \tau^{b-1} \exp(\tau) d\tau \leq K t^{a+b-1} (ct\mu)^{-a} \exp(ct) \quad \text{for} \ t > 0$,

and, if $c \geq c_0 + 1$, $c_0 > 0$, $\mu \geq 1$, then

(II) $\int_0^t \exp\{c_0(t - \tau)^\mu + cr\mu\} d\tau \leq K(c - c_0)^{-1} t^{1-\mu} \exp(ct\mu) \quad \text{for} \ t > 0$,

where $K$ is a constant independent of $t$ and $c$.

We have

$$
|(\xi^{-k\lambda} * f)(\xi)| \leq \|f\|_c \int_0^t |(t - \tau)^{-k\lambda/k} \exp(c|\tau|^\kappa/k) d\tau|, \quad \text{if} \ t = \xi^k, h < j.
$$

Now apply (I) with $\mu = \kappa k^{-1}$ and use $1 - \mu = \kappa(\kappa^{-1} - k^{-1}) = -\kappa k^{-1} < 0$. Hence we get for $\xi \in S^*$

$$
|(\xi^{-k\lambda} * f)(\xi)| \leq K c^{-1+k\lambda/k} \|f\|_c \exp(c|\xi|^\kappa).
$$

(3.3)
Here and in the following $K$ denotes a positive constant independent of $c$ and $\xi$ which may be different in different places. Similarly

$$\left\{ \begin{align*}
\|\xi^{-k_h} \alpha\| &\leq K\|\alpha\|_{c_0} \exp(c_0 |\xi|^\kappa) \quad \text{if } h < j, \\
|\xi^{-k_h-2k_h}(\xi^k f)| &\leq K c^{c-(k_h/k)} |\xi|^{k_h-(k_h-k)\kappa/k} \exp(c|\xi|^\kappa), \quad \text{if } h > j.
\end{align*} \right. $$

(3.4)

In the last expression we have

$$k_h - (k_h - k)\kappa/k = k_h\kappa\{\kappa^{-1} - k^{-1} + k_h^{-1}\} \leq 0 \quad \text{since } \kappa^{-1} = k^{-1} - k_{j+1}^{-1}.$$  

Hence

(3.5)

$$\|\xi^{k_h-2k_h}(\xi^k f)\|_c \leq K c^{c-(k_h/k)} \|f\|_c \quad \text{if } h > j.$$  

Finally we apply (II) to $\alpha f$ and $\xi^{-k_h} \alpha f, h < j$ using (3.4). Then we get

(3.6)

$$\|\alpha f\|_c, \|\xi^{-k_h} \alpha f\|_c \leq K(c - c_0)^{-1} \|f\|_c \quad \text{for } c > c_0 + 1, h < j.$$  

Combining (3.1), (3.3), (3.4), (3.5) and (3.6) we see that $\tilde{T}$ is a contraction on $W_c$ if $c$ is chosen sufficiently large. Therefore the unique solution $\psi$ of $\psi = \tilde{T}\psi + \tilde{S}$ satisfies $\psi|_{S^*} \in W_c$ for all sufficiently small $\delta > 0$. Hence $\psi \in E(\kappa, S(\theta^-, \theta^+))$

Finally we prove (I) and (II). We denote the lefthand sides of (I) and (II) by $I$ and $II$. Substituting $\tau = t(1 - s)^{1/\mu}$ we get

$$I = \mu^{-1} t^{a+b-1} \{\exp(ct^\mu)\} \int_0^1 \{1 - (1 - s)^{1/\mu}\} a-1(1-s)^{-1+b/\mu} \exp(-ct^\mu s)ds.$$  

Denote the integrand by $f(t)$. There is a constant $K_1 > 0$ independent of $c$ and $t$ such that

$$f(t) \leq K_1 s^{a-1} \exp(-ct^\mu s) \quad \text{if } 0 \leq s \leq \frac{1}{2},$$

$$f(t) \leq K_1 (1-s)^{-1+b/\mu} \exp(-\frac{1}{2}ct^\mu) \quad \text{if } \frac{1}{2} \leq s \leq 1.$$  

Hence

$$\int_0^1 f(t)dt \leq K_1 \{\Gamma(a)(ct^\mu)^{-a} + (\mu/b) \exp(-\frac{1}{2}ct^\mu)\}$$  

and (I) follows.
To prove (II) we use \( c_0(t - \tau)^\mu + c\tau^\mu \leq c_0 t^\mu + (c - c_0)\tau^\mu \), if \( 0 \leq \tau \leq t, \mu \geq 1 \). Using the same substitution as above we get

\[
II = \mu^{-1}t\{\exp(ct^\mu)\} \int_0^1 [\exp\{-(c - c_0)t^\mu s\}(1 - s)^{-1+1/\mu}]ds.
\]

Now \( \int_0^{1/2} \cdots \leq K_0(c - c_0)^{-1}t^{-\mu} \) and \( \int_{1/2}^1 \cdots \leq K_0 \exp\{-\frac{1}{2}(c - c_0)t^\mu\} \) for some \( K_0 \) independent of \( t \) and \( \mu \), and (II) follows.

**Proof of lemma 3.** If \( f \in C_0^\infty(d) \) we defined \( Q_j f \) by (2.5). Hence if

\[
(3.7) \quad \tilde{f} := A_{k_{j+1}, k_j} f
\]

then we have with (1.12):

\[
Q_{j+1} \tilde{f} = B_{k_{j+1}, M_{j+1}} D\mathcal{L}_{k_j} f = \{B_{k_{j+1}, (M_{j+1}^{-1})}\}_{k_{j+1}}^* B_{k_{j+1}, (\mathcal{L}_k, Q_j f)}.
\]

Hence

\[
Q_{j+1} \tilde{f} = \{B_{k_{j+1}, (M_{j+1}^{-1})}\}_{k_{j+1}}^* A_{k_{j+1}, k_j} Q_j f.
\]

This equality remains valid if \( f \in E(\kappa_j) \) in \( S(\theta^-_j, \theta^+_j) \) and \( f(x) = O(x^{-k_j}) \) as \( x \to 0 \) on this sector. This may be seen by approximating \( f \) on halflines \( d : \arg \xi = \theta \) with \( \theta \in (\theta^-_j, \theta^+_j) \) by elements of \( C_0^\infty(d) \).

For \( \tilde{\gamma} \) we have with (2.8) and (1.12) similarly

\[
\tilde{\gamma}_{j+1} = B_{k_{j+1}, M_{j+1}} g = \{B_{k_{j+1}, (M_{j+1}^{-1})}\}_{k_{j+1}}^* \{B_{k_{j+1}, M_j} g\} = \{B_{k_{j+1}, (M_{j+1}^{-1})}\}_{k_{j+1}}^* A_{k_{j+1}, k_j} \tilde{\gamma}_j.
\]

From this and (3.8) we deduce that \( Q_j \psi_j = \tilde{\gamma}_j \) with \( \psi_{j+1} = A_{k_{j+1}, k_j} \psi_j \) implies \( Q_{j+1} \psi_{j+1} = \tilde{\gamma}_{j+1} \).

**4. Fundamental matrices and multiscummability.** We consider the homogeneous linear differential equation (0.1) with \( A(x) \) an analytic \( n \times n \)-matrix in a
reduced neighborhood of 0 and a pole at \( x = 0 \). Turrittin [18] has shown that there exists an analytic transformation

\[
y = \left( \sum_{h=0}^{k} B_h t^h \right) w, \quad t = x^{1/p}, \quad k \in \mathbb{N}, p \in \mathbb{N}_+
\]

where the \( B_h \) are constant \( n \times n \)-matrices such that (0.1) is transformed into

\[
t \frac{dw}{dt} = (P(\frac{1}{t}) + N + tR(t))w,
\]

where \( P(t) \) is polynomial and diagonal, \( N \) nilpotent and \( R(t) \) an analytic \( n \times n \)-matrix in a full neighborhood of 0,

\[
P(t) = \oplus_{j=1}^{m} p_j(t)I_j, \quad N = \oplus_{j=1}^{m} N_j,
\]

where \( m \in \mathbb{N} \), \( I_j \) and \( N_j \) are \( n_j \times n_j \)-matrices, \( n_j \in \mathbb{N}, \sum_{j=1}^{m} n_j = n, I_j \) is identity matrix, \( N_j \) is a Jordan block nilpotent of order \( n_j \), and \( p_j(t) \) is a polynomial in \( t \) or \( p_j(t) \equiv 0 \).

Moreover, Turrittin showed that (4.2) has a formal fundamental matrix

\[
\begin{cases}
\hat{W}(t) = \hat{H}(t) t^{P(0) + N} \exp Q(t^{-1}), \text{ where} \\
Q(t) = -\int_{0}^{t} \{ P(\tau) - P(0) \} \frac{d\tau}{\tau} = \oplus_{j=1}^{m} q_j(t)I_j, \\
q_j(t) = -\int_{0}^{t} \{ p_j(\tau) - p_j(0) \} \frac{d\tau}{\tau}, \hat{H}(t) \in GL(n; \mathbb{C}[t]).
\end{cases}
\]

So the matrices \( P(0), N \) and \( Q(t) \) commute. Correspondingly, (0.1) has the formal fundamental matrix (0.2) with \( A = P(0) + N \) and

\[
\hat{F}(t) = \left( \sum_{k=0}^{k} B_h t^h \right) \hat{H}(t).
\]

We now substitute for \( w \) in (4.2) an \( n \times n \)-matrix \( W \) and put

\[
W(t) = H(t) t^{A} \exp Q(t^{-1}).
\]

Then the equivalent differential equation for \( H \) is:

\[
t \frac{dH}{dt} = (P(\frac{1}{t}) + N)H - H(P(\frac{1}{t}) + N) + tR(t)H.
\]
Now we use the partition of \( H \) into blocks \( H_{hj} \) which corresponds with the partition of \( P(t) \). Also we denote by \( H_h \) the \( n_h \times n \)-matrix consisting of the row of blocks \( H_{h1}, \ldots, H_{hm} \) and by \( H_j \) the \( n \times n_j \)-matrix consisting of the column of blocks \( H_{1j}, \ldots, H_{mj} \). Then we have

\[
(4.7) \quad \frac{dH_{hj}}{dt} = (p_h - p_j)(\frac{1}{t})H_{hj} + N_hH_{hj} - H_{hj}N_j + tR_{hj}(t)H_{j}, \quad j, h \in \{1, \ldots, m\}.
\]

Let \( k_{hj} \) be the degree of \( (p_h - p_j)(t) \) if \( p_h \neq p_j \) and otherwise \( k_{hj} = 0 \). We assume that at least one \( k_{hj} \) differs from zero. Otherwise \( P \) is constant and \( t = 0 \) is a regular singular point of (4.2) which implies that \( \hat{H}(t) \) converges.

We have

\[
(4.8) \quad (p_h - p_j)(t) = c_{hj} t^{k_{hj}} (1 + o(1)) \quad \text{as} \quad t \to \infty, c_{hj} \neq 0 \quad \text{if} \quad p_h \neq p_j.
\]

So the righthand side of (4.7) is of the form

\[
t^{-k_{hj}} \{c_{hj}H_{hj} + t\tilde{R}_{hj}(t)H_{j}\} \quad \text{if} \quad k_{hj} \neq 0
\]

and

\[
(p_h - p_j)(0)H_{hj} + N_hH_{hj} - H_{hj}N_j + tR_{hj}(t)H_{j} \quad \text{if} \quad k_{hj} = 0.
\]

Here \( \tilde{R}_{hj}(t) \) is an \( n_h \times n \)-matrix analytic in a neighborhood of \( 0 \). From this we see that (4.7) with \( h = 1, \ldots, m \) constitute a differential equation for \( H_{j} \) of the type (2.1) with \( g = 0 \), and with a formal solution \( \hat{H}_{j}(t) \), to which theorem 1 is applicable.

To formulate the result we first consider the **levels** and singular values associated with (4.7). The degrees \( k_{hj} \) of \( p_h - p_j \) with \( k_{hj} > 0 \) are the levels of (4.7). For fixed \( j \) we arrange these in increasing order: \( m_{1j} < \cdots < m_{r(j),j} \) so that \( m_{\mu j} \) is the \( \mu \)-th level associated with \( j \) in (4.7). If we arrange the levels \( k_{hj} > 0 \) in increasing order with both \( h \) and \( j \) varying we obtain the levels \( m_1 < \cdots < m_r \) of (4.7).

A singular value of level \( k_{hj} \) of (4.7) is a number \( \rho_{hj} \neq 0 \) such that in (4.8) we have

\[
c_{hj} = k_{hj}(\rho_{hj})^{k_{hj}}.
\]
or equivalently

\[(q_h - q_j)(t) = -(\rho_{hj} t)^{kh_j} (1 + o(1)) \text{ as } t \to \infty.\]

Let

\[(4.10) \quad V_{gj} := \{\rho_{hj} | \ k_{hj} = m_{gj}\}.\]

If \(h \neq j\), then \(\arg \rho_{hj} \pm (2k_{hj})^{-1} \pi\) will be called a **Stokes direction** and the pair

\[
\arg \rho_{hj} - (2k_{hj})^{-1} \pi, \ arg \rho_{hj} + (2k_{hj})^{-1} \pi
\]

a **Stokes pair** associated with \(q_h - q_j\). In these directions \(\text{Re}(q_h - q_j)(t^{-1})\) changes sign at 0. If \(\theta_0\) is a Stokes direction then the ray \(\arg \xi = \theta_0\) is called a **Stokes ray**.

With these definitions we immediately deduce from theorem 1:

**Theorem 2.** Let the assumptions above be fulfilled: Let \(\tau^-\) and \(\tau^+\) be two Stokes directions such that \(\tau^- < \tau^+\) and such that \((\tau^-, \tau^+)\) does not contain any Stokes pair.

Then (0.1) has a fundamental matrix \(Y(x)\) as in (0.3) with \(\Lambda = P(0) + N\), (4.4) and where \(F(t)\) is the \((m_r, ..., m_1)\)-sum of \(\hat{F}(t)\) on \(S(\tau^-, \tau^+)\). In particular, if \(j \in \{1, ..., m\}\) then \(\hat{F}_j(t)\)

is \((m_{r(j)}, ..., m_{s(j)})\)-summable with sum \(F_j(t)\) on \(S(\tau_j^-, \tau_j^+)\) if \(\tau_j^- < \tau_j^+\) such that \((\tau_j^-, \tau_j^+)\) does not contain any Stokes pair associated with \(q_h - q_j\), \(h \in \{1, ..., m\}\).

**Remark 1.** If \(\varphi_{j1} := \hat{B}_{m_{ij}} \hat{F}_{j}, \varphi_{t+1,j} := A_{m_{t+1,i} m_{tj}} \varphi_{tj}\) for \(t = 1, \cdots, r(j) - 1\), then the singularities of \(\varphi_{tj}\) are the singular values \(\rho_{hj}\) of level \(k_{hj} = m_{tj}\), and 0. The widths of the asymptotic sectors for \(F\) correspond with those studied by Jurkat [13], §7, \(f : (q_h - q_j)(t^{-1})\) has at most one dominance change in \(S(\tau_j^-, \tau_j^+)\). The levels \(m_h\) divided by \(p\) are the slopes of the Newton polygon of (0.1).

5. **Decompositions of multisums.** In the discussion of Stokes multipliers we need decompositions of the sum \(f\) of a \((k_r, ..., k_1)\)-summable formal series \(\hat{f}\). We will use the notation of section 1, (1.6), (1.16), (1.20), (1.27)-(1.31), and

\[(5.1) \quad A_j := A_{k_j+1, k_j, s_j}, \ A^*_j := A^*_{k_j+1, k_j}, \ \tilde{A}_j := A_j - A^*_j, \ j \in \{1, ..., r - 1\},\]

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where \( z_j \in S_j, z_j \neq 0, j = 1, \ldots, r - 1 \). The path of integration in \( A_j^* \) is in \( S_j \). So for \( \tilde{A}_j \) we replace the path of integration in the definition of \( A_j \) (cf. (1.16)) by a path from \( z_j \) to \( \infty \) in \( S_j \).

Then \( \tilde{A}_j \varphi_j = \varphi_{j+1} - A_j^* \varphi_j \) exists on \( S_{j+1} \) and belongs to \( E(\kappa_{j+1}, S_{j+1}) \) since \( (A_j^* \varphi_j)(\xi) \) is \( \xi^{-k_{j+1}} \) times an entire function of \( \xi^{-k_j} \) (cf. (1.20)). Now we define

\[
\begin{cases}
\tilde{\varphi}_1 := \varphi_1, & \tilde{\varphi}_{j+1} = \tilde{A}_j \tilde{\varphi}_j \text{ if } j = 1, \ldots, r - 1 \text{ and } \\
u_j = L_{k_j}^{z_j} \tilde{\varphi}_j \text{ if } j = 1, \ldots, r \text{ where } z_r \in S_r, |z_r| = \infty.
\end{cases}
\]

Then we have

**Lemma 4.** The function \( \tilde{\varphi}_j \) exists on \( S_j \), is analytic on \( S_j \) and \( \tilde{\varphi}_j \in E(\kappa_j, S_j) \). Moreover, \( \tilde{\varphi}_j - \varphi_j \) is analytic on \( C_\infty \) and bounded on any sector \( S(\theta_-, \theta_+) \) in a neighborhood of \( \infty \). The function \( u_j(x) \) is an entire function of \( x^{-k_j} \) if \( j = 1, \ldots, r - 1 \), and

\[
f = u_1 + \cdots + u_r, \quad u_1(x) \sim \tilde{f}(x) \text{ as } x \to 0 \text{ on } |\arg(x/z_1)| < \pi/(2k_1).
\]

If \( r \geq 2 \) and \( \delta > 0 \) there exist positive constants \( c_j(\delta) \) and \( \tilde{c}_j(\delta) \) such that

\[
\begin{cases}
u_j(x) = O(1) \exp\{-c_j(\delta)(z_{j-1}/x)^{k_{j-1}}\} \text{ as } x \to 0 \\
o \text{ on } |\arg(x/z_{j-1})| \leq \pi/(2k_{j-1}) - \delta,
\end{cases}
\]

\[
\begin{cases}
\tilde{\varphi}_j(\xi) = O(1) \exp\{-\tilde{c}_j(\delta)(z_{j-1}/\xi)^{k_{j-1}}\} \text{ as } \xi \to 0 \\
o \text{ on } |\arg(\xi/z_{j-1})| \leq \pi/(2\kappa_{j-1}) - \delta,
\end{cases}
\]

for \( j = 2, \ldots, r \).

**Proof:** First we prove by induction that \( \tilde{\varphi}_j - \varphi_j \) is analytic on \( C_\infty \) and bounded on any sector \( S(\theta_-, \theta_+) \), and

\[
\varphi_j = B_{k_j}(u_1 + \cdots + u_{j-1}) + \tilde{\varphi}_j.
\]

Since \( \varphi_1 = \tilde{\varphi}_1 \), the assertion for \( j = 1 \) is obvious. Suppose it holds for \( j \). We have

\[
\tilde{A}_j(\tilde{\varphi}_j - \varphi_j)(\xi) = \xi^{-k_{j+1}} \int_{z_j}^{\infty} \exp(i\theta) C_{k_{j+1}/k_j}((t/\xi)^{k_j})(\tilde{\varphi}_j - \varphi_j)(t) d(t^{k_j}).
\]

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Using (1.18) and (1.19) we see that the righthand side exists, is analytic on $C_{\infty}$ and bounded on any $S(\theta_-,\theta_+)$ as $\xi \to \infty$. Since $\tilde{A}_j\varphi_j$ exists on $S_{j+1}$ and belongs to $E(\kappa_{j+1}, S_{j+1})$ as we saw above, now also $\tilde{\varphi}_{j+1} = \tilde{A}_j \tilde{\varphi}_j$ has this property. Furthermore,

$$\tilde{\varphi}_{j+1} - \varphi_{j+1} = \tilde{A}_j (\varphi_j - \varphi_j) - A_j^* \varphi_j,$$

and so both sides are analytic on $C_{\infty}$ and bounded on any $S(\theta_-,\theta_+)$ in a neighborhood of $\infty$. From (1.21) we deduce $A_j^* \tilde{\varphi}_j = B_{k_{j+1}} \tilde{L}_j \tilde{\varphi}_j = B_{k_{j+1}} u_j$. So (5.5) and (1.21) imply

$$\begin{align*}
\varphi_{j+1} &= A_j \varphi_j = A_j B_{k_j} (u_1 + \cdots + u_{j-1}) + A_j^* \tilde{\varphi}_j + \tilde{A}_j \tilde{\varphi}_j = \\
&= B_{k_{j+1}} (u_1 + \cdots + u_j) + \tilde{\varphi}_{j+1}.
\end{align*}$$

Hence (5.5) holds for $j = 1, \ldots, r$ and $f = L_k \varphi_r = u_1 + \cdots + u_r$.

From the definition of $u_1$ we immediately deduce $u_1(x) \sim \tilde{f}(x)$ as $x \to 0$ on $|\arg(x/z_1)| < \pi/(2k_1)$. From the definition of $\tilde{\varphi}_j$ and (1.19) we readily get the second formula in (5.4). This implies

$$u_j(x) = \int_0^{z_j} O(1) \exp\{-{(\xi/x)}^{k_j} - \tilde{c}_j(\delta)(z_j-1/\xi)^{\kappa_j-1}\} d(\xi^{k_j}).$$

The saddle point $\xi_0$ of the exponential term in the integrand satisfies $\xi_0^{\kappa_j+k} = (\kappa/k)cz_j^{\kappa_j-1}x^k$ where $\kappa = \kappa_j-1$, $k = k_j$, $c = \tilde{c}_j(\delta)$. The value of the exponential term in $\xi_0$ is

$$\exp\left[-c (\frac{\kappa}{k} + 1) \left(\frac{k}{\kappa c}\right)^{\kappa/(k+\kappa)} \left(\frac{z_j-1}{x}\right)^{\kappa j/(k+\kappa)} \right],$$

from which we may deduce the first part of (5.4).

**Remark 2.** In case $\hat{y}$ is a formal solution of $D[y] = g$ as in section 2 we may construct the corresponding multisum $y$ and its decomposition from lemma 4 directly without using the construction of section 2 as follows. Start with the convolution equation $Q_1 \psi = \gamma_1$ and obtain the solution $\varphi_1$ as Borel transform of order $k_1$ of $\hat{y}$. Then introduce $u_1$ as in (5.2). Now $D[u_1] = b_1$ where $b_1$ is exponentially small of order $k_1$. Next construct an exponentially small solution $\varphi_2$ of the convolution equation $Q_2 \varphi = \eta_2$ which corresponds to $D[u] = -b_1$. Then define $u_2 = L^2_{k_2} \varphi_2$. Now $u_2$ is exponentially small of order $k_1$ and
\[ D[u_2] = -b_1 + b_2, \text{ where } b_2 \text{ also is exponentially small of order } k_2 \text{ etc.} \] Finally we solve \( D[u_r] = -b_{r-1} \) with \( u_r = L_{\kappa_r} \varphi_r, \varphi_r \text{ exponentially small solution of } Q_r \varphi = \eta_r. \) Then \( y = u_1 + \cdots + u_r \) is the solution of \( D[y] = g \) which corresponds to the solution of section 2. This method is due to Ecalle.

In the discussion of Stokes multipliers in section 6 we need some related decompositions of solutions of (0.1) considered in theorem 2. We derive those here using the notation of section 4.

Let \( d \) be a halfline \( \arg \xi = \theta_0, S_0 := S(\theta_0 - \delta, \theta_0 + \delta), S_- := S(\theta_0 - \delta, \theta_0), S_+ := S(\theta_0, \theta_0 + \delta) \) and \( S = S_- \text{ or } S_+. \) Here \( \delta > 0 \) is so small that \( S_0 \setminus d \) does not contain singular values of (4.7): \( V_j \cap (S_0 \setminus d) = \emptyset \) for all \( j \) and \( \ell \) (cf. (4.10)).

Let \( H \) be the multiset associated with \( \mathcal{H} \) such that in its definition all \( S_h \) include \( S \) (cf. (1.29)). If \( S = S_- \) and \( S = S_+ \), we denote \( H \) also by \( H^- \) and \( H^+ \) respectively. Choose \( \lambda_0 \in \mathbb{C} \) such that

\[
(5.6) \quad \text{Re } \lambda_h > 0 \text{ if } \lambda_h = p_h(0) + \lambda_0, \quad h = 1, \ldots, m.
\]

In the discussion in section 6 we will use

\[
(5.7) \quad \begin{cases}
G\cdot j(t) := H\cdot j(t)^{N_j + \lambda_j L_j} \text{ if } \Lambda \neq 0, & G\cdot j(t) := H\cdot j(t) \text{ if } \Lambda = 0, \\
Z\cdot h\cdot j(t) := G\cdot h\cdot j(t) \exp(q_h - q_j)(t^{-1}).
\end{cases}
\]

Here \( G\cdot j = G^- \cdot j \) or \( G^+ \cdot h \cdot j \) according as \( S = S^- \) or \( S^+ \). Here we give a decomposition of these functions. Let

\[
(5.8) \quad B_{m\cdot sj}, S_0 t^{N_j + \lambda_j L_j} =: \eta\cdot g\cdot j, \quad B_{m\cdot sj}, S H\cdot j =: \psi\cdot g\cdot j, \quad g \in \{1, \ldots, r(j)\}
\]

in the sense of (1.31). Then

\[
(5.9) \quad B_{m\cdot sj}, S G\cdot j =: \varphi\cdot g\cdot j, \quad \text{where } \begin{cases}
\varphi\cdot g\cdot j = \psi\cdot g\cdot j \text{ if } \Lambda = 0 \text{ and } \\
\varphi\cdot g\cdot j = H\cdot j(0)\eta\cdot g\cdot j + \psi\cdot g\cdot j \cdot \eta\cdot g\cdot j \text{ if } \Lambda \neq 0.
\end{cases}
\]

Next define functions \( \tilde{\varphi} \) and \( u \) as in (5.2)

\[
(5.10) \quad \tilde{\varphi}_{1\cdot j} := \varphi_{1\cdot j}, \quad \tilde{\varphi}_{g+1\cdot j} := \tilde{A}_g \tilde{\varphi}_{g\cdot j}, \quad \tilde{u}_{g\cdot j} =: L_{m\cdot sj} \tilde{\varphi}_{g\cdot j},
\]

\[ 23 \]
where $\hat{A}_g$ is defined as in (5.1) with $j$, $k_j$ and $k_{j+1}$ replaced by $g$, $m_{gj}$ and $m_{g+1,j}$ and $z_g \in d$ such that

$$|z_g| > \max\{|\rho_{th}|, \quad \rho_{th} \in V_{gj} \cap S_0\}.$$  

(5.11)

The paths of integration in the integrals defining $\hat{A}_g\tilde{\phi}$ and $u$ are in $S$. Then

$$G_{j} = \sum_{g=1}^{r(j)} u_{gj}, \quad u_{1j}(t) \sim \hat{F}_{j}(t) t^{N_j + \lambda_j I_j}, \quad t \to 0 \text{ on } S_0$$

$$u_{gj}(t) = O(1) \exp\{-c_g(z_{g-1}/t)^{m_{g-1,j}}\}, \quad t \to 0 \text{ on } S_0, \quad g \geq 2$$

(5.12)

for some $c_g > 0$ and $\delta > 0$ sufficiently small. Moreover, $\tilde{\phi}_{gj} - \phi_{gj}$ is analytic on $C_\infty$.

Next we consider $Z_{hj}$ in case $\rho_{hj} \in d$ (cf. (5.7) and (4.9)). First define

$$\gamma_{j}^{(h)}(\xi) := 2\pi i \int_{0}^{\infty} \exp(i\theta) \exp\{(\xi/t)^{kh_j} + (q_h - q_j)(t^{-1})\}d(t^{-kh_j})$$

$$= 2\pi i \{L_{kh_j,s} \exp(q_h - q_j)\}(\xi^{-1} \exp(-\pi i/k_{hj})).$$

(5.13)

Here we first choose $\xi$ and $\theta$ such that $\text{Re} \,(\xi/\rho)^k < 1$, $\theta = \arg \rho$ where $k := kh_j, \rho := \rho_{hj}$. By varying $\xi$ and $\theta$ we see that $\gamma_{j}^{(h)}(\xi)$ may be continued analytically for $\xi^k \neq \rho^k$, and $\gamma_{j}^{(h)}(\xi) \to 0$ as $\xi \to \infty$ on any sector of finite opening. By inversion of (5.13) we get (cf. (1.18)):

$$\exp(q_h - q_j)(t^{-1}) = \int_{C(\rho_{hj})} \exp\{-(\xi/t)^{kh_j}\}\gamma_{j}^{(h)}(\xi)d(\xi^{kh_j}),$$

(5.14)

where the contour is a loop from $\infty e^{i\arg t}$ around $\rho_{hj}$ in negative sense to $\infty e^{i\arg t}$ which does not enclose $\rho_{hj} \exp(2\pi i/k_{hj}), g = 1, ..., k_{hj} - 1$. We also may rewrite (5.14) as

$$\exp(q_h - q_j)(t^{-1}) = \{L_{kh_j,s_+} \gamma_{j}^{(h)} - L_{kh_j,s_-} \gamma_{j}^{(h)}\}(t)$$

(5.15)

where $t \in S_0$. In $L_{S_+}$ we may deform the path of integration without passing thru $\rho_{hj}$, and similarly for $L_{S_-}$. The accelerates $A_{\ell,k}$ with $\ell > k$ of $\gamma_{j}^{(h)}$ exist and are analytic on $C_\infty$.

Let $k_{hj}$ be the $g$th level $m_{gj}$ associated with $j$ in (4.7). Then define for $\mu \geq g$ in the sense of (1.31):

$$\varphi_{\mu j}^{(h)} := B_{m_{\mu j},s_-} G_{\mu h}^{(h)}, \quad \gamma_{\mu j}^{(h)\pm} := A_{m_{\mu j},m_{gj},s_{\pm}} \gamma_{j}^{(h)}.$$  

(5.16)
So $\gamma_{\mu j}^{(h)}$ is analytic if $\mu > g$. Furthermore $\psi_{\mu j}^{(h)}$ is analytic in $S_0$ except for singular (branch-) points in

\[(5.17) \quad V(h, \mu, j) := \{ \rho_{\ell h} \in d \mid k_{\ell h} = m_{\mu j} \}.
\]

We make the additional assumption that for fixed $j$ and $h$ with $\rho_{h j} \in d$ and $k_{h j} = m_{g j}$ as above we have (cf. (4.10))

\[(A_{h j}) \quad \rho_{\ell h} \neq \rho_{h j} \text{ if } k_{\ell h} = k_{h j}, \text{ i.e. } \rho_{h j} \notin V(h, g, j).
\]

Let $S_{+}^{h \mu}$ be the sector $S_+$ with cuts from $V(h, \mu, j)$ to points on $\arg \xi = \theta_0 + \delta$, and with added small neighborhoods of $V(h, \mu, j)$ in $S_-$. Let $S_{-}^{h \mu} := S_-.$

Then define

\[(5.18) \quad \begin{aligned}
\chi_{g j}^{(h)} &= G_{\mu j}(0)\gamma_{j}^{(h)} + \psi_{g j}^{(h)} \ast m_{g j}, \\
\chi_{\mu j}^{(h) \pm} &= G_{\mu j}(0)\gamma_{\mu j}^{(h) \pm} + \psi_{\mu j}^{(h)} \ast m_{\mu j} 
\end{aligned}
\]

In case of the upper sign we choose the path of integration in $S_{+}^{h \mu}$ and in case of the lower sign in $S_-$. Then $\chi_{g j}^{(h)+}$ is analytic in $S_{+}^{h g}$ and can be continued analytically into $S_-$ except for a branch point at $\rho_{h j}$. Similarly for $\chi_{g j}^{(h)-}$. Furthermore, if $\mu > g$ we have that $\chi_{\mu j}^{(h) \pm}$ is analytic in $S_{+}^{h \mu} \cup S_-$ with singular (branch-) points in $\rho_{\ell h} \in d$ with $k_{\ell h} = m_{\mu j}$, so in $V(h, \mu, j)$. Finally, from (1.12), (1.22), (5.7), (5.15), (5.16) and (5.18) we deduce

\[(5.19) \quad Z_{h j} = L_{k_{r j}, S_+^r} \chi_{r j}^{(h)+} - L_{k_{r j}, S_-} \chi_{r j}^{(h)-}, \text{ where } r = r(j).
\]

Analogous to (5.2) we define

\[(5.20) \quad \tilde{\chi}_{g j}^{(h) \pm} := \chi_{g j}^{(h) \pm}, \quad \tilde{\chi}_{\mu j}^{(h) \pm} := \tilde{A}^{\pm} \chi_{\mu j}^{(h) \pm} \text{ if } \mu \geq g,
\]

where $\tilde{A}^{\pm} := \tilde{A}_{m_{\mu + 1, j}, m_{\mu j}, S_{\pm}}$ as in (5.10) and (5.1), but now the index $S_{+}^{h \mu}$ means that the path of integration for the integral in (1.16) is taken in $S_{+}^{h \mu}$. Then analogous to the previous case in lemma 4, we have $\tilde{\chi}_{\mu j}^{(h) \pm} = \chi_{\mu j}^{(h)}$ is analytic in $C_\infty$. Analogous to (5.2) we also define

\[(5.21) \quad v_{\mu j}^{(h)}(t) := \int_{\Gamma(z_{\mu})} \{ \exp - (\xi/t)^{m_{j}} \} \tilde{\chi}_{\mu j}^{(h)}(\xi) d(\zeta^{m_{j}}), \mu \geq g.
\]
where $\Gamma(z_\mu)$ runs from $z_\mu$ to 0 in $S_-$ with $\tilde{\chi}^{(h)} = \tilde{\chi}^{(h)-}$ and then from 0 to $z_\mu$ in $S_+^{h \mu}$ with $\tilde{\chi}^{(h)} = \tilde{\chi}^{(h)+}$. If $\mu = g$, we have $\tilde{\chi}_{gj}^{(h)\pm} = \chi_{gj}^{(h)}$ (c.f. (5.20) and (5.18)), which has inside $\Gamma(z_g)$ only a singularity in $\rho_{hj}$.

In the same way as in lemma 4 we now may prove using (5.19):

$$
\begin{align*}
Z_{hj} &= \sum_{\mu=g}^{r(j)} v^{(h)}_{\mu j}, \text{ where } k_{hj} = m_{gj} \\
v^{(h)}_{gj}(t) &\sim \hat{G}_{-h}(t) \exp(q_h - q_j)(t^{-1}), \quad t \to 0 \text{ on } S_0, \\
v^{(h)}_{\mu j}(t) &= O(1) \exp\{-c_\mu(z_{\mu-1}/t)^{m_{\mu-1,j}}\}, \quad t \to 0 \text{ on } S_0, \quad \mu > g,
\end{align*}
$$

(5.22)

where $c_\mu > 0$ is some constant.

6. Relations for Stokes multipliers. Here we consider two fundamental matrices $Y^+$ and $Y^-$ of (0.1) in overlapping sectors $S^+$ and $S^-$ as constructed in theorem 2. Then $Y^+(x) = Y^-(x)C$, where $C \in GL(n; \mathbb{C})$ is the corresponding Stokes matrix. We derive relations of $C$ with accelerates of the $B_{m_1}$-transform of $\hat{H}$ of theorem 2.

Since (0.1) corresponds via (4.1) with (4.2) we have

$$
Y^\pm(x) = \left( \sum_{h=0}^{k} B_{h}^{t^h} \right) W^\pm(t), \quad t = x^{1/p},
$$

where $W^+(t)$ and $W^-(t)$ are fundamental matrices of (4.2) on $S^+$ and $S^-$. So $W^+(t) = W^-(t)C$.

Let

$$
\begin{align*}
W^\pm(t) &= H^\pm(t)t^\Lambda \exp Q(t^{-1}), \Lambda = P(0) + N, \quad H^\pm(t) \sim \hat{H}(t) \text{ on } S^\pm \\
G^\pm(t) &= H^\pm(t)t^{\Lambda + \lambda_0} \text{ if } \Lambda \neq 0; \quad \text{and } G^\pm(t) = H^\pm(t) \text{ if } \Lambda = 0.
\end{align*}
$$

(6.1)

where $\lambda_0$ is such that $\text{Re}\lambda_j > 0$ as in (5.6). Using the block notation as introduced after (4.6) we have if $j \in \{1, \ldots, m\}$:

$$
G^+_j(t) = \sum_h G^-_{jh}(t)\{\exp(q_h - q_j)(t^{-1})\}C_{hj} = \sum_h Z_{hj}(t)C_{hj}
$$

(6.2)

where $Z_{hj}$ is defined as in (5.7). It is sufficient to consider relations for $C_{hj}$ with fixed $j$. 

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So we consider multisums of $\hat{H}_j(t)$ on two consecutive sectors $S_j^-$ and $S_j^+$ as in theorem 2, where we choose these sectors with maximal opening and $S^- \subset S_j^-$, $S^+ \subset S_j^+$. Then it is easy to see that

$$S_j^- = S(\tau_1, \tau_+), \quad S_j^+ = (\tau_-, \tau_2),$$

where $\tau_1$, $\tau_2$ are Stokes directions, $(\tau_-, \tau_+)$ is a Stokes pair associated with some $g_\lambda - q_j$ and $\tau_1 < \tau_-, \tau_+ < \tau_2$. Hence

$$(6.3) \quad S_0 := S_j^- \cap S_j^+ = S\left(\theta_0 - \frac{\pi}{2k}, \theta_0 + \frac{\pi}{2k}\right) \text{ if } \theta_0 := \arg \rho_{\lambda j}, \quad k := k_{\lambda j}.$$ 

Let $k_{\lambda j}$ be the $\sigma$-th level $m_{\sigma j}$ associated with $j$ in (4.7) and $d$ be the halfline $\arg \xi = \theta_0$. Because $S_0$ cannot contain a pair of Stokes rays associated with $q_h - q_j$ for any $h$ we have $\rho_{h j} \notin d$ if $k_{h j} > k_{\lambda j} = m_{\sigma j}$. Define for $g \in \{1, \ldots, \sigma\}$

$$(6.4) \quad I_{g j} := \{h \in \{1, \ldots, m\} \mid \rho_{h j} \epsilon d, \ k_{h j} = m_{g j}\}$$

or equivalently $I_{g j} := \{h \mid \rho_{h j} \epsilon V_{g j} \cap d\}$ (cf. (4.10)). Furthermore let $S_+ := S(\theta_0, \theta_0 + \delta)$ and $S_- := S(\theta_0 - \delta, \theta_0)$, where $\delta > 0$ will be chosen sufficiently small.

Let $\chi_{gj}^{(h)}$ be defined by (5.18), (5.16) and (5.13). Let $\varphi_{gj}^\pm, \psi_{gj}^\pm$ be defined by (5.9) and (5.8) with $S := S_{\pm}, H := H^\pm, G := G^\pm$. So

$$(6.5) \quad \varphi_{1 j}^\pm := \varphi_{1 j}, \ \varphi_{g+1, j}^\pm := A_g^\pm \varphi_{g j}^\pm \text{ where } A_g^\pm := A_{m_{g+1, j}, m_{g j}, S_\pm}.$$ 

Furthermore we use assumption $(A_{h j})$ from section 5 if $\rho_{h j} \epsilon d, k_{h j} \leq m_{\sigma j}$, or equivalently:

$$(A_j) \quad \rho_{l h} \neq \rho_{h j} \text{ if } \rho_{h j} \epsilon d \text{ and } k_{l h} = k_{h j} \leq m_{\sigma j}.$$ 

Then we have

**Theorem 3.** Let $Y^-$ and $Y^+$ be the fundamental matrices of (0.1) given by Theorem 2 on two consecutive, overlapping sectors $S^-$ and $S^+$, and let $Y^+ := Y^- C$. Let $j \in \{1, \ldots, m\}$ and let the corresponding multisums $H_j^-$ and $H_j^+$ (cf. (6.1)) be defined on $S_j^-$ and $S_j^+$.
with $S^\pm \subset S_j^\pm$ and (6.3). Let $k_{hj} = m_{\sigma j}$ and $d$ be the halfline $\arg \xi = \theta_0$, and assume $(A_j)$ is satisfied. Then

$$C_{hj} = 0 \text{ if } h \notin \bigcup_{g=1}^{\sigma} I_{gj},$$

and

$$K_{gj}(\xi) := \varphi_{gj}^+(\xi) - \sum_{h \in I_{gj}} \chi_{gj}^{(h)}(\xi) C_{hj}$$

is analytic at the singular points of $\varphi_{gj}$ on $d$, $g \in \{1, \ldots, \sigma\}$.

**Remark 3.** Theorem 3 gives a possibility to obtain limit formulas for Stokes multipliers $C_{hj}$. Because of (5.18) the behavior of $\chi_{gj}^{(h)}$ near the singular point $\rho_{hj}$ of $\varphi_{gj}$ on $d$ with $k_{hj} = m_{gj}$ is determined by the behavior of $\gamma_j^{(h)}$ near $\rho_{hj}$ (cf. (5.13)) and the behavior of $\varphi_{gj}^{(h)}$ near 0:

$$\varphi_{gj}^{(h)} \sim \hat{B}_{m_{gj}}(\hat{H}_h t^{N_h + \lambda_h I_h}) \text{ if } \Lambda \neq 0, \quad \chi_{gj}^{(h)} \sim H_{-h}(0) \gamma_j^{(h)} \text{ if } \Lambda = 0.$$  

The function $\varphi_{gj}^+$ in (6.6) can be constructed by acceleration of the Borel transform of order $m_{1j}$ of $\hat{H}_{-h}$. In particular, $\varphi_{gj}^+$ is the $(m_{1j}^{-1} - m_{2j}^{-1})^{-1}$-sum of $\varphi_{2j} = \hat{B}_{m_{2j}} \hat{H}_{-h}$ (cf. section 1, remark after (1.30)).

**Proof:** From (5.12) we deduce

$$G_{rj}^+(t) - G_{rj}^-(t) = \sum_{h=1}^{r(j)} (u_{hj}^+ - u_{hj}^-)(t),$$

where in (5.10) we choose $z_h \in d$ with (5.11), and the upper and lower sign correspond to the path of integration in the Laplace integral in (5.10) lying in $S_+$ and $S_-$ respectively. In (5.9) $\psi_{1j}$ and $\varphi_{1j}$ are analytic in $S(\theta_0 - \delta, \theta_0 + \delta)$ for some $\delta > 0$ except possibly at singular points $\varphi_{hj} \in d$, with $k_{hj} = m_{1j}$. Hence

$$(u_{1j}^+ - u_{1j}^-)(t) = \int_{C_1} \{\exp - (\xi/t)^{m_{1j}} \} \varphi_{1j}(\xi) d(\xi^{m_{1j}}),$$

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where the contour $C_1$ lies in $S(\theta_0 - \delta, \theta_0 + \delta)$ and runs from $z_1$ to $z_1$ around the singular points of $\varphi_{1j}$ in negative sense. Here $0 < \delta < \pi/(2k\lambda_j)$ and $\delta$ will be chosen sufficiently small. In particular we deduce from (6.8), (6.9) and (5.12) that

\begin{equation}
G_{j}^{+}(t) - G_{j}^{-}(t) = O(1) \exp(-c|t|^{-m_{ij}}) \text{ as } t \to 0 \text{ on } S(\theta_0 - \delta, \theta_0 + \delta)
\end{equation}

for some $c > 0$.

The multisums in the righthand side of (6.2) are defined in $S(\theta_0 - \delta, \theta_0 + \delta) \subset S_0$ for some $\delta > 0$. Hence we obtain with (6.10)

\begin{equation}
G_{j}^{+}(t) - G_{j}^{-}(t) = \sum_{h \in I} Z_{hj}(t)C_{hj},
\end{equation}

where $I = \{h \in \{1, \ldots, m\} \mid h \neq j, C_{hj} \neq 0\}$ and (6.12)

\begin{equation}
\text{Re}(q_h - q_j)(t^{-1}) < 0, \text{ if } t \in S(\theta_0 - \delta, \theta_0 + \delta) \cap \Delta(0; \varepsilon) \text{ and } h \in I.
\end{equation}

Here $\varepsilon$ is some positive number.

Let

\begin{equation}
X_1(t) := \sum_{h \in I \setminus I_{1j}} Z_{hj}(t)C_{hj}.
\end{equation}

Then combining (6.11), (6.8), (5.22) and (5.21) we obtain

\begin{equation}
N_1(t) = \int_{\Gamma(z_1)} \{\text{exp}(-\xi/t)^m\} \{\varphi_{1j}(\xi) - \sum_{h \in I_{1j}} \chi_{1j}^{(h)}(\xi)C_{hj}\}d(\xi)^m
\end{equation}

\begin{equation}
= \sum_{h=2}^{r(j)} (u_{hj}^{+} - u_{hj}^{-})(t) + \sum_{h \in I_{1j}} \sum_{\ell=2}^{r(j)} v_{l_j}^{(h)}(t)C_{hj} + X_1(t).
\end{equation}

Here $\Gamma(z_1)$ is as in (5.21) and such that $V(h, 1, j)$ is outside $\Gamma(z_1)$ if $h \in I_{1j}$ and $V_{1j} \cap d$ is inside $\Gamma(z_1)$ (cf. (4.10)). This is possible because of assumption $(A_j)$. We may deform $\Gamma(z_1)$ such that 0 lies outside $\Gamma(z_1)$ since the integrand is analytic near 0 in $S(\theta_0 - \delta, \theta_0 + \delta)$.

Now consider

\begin{equation}
f(s) := \int_0^t \{\text{exp}(s/t)^m\}N_1(t)d(t^{-m})
\end{equation}

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where \( t_1 \in d, \ |t_1| \) sufficiently small. From the behavior of the last member of (6.14) near \( t = 0 \) (cf. (5.12), (5.22), (6.13) and (6.12)) it follows that \( f(s) \) is analytic for \( s \in d, \ |s| < R \) were \( R \rightarrow \infty \), as \( z_1 \rightarrow \infty \). From (6.14) and (6.6) we also deduce

\[
f(s) = \int_{\Gamma(z_1)} \left[ \exp\{(s/t_1)^{m_{1j}} - (\xi/t_1)^{m_{1j}}\} \right] K_{1j}(\xi) \frac{d(\xi^{m_{1j}})}{\xi^{m_{1j}} - s^{m_{1j}}},
\]

if \( \text{Re}\{(\xi e^{-i\theta_0})^{m_{1j}} - (se^{-i\theta_0})^{m_{1j}}\} > 0 \) for all \( \xi \in \Gamma(z_1) \).

By using a contour \( \Gamma' \) from \( z_1 \) to \( z_1 \) which includes \( s \) but not \( 0 \) we see that \( f(s) = -2\pi i K_{1j}(s) + \int_{\Gamma'}, \) where the last integral is analytic inside \( \Gamma' \). Hence \( K_{1j}(s) \) is analytic inside \( \Gamma(z_1) \) and \( N_1(t) = 0 \) in (6.14).

From (6.14) and the behavior of \( u_{hj}^\pm(t) \) and \( v_{hj}^{(k)}(t) \) near \( t = 0 \) (cf. (5.12) and (5.22)) it follows that \( I \setminus I_{1j} = J_1 \), where

\[
J_\ell := \{ h \in I \ | \ k_{hj} > m_{\ell j} \}, \quad \ell = 1, \ldots, r(j).
\]

Since \( K_{1j} \) is analytic inside \( \Gamma(z_1) \) we have \( \tilde{\mathcal{A}}^-_{1} K_{1j} = \tilde{\mathcal{A}}^+_{1} K_{1j} \) (cf. the definition of \( \tilde{\mathcal{A}}^\pm \) after (5.20)). Hence if \( \xi \in S_0 \) and \( |\xi| \) is sufficiently small then

\[
\tilde{\varphi}^+_{2j}(\xi) - \sum_{h \in I_{1j}} \tilde{\chi}_{2j}^{(h+)}(\xi) C_{hj} = \tilde{\varphi}^+_{2j}(\xi) - \sum_{h \in I_{1j}} \tilde{\chi}_{2j}^{(h-)}(\xi) C_{hj},
\]

where \( \tilde{\varphi}^\pm \) is defined as in (6.5) with tildes added to \( \varphi^\pm \) and \( \mathcal{A}^\pm \). Hence (5.21) implies

\[
\sum_{h \in I_{1j}} v_{2j}^{(k)}(t) C_{hj} = \int_{\Gamma(z_2)} \{ \exp - (\xi/t)^{m_{2j}} \} \tilde{\varphi}_{2j}(\xi) d(\xi^{m_{2j}}),
\]

where \( \Gamma(z_2) \) runs from \( z_2 \) to \( 0 \) in \( S_- \) with \( \varphi = \tilde{\varphi}^- \) and from \( 0 \) to \( z_2 \) in \( S_+^{h_2} \) with \( \varphi = \tilde{\varphi}^+ \).

It now follows from (5.10), (5.21) and (6.17) that

\[
N_2(t) := (u_{2j}^+ - u_{2j}^-(t)) - \sum_{\ell=1}^2 \sum_{h \in I_{1j}} v_{2j}^{(k)}(t) C_{hj} =
\]

\[
= \int_{\Gamma} \{ \exp - (\xi/t)^{m_{2j}} \} \tilde{K}_{2j}(\xi) d(\xi^{m_{2j}}),
\]
where $\tilde{K}_{2j}$ is defined by (6.6) with tildes added, and $\Gamma$ runs from $z_2$ to $z_2$ once around $V(2,j) \cap d$ in negative sense and with $V(h,2,j)$ outside $\Gamma$ if $h \in I_{2j}$. This is possible because of assumption $(A_j)$. Since $\tilde{\varphi} - \varphi$ and $\tilde{\chi} - \chi$ are analytic according to section 5, we may omit the tilde in (6.18). Since $N_1(t) = 0$, we obtain from (6.14):

\begin{equation}
N_2(t) = - \sum_{h=3}^{r(j)} (u_{h,j}^+ - u_{h,j}^-)(t) + \sum_{t=1}^{2} \sum_{h \in I_{ij}} \sum_{\alpha=3}^{r(j)} v_{\alpha j}^{(h)}(t) + X_2(t),
\end{equation}

where

$$X_2(t) := \sum_{h \in I_1 \setminus I_{2j}} Z_{h,j} C_{h,j}.$$ 

To (6.18) and (6.19) we may apply the same arguments made above to $N_1(t)$. Thus we obtain: $K_{2j}$ is analytic inside $\Gamma$, $N_2(t) = 0$ and $J_1 \setminus I_{2j} = J_2$, so $I \subset J_2 \cup I_{ij} \cup I_{2j}$.

Repeating this procedure we obtain the result concerning $K_{gj}$ for $g \in \{1, \ldots, \sigma\}$ in the theorem. Moreover, we see that

$$I \subset \bigcup_{g=1}^{\sigma} I_{gj},$$

and so we get $C_{h,j} = 0$ if $h \notin \bigcup_{g=1}^{\sigma} I_{gj}$.

**Remark 4.** Another method to obtain relations for Stokes multipliers has recently been given by Immink [12]. In the case of one level the problem has received a lot of attention, for example see Balser, Jurkat, Lutz [2], [3] and R. Schäfke [16].

**Example 1.** We consider the following special case of (4.2):

$$t \frac{d w}{d t} = \text{diag} \{0, at^{-1}, 2t^{-2}\} + t R(t)\} w,$$

where $R(t)$ is an analytic $3 \times 3$-matrix in a neighborhood of 0. We now have two levels so that the functions $\varphi_{gj}^+$ in theorem 3 can be constructed using single $k$-summability instead of accelerations.
From (4.4) we see \( q_1(t) = 0, q_2(t) = -at, q_3(t) = -t^2 \), and there is a formal fundamental matrix

\[
\hat{H}(t) \text{ diag } \{ 1, \exp(-at^{-1}), \exp(-t^{-2}) \}, \quad \hat{H} = \sum_{0}^{\infty} H_k t^k, H_0 = I.
\]

From (4.9) we get \( \rho_{21} = a, \rho_{12} = -a, \rho_{31} = \pm 1, \rho_{13} = \pm i, \rho_{32} = \pm i, \rho_{23} = \pm i, k_{12} = 1, k_{13} = k_{23} = 2, m_{11} = m_{12} = 1, m_{21} = m_{22} = m_{13} = 2, r(1) = r(2) = 2, r(3) = 1. \)

Suppose \( \alpha := \arg \alpha \in (0, \frac{1}{2} \pi) \). From theorem 2 with \( j = 1 \) we see that \( \hat{w}_1(t) := \hat{H}_1(t) \) is (2,1)-summable in \( S^- := S(-\frac{5}{4} \pi, \frac{1}{4} \pi) \) and \( S^+ = S(-\frac{1}{4} \pi, \frac{5}{4} \pi) \) with sums \( w^{-1}_1(t) \) and \( w^+_1(t) \) respectively which satisfy (4.2). Now \( S^- \cap S^+ = S(-\frac{\pi}{4}, \frac{\pi}{4}) \) and so in (6.3) we have \( \theta_0 = 0 = \arg \rho_{31}, d \) is the halfline \( \arg \xi = 0, k_{31} = 2 = m_{21} \), \( \sigma = 2. \)

The condition \( (A_1) \) now becomes: \( p_{th} \neq \rho_{h1} \) if \( \rho_{h1} > 0 \) and \( k_{th} = k_{h1} \leq 2. \) Since \( \rho_{h1} > 0 \) implies \( h = 3 \) we see that assumption \( (A_1) \) is satisfied. Furthermore \( I_{11} = 0, I_{21} = \{ 3 \} \) (cf. (6.4)). Hence theorem 3 gives for the Stokes multipliers \( C_{h1} \): \( C_{h1} = 0 \) if \( h = 1, 2 \) and \( C_{31} \) follows from: \( \varphi^{+}_{21}(\xi) - \chi^{(3)}_{21}(\xi)C_{31} \) is analytic at \( \xi = \rho_{31} = 1. \) Here we have by (5.13)

\[
\gamma^{(3)}_{1}(\xi) = 2\pi i \int_{0}^{\infty} \exp\{(\xi/t)^2 - t^{-2}\} dt^{-2} = 2\pi i(1 - \xi^2)^{-1},
\]

and so by (5.18):

\[
\chi^{(3)}_{21}(\xi) \sim 2\pi i(1 - \xi^2)^{-1} \varepsilon_3, \quad \xi \to 1.
\]

Here we denote the \( k \)-th unit vector by \( \varepsilon_k \), so \( \varepsilon_3 = (0, 0, 1)^T \) where \( T \) denotes transposition. Hence we obtain

\[
C_{31} = \frac{1}{\pi i} \lim_{\xi \to 1} (1 - \xi) \varepsilon_3^T \varphi^{+}_{21}(\xi).
\]

According to remark 3 after theorem 3 we have that \( \varphi^{+}_{21} \) is the 2-sum of

\[
\sum_{1}^{\infty} H_p \varepsilon_1 \xi^{p-2}/\Gamma(\frac{1}{2} p):
\]

\[
\varphi^{+}_{21}(\xi) = H_1 \varepsilon_1/(\sqrt{\pi} \xi) + H_2 \varepsilon_1 + (L_2, S_1, f_1)(\xi),
\]

where \( f_1 = \hat{B}_2 \sum_{3}^{\infty} H_k \varepsilon_1 \xi^{k-2}/\Gamma(\frac{1}{2} k), S_1 = S(\alpha - 2\pi, \alpha) \), and \( |\xi| \) is sufficiently small. We may rewrite this formula as \( \varphi^{+}_{21}(\xi) = \xi^{-2}(L_2, S_1, f)(\xi) \), where \( f(t) = \sum_{1}^{\infty} H_k \varepsilon_1 t^{k-2}/\Gamma^2(\frac{1}{2} k). \)
Now $\varphi_{21}^+(\xi)$ has an analytic continuation outside the points $\pm 1$; we denote this continuation also by $\varphi_{21}^+(\xi)$ if $\text{Im } \xi > 0$. Its behavior near $\xi = 1$ determines $C_{31}$ as we saw above.

**Example 2.** A simple example with three levels is given by

$$t \frac{dw}{dt} = [\text{diag } \{0, at^{-1}, 2b^2 t^{-2}, -3t^{-3}\} + tR(t)]w,$$

where $R(t)$ is a $3 \times 3$-matrix analytic at $0$, $0 < \arg a < \frac{1}{2} \pi$, $b > 0$, $b \neq 1$.

Now $q_1 = 0, q_2(t) = -at, q_3(t) = -b^2 t^2, q_4(t) = t^3$, and there exists a formal fundamental matrix $\hat{H}(t)$ diag $\{1, \exp(-at^{-1}), \exp(-b^2 t^{-2}), \exp t^{-3}\}$ where $\hat{H}(t) = \sum_{k=0}^{\infty} H_k t^k, H_0 = I$.

From (4.9) we deduce $k_{21} = 1, \rho_{21} = a, k_{31} = 2, \rho_{31} = \pm b, k_{41} = 3, \rho_{41} = -1, \exp(\pm \pi i/3)$. Now $\hat{H}_1(t)$ is $(3, 2, 1)$-summable in $S(\frac{3\pi}{6}, \frac{11\pi}{6})$ and $S(\frac{\pi}{6}, \frac{7\pi}{6})$ and we consider the Stokes multipliers $C_{h1}$ associated with the corresponding fundamental solutions.

The intersection of these sectors is $S(\frac{5\pi}{6}, \frac{7\pi}{6})$ and so we have (6.3) with $\theta_0 = \pi, k = 3 = k_{41} = m_{31}, \sigma = 3$. Hence $d : \arg \xi = \pi$ contains the singular values $\rho_{31} = -b$ and $\rho_{41} = -1$. Furthermore it is easily verified that $(A_1)$ is satisfied and $I_{11} = \emptyset, I_{21} = \{3\}, I_{31} = \{4\}$. Hence theorem 3 implies $C_{11} = C_{21} = 0$ and

$$\varphi_{21}^+(\xi) - \chi_{21}^{(3)}(\xi) C_{31} \text{ is analytic at } \rho_{31} = -b,$$

$$\varphi_{31}^+(\xi) - \chi_{31}^{(4)}(\xi) C_{41} \text{ is analytic at } \rho_{41} = -1.$$

According to (5.13) we have

$$\gamma_1^{(3)}(\xi) = 2\pi i \int_0^{\infty} \exp\{(\xi/t)^2 - b^2 t^{-2}\} d(t^{-2}) = \frac{2\pi i}{b^2 - \xi^2}$$

and

$$\gamma_1^{(4)}(\xi) = 2\pi i \int_0^{\infty} \exp(\pi i) \exp\{(\xi/t)^3 + t^{-3}\} d(t^{-3}) = \frac{-2\pi i}{\xi^3 + 1}.$$

Hence (5.18) implies

$$\chi_{21}^{(3)}(\xi) \sim \frac{\pi i}{b(\xi + b)} e_3, \xi \rightarrow -b$$

$$\chi_{31}^{(4)}(\xi) \sim \frac{-2\pi i}{3(\xi + 1)} e_4, \xi \rightarrow -1.$$
Therefore,

\[ C_{31} = \frac{b}{\pi i} \lim_{\xi \to b} (\xi + b) e^{T} \varphi^+_{21}(\xi), \]

\[ C_{41} = \frac{-3}{2\pi i} \lim_{\xi \to -1} (\xi + 1) e^{T} \varphi^+_{31}(\xi). \]

Here we have the same expression for \( \varphi^+_{21}(\xi) \) as in example 1 for small \(|\xi|\), and \( \varphi^+_{31}(\xi) \) is continued analytically on the lower half plane \( S_2 : \text{Im } \xi < 0 \). Furthermore, \( \varphi^+_{31} = A_{3,2,S_3} \varphi^+_{21} \) in a neighborhood of 0 in \( S_2 \), and \( \varphi^+_{31} \) is continued analytically on \( S_2 \).

References


8. __________, L'accélération des fonctions résurgentes, manuscript 1987.


