AN UPWIND FINITE-VOLUME SCHEME
WITH A TRIANGULAR MESH
FOR CONSERVATION LAWS

By

San-Yih Lin

and

Yan-Shin Chin

IMA Preprint Series # 726

November 1990
AN UPWIND FINITE-VOLUME SCHEME
WITH A TRIANGULAR MESH
FOR CONSERVATION LAWS

By

San-Yih Lin
and
Yan-Shin Chin

IMA Preprint Series # 726
November 1990
AN UPWIND FINITE-VOLUME SCHEME WITH A TRIANGULAR MESH FOR CONSERVATION LAWS

SAN-YIH LIN† AND YAN-SHIN CHIN‡

Abstract. A new numerical scheme has been developed and analyzed for finite-volume solution of a conservation law on triangular meshes using an upwind method. The scheme is formally uniformly second order accurate and satisfies maximum principles. Preliminary numerical results showing the performance of the scheme on a variety of initial-boundary value problems are shown.

AMS(MOS) subject classifications. 65M60, 65N30, 35L65.

1. Introduction.

The new numerical scheme is developed to compute approximations of the physically relevant solution of the initial-boundary value problem associated with the hyperbolic conservation law

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} &= 0 \quad \text{in } \Omega \times (0, T) \\
 u(x, 0) &= u_0(x) \quad \text{in } \Omega \\
 u(x, t) &= r(x, t) \quad \text{on } \partial \Omega \times (0, T)
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \), \( u = (u_1, \ldots, u_m)^t \), and \((f, g)\) is a flux such that any real combination of the Jacobian matrices \( n_1 \partial f \partial u + n_2 \partial g \partial u \) has \( m \) real eigenvalues and a complete set of eigenvectors.

In this paper, we only concentrate on the scheme for scalar conservation laws, \( m = 1 \). The scheme stores the flow variables at the barycenters of the triangles. Fluid fluxes are integrated about the edges of the triangle containing the barycenter using trapezoidal integration. Flow solutions are computed using T.V.D. Runge-Kutta time integration [2,10]. We use a simple second order extrapolation to evaluate the flow variables at the edges and apply a local projection to assure that the scheme satisfies maximum principles.

Research has been conducted using finite-element schemes [3,8,12] for the conservation laws on totally unstructured meshes. Finite-element schemes require the storage of mesh connectivity and flux variables. Increased computer storage and indirect addressing imply that any solution technique used is going to run slow in a finite-element code.

A. Jameson and co-workers are applied centrally differenced, finite-volume schemes [5,6] to meshes composed of triangles and tetrahedrons. These schemes use a scalar dissipation

†Institute of Aeronautics and Astronautics, National Cheng Kung University, Tainan, Taiwan 70101, R.O.C.
*Associate Professor, supported in part by the NSC, Taiwan, R.O.C.
**Graduate Student, supported in part by the NSC, Taiwan, R.O.C.
coefficient to ensure convergence and prevent spurious oscillations. In last few years, extensive research has been done using upwind schemes to solve the conservation laws [1,9,14]. In general, upwind schemes are much more accurate than centrally differenced methods with a scalar dissipation coefficient, but one needs to do a lot of work to achieve high order upwind schemes in two-dimensional space.

The object of present paper is to introduce and analyze a second order, upwind, finite-volume scheme. In Section 2 we present and analyze our scheme. The main results are the definitions of a second order extrapolation and a local project in which does not destroy the high order accuracy of the scheme while enforcing a local maximum principle. In Section 3 we present several numerical results. Finally concluding remarks are presented in Section 4.

2. General Formulation.

2.1 Semi-discretization formulation. The conservative, integral form of the eqn (1.1a) is given below:

\[
\frac{\partial}{\partial t} \int_{\Omega} u dS + \int_{\Gamma} (f, g) \cdot n d\ell = 0
\]

where \( \Omega \) is the domain, \( \Gamma \) is the boundary of \( \Omega \) and \( n \) is the outer unit normal. We assume the two-dimensional domain \( \Omega \) can be discretized into a group of triangular polygons, \( T_h \). The vertices, barycenters and edges of the triangles \( K_i \), are denoted by \( V_{ij}, C_i \) and \( e_{ij}, j = 1, 2, 3 \), respectively.

In each triangle \( K_i \), flow variables are stored at the barycenter \( C_i \) and conservation is enforced the boundary \( \partial K_i \). If we assume that the triangular mesh is geometrically time invariant and the flow variables stored at the barycenter \( C_i \) are an area average of the integrated flow variables in the triangle \( K_i \), then the eqn (2.1) can be written as

\[
A(K_i) \frac{\partial u_i}{\partial t} = -\int_{\partial K_i} (f, g) \cdot n d\ell
\]

where \( A(K_i) \) is the area of \( K - i \).

To evaluate the right side of (2.2), we sum all the flux vectors on the three edges of \( K_i \).

\[
\int_{\partial K_i} (f, g) \cdot n d\ell = \sum_{j=1}^{3} F_{ij} \cdot |e_{ij}|
\]

where \( F_{ij} \) is the numerical approximation for the flux associated with the edge \( e_{ij} \) and \( |e_{ij}| \) is the length of the edge \( e_{ij} \).

In order to evaluate \( F_{ij} \) using an upwind scheme, it is necessary to have two fluid dynamic states, \( u_{ij,L} \) and \( u_{ij,R} \). Observing Figure 1, let point \( M_{ij} \) is the midpoint of
the edge $e_{ij}$. For given two variables $u_{ij,L}$ and $u_{ij,R}$, we can define the flux function $h_{ij}(u_{ij,L},u_{ij,R})$ such that $h_{ij}(\cdot,\cdot)$ is any function verifying the follow conditions:

\begin{align}
(2.4a) & \quad h_{ij}(u,u) = (f,g) \cdot n_{ij} \\
(2.4b) & \quad h_{ij}(u,v) \text{ is nondecreasing in } u \text{ and nonincreasing in } v, \\
(2.4c) & \quad h_{ij}(\cdot,\cdot) \text{ is Lipschitz} \\
(2.4d) & \quad h_{ij}(u_{ij,L},u_{ij,R}) = -h_{ij}(u_{ij,R},u_{ij,L}).
\end{align}

where $n_{ij}$ the outer unit normal of the edge $e_{ij}$ corresponding to the triangle $K_i$. The last property justifies $h_{ij}(\cdot,\cdot)$ as a flux. Examples of $h$ can be found in [4]. In this way, we obtain

\begin{equation}
(2.5) \quad A(K_i) \frac{du_i}{dt} = -\sum_{j=1}^{3} h_{ij}(u_{ij,L},u_{ij,R}) \cdot |e_{ij}|, \text{ for all } i
\end{equation}

![Figure 1. Representative variables of two fluid dynamics states $u_{ij,L}$ and $u_{ij,R}$.](image)

Now, we introduce a second order extrapolation to evaluate the left and right Riemann states for the upwind solver. Observing Figure 2, we use the three variables $u_0, u_2, u_3$ corresponding to barycenters $C_0, C_2, C_3$ to extrapolate the left variables $u_{01,L}$. Since the three values of three points determine a linear function in two-dimensional space, the linear function is given by

\begin{equation}
(2.6) \quad u \sim (u_2 - u_0)\lambda_2 + (u_3 - u_0)\lambda_3 + u_0
\end{equation}

where $\lambda_j$ is the barycentric coordinates such that $\lambda_j(C_k) = \delta_{jk}, k = 0,2,3$. 

3
Figure 2. Representative variable \( u_{01,L} \) at point \( M_1 \).

Therefore, the variable \( u_{0,1,L} \) is approximated by

\[
(2.7) \quad u_{01,L} \sim (u_2 - u_0) \lambda_2(M_1) + (u_3 - u_0) \lambda_3(M_1) + u_0
\]

Similarly, we can evaluate the right Riemann state \( u_{01,R} \). The signs of \( \lambda_j(M_1) \) are nonpositive for general triangulations.

2.2 The maximum principle, the local projection and TVD triangulations. From Section 2.1, we can see that the scheme can be expressed in the form

\[
(2.8) \quad \frac{d u_i}{d t} = \sum_{j=1}^{3} c_{ij} (u_{ij} - u_i)
\]

where \( u_{ij} \) are the variables of barycenters \( c_{ij} \) of the triangle \( K_{ij} \) with \( K_{ij} \cap K_i = e_{ij} \). Then we require all the coefficients to be nonnegative [5]:

\[
(2.9) \quad c_{ij} \geq 0, \quad j = 1, 2, 3.
\]

This condition on the signs of the coefficients, which is a direct generalization of the condition for a one-dimensional three point scheme to be TVD, assures that a maximum cannot increase.

From the definition of the flux \( h \), we have

\[
(2.10) \quad \sum_{j=1}^{3} h_{ij}(u_i, u_i) \cdot |e_{ij}| = 0 \quad \text{for all } i.
\]
We can write the right side of (2.5) as follows:

\[- \sum_{j=1}^{3} h_{ij}(u_{ij,L}, u_{ij,R}) \cdot |e_{ij}| = - \sum_{j=1}^{3} [h_{ij}(u_{ij,L}, u_{ij,R}) - h_{ij}(u_i, u_i)] \cdot |e_{ij}| = - \sum_{j=1}^{3} \{[h_{ij}(u_{ij,L}, u_{ij,R}) - h_{ij}(u_i, u_i)] \cdot |e_{ij}| \}
\]

\[+ [h_{ij}(u_i, u_{ij,R}) - h_{ij}(u_i, u_i)] \cdot |e_{ij}| \}
\]

\[= \sum_{j=1}^{3} -|e_{ij}| \cdot h_{ij,1} \cdot (u_{ij,L} - u_i) + \sum_{j=1}^{3} -|e_{ij}| \cdot h_{ij,2} \cdot (u_{ij,R} - u_i)
\]

where $h_{ij,1}$ is the $u$-derivative of $h_{ij}$ evaluating at some points and $h_{ij,2}$ is the $v$-derivative of $h_{ij}$ evaluating at some points. From (2.4b) we have $h_{ij,1} \geq 0$ and $h_{ij,2} \leq 0$. Now, we first study the first right term. For fixed $j = 1$, we have

\[- |e_{i1}| \cdot h_{ij,1} \cdot (u_{i1,L} - u_i)
\]

\[= - |e_{i1}| \cdot h_{i1,1} \cdot [(u_{i2} - u_i) \lambda_{i2}(M_{i1}) + (u_{i3} - u_i) \lambda_{i3}(M_{i1})]
\]

\[= - |e_{i1}| \cdot h_{i1,1} \cdot \lambda_{i2}(M_{i1}) \cdot (u_{i2} - u_i) - |e_{i1}| \cdot h_{i1,1} \cdot \lambda_{i3}(M_{i1}) \cdot (u_{i3} - u_i)
\]

Then the first condition to achieve the maximum principle is following

\[\lambda_{ij}(M_{ik}) \leq 0, \quad j, k = 1, 2, 3 \text{ and } j \neq k, \quad \text{for all } i
\]

For the second right term of eqn (2.11), if we ask that

\[\frac{u_{ij,R} - u_i}{u_{ij} - u_i} \geq 0
\]

then a sufficient condition to verify the condition (2.14a) is

\[0 \leq \frac{u_{ij,R} - u_{ij}}{u_{ij} - u_i} \leq 1
\]

Combing the conditions (2.13) and (2.14a) (or (2.14b)) [7], we prove that the semi-discretization scheme (2.5) satisfies the maximum principle. From eqn (2.13), we have to restrict ourselves to consider a special class of triangulation $T_h$ that we introduce next.
DEFINITION 2.1. A triangulation $T_h$ is said to be TVD-Triangulation if for each triangle $K_i$, the condition (2.13) is satisfied.

From the condition (2.14b), we can define the local projection $P^1_{ij}$ as follows:

\[(2.15a)\quad P^1_{ij} : R \to R\]

such that

\[(2.15b)\quad u^\text{new}_{ij,L} = P^1_{ij}(u_{ij,L}) = \min \text{mod}(u_{ij,L} - u_i, b \cdot (u_{ij} - u_i)) + u_i\]

where $b$ is some positive constant such that $b \geq 1$, and $\min \text{mod}$ is the function:

$$
\min \text{mod}(a, b) = \begin{cases} 
\min(|a|, |b|) \cdot \text{sign } a & \text{if sign } a = \text{sign } b \\
0 & \text{otherwise}
\end{cases}
$$

Similarly, we update the variable $u_{ij,R}$ by looking for the triangle $K^*$ containing the barycenter $C_{ij}$. Combining the definition 2.1 and the local projection $P^1_{ij}$, we have proven the following result:

THEOREM 2.2. Let $P^1_{ij}$ be the projections defined by (2.15) and let $T_h$ be TVD-triangulation, then the scheme (2.5) satisfies the maximum principle.

Remark 2.3. For some TVD triangulations, the projection $P^1_{ij}$ will kill second order accuracy. Here, we introduce the other projection $P^2_{ij}$. From the second right term of eqn (2.11), if the quantity $U_{ij,R} - u_i$ can be written as positive linear combination of $u_{ik} - u_i, k = 1, 2, 3$. Then the scheme (2.5) still satisfies the maximum principle. Observing Figure 3, vector $\overrightarrow{C_iM_1}$ can be expressed as positive linear combination of vectors $\overrightarrow{C_iC_{i1}}$ and $\overrightarrow{C_iC_{i2}}$ as follows

$$
\overrightarrow{C_iM_1} = \theta_1 \overrightarrow{C_iC_{i1}} + \theta_2 \overrightarrow{C_iC_{i2}}
$$

where $\theta_1, \theta_2 \geq 0$, and define

$$
U_{i1} = \theta_1 \cdot (u_{i1} - u_i) + \theta_2 \cdot (u_{i2} - u_i).
$$
Figure 3. Vector $\overrightarrow{C_iM_i}$ can be expressed as positive linear combination of vectors $\overrightarrow{C_iC_{i1}}$ and $\overrightarrow{C_iC_{i2}}$.

Then we define the second projection $P_{i1}^2$ as follows:

$$P_{i1}^2 : R \rightarrow R$$

such that

$$n_{i1,R}^{\text{new}} \equiv P_{i1}^2(u_{i1,R}) = \minmod(u_{i1,R} - u_i, b' \cdot U_i) + u_i$$

where $b' \geq 2$. In the same way, we can define the whole local projection $P_{ij}^2$. One can see that this local projection will keep second order accurate for smooth solutions. One point we need to mention is this projection depends on the structure of triangulations, but no extra condition restricts on triangulations. In this way, our TVD Triangulations actually contain B Triangulations [3].

2.3 TVD Runke-Kutta time integration. A second order TVD Runge-Kutta time integration scheme [2,10] was used to integrate the ODE (2.5). Runge-Kutta time integration is an explicit scheme which is very fast to evaluate and simple to implement.

Define a residual as:

$$(2.17) \quad R_i = \frac{-1}{A(K_i)} \sum_{j=1}^{3} h_{ij}(u_{ij,L}, u_{ij,R}) \cdot |e_{ij}| \text{ for all } i$$

The second order TVD Runge-Kutta time stepping scheme with $CFL \leq 1/4$ was then:

$$
\begin{align*}
(2.18a) \quad u_i^{(0)} &= u_i^n \\
(2.18b) \quad u_i^{(1)} &= u_i^{(0)} + \Delta t R_i^{(0)} \\
(2.18c) \quad u_i^{(2)} &= u_i^{(1)} + \Delta t R_i^{(1)} \\
(2.18d) \quad u_i^{n+1} &= 0.5(u_i^{(0)} + u_i^{(2)})
\end{align*}
$$
where

\[(2.19a) \quad CFL = \sup_{e \in \partial k, K \in T_t} \Delta t \frac{|e|}{|K|} \| f' \cdot n_{e,k} \|_{L^\infty[a_0,b_0]} \]

\[(2.19b) \quad a_0 = \inf \left\{ \inf_{(x,y) \in \Omega} u_0(x,y), \inf_{(x,y) \in \partial \Omega} r(x,y,t) \right\} \]

\[(2.19c) \quad b_0 = \sup \left\{ \sup_{(x,y,\Omega)} u_0(x,y), \sup_{(x,y) \in \partial \Omega} r(x,y,t) \right\} \]

It is not difficulty to see that the fully discretized scheme (2.18) still verifies the maximum principle and is second order accurate in space and time.


In this section we test our scheme in some examples. The used grid systems are made of rectangles such that the domain is divided by $M \times N$ rectangles then each rectangle is cut by two along the diagonal, see Figure 4. It is easy to check that this inform triangulation is TVD-Triangulation. For some information about TVD Triangulations, one can see the paper. We take the Lax-Friedrichs or Godunov flux as the flux $h_{ij}$, and compare the difference between them. It is pointed out we compute the $L^\infty$-error on the triangle by evaluating the error at the barycenter. The $L^1$-error is obtained by multiplying that value by the area of the triangle.

Example 1. In this problem, we consider the 2D version of Burger's equation with periodic boundary condition:

\[(3.1a) \quad u_t + \left( \frac{u^2}{2} \right)_x + \left( \frac{u^2}{2} \right)_y = 0, \quad \text{in} \quad (0,T) \times \Omega \]

\[(3.1b) \quad u(t=0, x, y) = \frac{1}{4} + \frac{1}{2} \sin(\pi(x + y)), \quad (x, y) \in \Omega \]

where the domain $\Omega$ is the square $(-1,1) \times (-1,1)$. We use $M \times M \times 2$ triangular grid.
At $T = 0.1$ the solution is smooth. Figure 5 shows the curve cut along the diagonal computing with Lax-Friedrichs flux. The $L^1, L^\infty$ errors and the order of the solutions are displayed on Table 1 and 2 for the schemes by using Lax-Friedrichs and Godunov flux, respectively.

<table>
<thead>
<tr>
<th>M</th>
<th>$L_1$ error</th>
<th>order</th>
<th>$L_\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4.331E-02</td>
<td></td>
<td>3.155E-02</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1.340E-02</td>
<td>1.63</td>
<td>1.133E-02</td>
<td>1.48</td>
</tr>
<tr>
<td>32</td>
<td>4.138E-03</td>
<td>1.76</td>
<td>4.263E-03</td>
<td>1.41</td>
</tr>
<tr>
<td>64</td>
<td>1.238E-03</td>
<td>1.74</td>
<td>1.973E-03</td>
<td>1.11</td>
</tr>
</tbody>
</table>

Table 1. Smooth solution for $T = 0.1$ with Lax-Friedrichs flux.
<table>
<thead>
<tr>
<th>M</th>
<th>$L_1$ error</th>
<th>order</th>
<th>$L_\infty$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4.495E-02</td>
<td>1.67</td>
<td>3.395E-02</td>
<td>1.58</td>
</tr>
<tr>
<td>16</td>
<td>1.410E-02</td>
<td>1.76</td>
<td>1.133E-02</td>
<td>1.41</td>
</tr>
<tr>
<td>32</td>
<td>4.161E-03</td>
<td>1.75</td>
<td>4.262E-03</td>
<td>1.41</td>
</tr>
<tr>
<td>64</td>
<td>1.234E-03</td>
<td>1.75</td>
<td>1.981E-04</td>
<td>1.21</td>
</tr>
</tbody>
</table>

Table 2. Smooth solution for $T = 0.1$ with Godunov flux.

Figure 5. Examples 3.1 with Lax-Friedrichs flux, $T = 0.1$, $CFL = 0.4$
$32 \times 32 \times 2$ Triangles. The solution cut along the diagonal.

At $T = 0.5$, solution presents a discontinuity curve, see Figure 6, which is cut along the diagonal computing with Lax-Friedrichs flux. We can see how the discontinuity has been captured within a single element. Similarly, errors and orders are displayed on Table 3 and 4, for the two fluxes.
Figure 6. Example 3.1 with Lax-Friedrichs flux, $T = 0.5$, $CFL = 0.4$
$32 \times 32 \times 2$ Triangles. The solution cut along the diagonal.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$L_1$ error</th>
<th>order</th>
<th>$L_{\infty}$ error</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.426E-03</td>
<td></td>
<td>7.548E-03</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>6.436E-04</td>
<td>1.15</td>
<td>2.531E-03</td>
<td>1.58</td>
</tr>
<tr>
<td>32</td>
<td>1.837E-04</td>
<td>1.81</td>
<td>1.477E-03</td>
<td>0.78</td>
</tr>
<tr>
<td>64</td>
<td>5.212E-05</td>
<td>1.82</td>
<td>6.391E-04</td>
<td>1.21</td>
</tr>
</tbody>
</table>

Table 3. Discontinuity solution for $T = 0.5$ and Lax-Friedrichs flux.
Table 4. Discontinuity solution for $T = 0.5$ with Godunov flux computing domain: $[-0.25, 0.5] \times [-0.25, 0.5]$.

When the computing solution is smooth, the performance between Lax-Friedrichs and Godunov flux are almost equal, see tables 1, 3. On the other hand, one can see the difference between them when the computing solution has discontinuity, see table 5. The scheme is based on TVD idea, so it will lose some accuracy on the critical points of the solution.

Table 5. Discontinuity solution for $T = 0.5$. Computing Domain: $[-1, 1] \times [-1, 1]$.

Example 2. In this problem we consider the initial-boundary value problem of the 2D Burger’s equation:

\begin{align*}
(3.2a) & \quad u_t + \left( \frac{u^2}{2} \right)_x + \left( \frac{u^2}{2} \right)_y = 0, \quad \text{in } (0,T) \times \Omega \\
(3.2b) & \quad u(t=0,x,y) = \begin{cases} 
-0.2, & \text{for } x > 0, y > 0 \\
-1.0, & \text{for } x < 0, y > 0 \\
0.5, & \text{for } x < 0, y < 0 \ (x,y) \in \Omega, \\
0.8, & \text{for } x > 0, y < 0 
\end{cases} \\
(3.2c) & \quad u(t,x,y) = v(t,x,y), (x,y) \in \partial \Omega
\end{align*}
where \( v \) is the exact solution of problem (3.2), and \( \Omega \) is the square \([-1, 1] \times [-1, 1]\). On Figure 7 and Figure 9, we display the level curves of the approximate solution and curves cut along \( y = -1./96 \) at \( T = 1.0 \) with Lax-Friedrichs flux. And similarly for Figure 8 and Figure 10 but with Godunov flux. Here, \( 64 \times 64 \times 2 \) triangle meshes are used. Again, we can see Godunov presents better solutions than Lax-Friedrichs.

Figure 7. Example 3.2 with Lax-Friedrichs flux, \( T = 1.0 \), \( CFL = 0.4 \) \( 64 \times 64 \times 2 \) Triangles. The level curves.
Figure 8. Example 3.2 with Godunov flux, $T = 1.0$, $CFL = 0.4$
$64 \times 64 \times 2$ Triangles. The level curves.
Figure 9. Example 3.2 with Lax-Friedrichs flux. The solutions cut along $y = 1/96$. 
Figure 10. Example 3.2 with Godunov flux. The solution cut along $y = 1/96$.

Example 3. The last problem we consider a two dimensional boundary layer problem.

\begin{align}
(3.3a) & \quad u_t + \left( \frac{u^2}{2} \right)_x + u_y = 0, \quad 0 \leq x \leq 1., \quad 0 \leq y \leq 1. \\
(3.3b) & \quad u(x, 0, t) = a + b \sin(2\pi x), \\
(3.3c) & \quad u \text{ is periodic in } x
\end{align}

We solve the problem to steady stat that $\|u^{n+1} - u^n\|_{L^1} < 10^{-1}$, with initial condition $u(x, y, 0) = a + b \sin(2\pi x)$ and periodic boundary condition in $x$. Enforce (3.3b) at $y = 0$, and imposed no boundary condition at the outflow boundary $y = 1$. Figure 11 and Figure 12 contain the level curves for $a = 0.5, b = 1$, and $a = 0, b = 1$, respectively. In this case, the scheme use Lax-Friedrichs flux with $32 \times 32 \times 2$ triangular meshes. On the second case, around points $(0.5, 0.2)$ and $(0.5, 0.4)$ in Figure 12, the numerical solution smears out more points at the right side of the shock. We do not find out which reason to produce this phenomenon, but we doubt that is triangle-mesh effect.
Figure 11. Example 3.3 with $a = 0.5$, $b = 1.0$ with Lax-Friedrichs flux, $CFL = 0.04$, $32 \times 32 \times 2$ Triangles. The level curves.
4. Conclusion.

A numerical scheme has been developed and analyzed for the finite-volume solution of scalar conservation Laws on triangular meshes. The scheme satisfies the maximum principle for general nonlinear fluxes if the triangulations are TVD triangulations. We have investigated and compared two kinds of flux, Lax-Friedrichs and Godunov fluxes. The scheme with Godunov flux is a litter better than with Lax-Friedrichs flux, but schemes with Lax-Friedrich flux seem good enough. This indicates that we can extend our scheme with Lax-Friedrichs flux to compute Two-Dimensional Euler Equations. Extensions to schemes based on TVB idea to improve the order of accuracy, as well as to two dimensional systems are ongoing works. Mesh adaptation remains to be implemented but promises to greatly increase the power of the method.
REFERENCES


<table>
<thead>
<tr>
<th>#</th>
<th>Author/s</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>637</td>
<td>Philip Korman</td>
<td>Dynamics of the Lotka–Volterra systems with diffusion</td>
</tr>
<tr>
<td>638</td>
<td>Harlan W. Stech</td>
<td>Generic Hopf bifurcation in a class of integro-differential equations</td>
</tr>
<tr>
<td>639</td>
<td>Stepanache Laederich</td>
<td>Periodic solutions of non linear differential difference equations</td>
</tr>
<tr>
<td>640</td>
<td>Peter J. Olver</td>
<td>Canonical Forms and Integrability of BiHamiltonian Systems</td>
</tr>
<tr>
<td>641</td>
<td>S.A. van Gils, M.P. Krupa and W.F. Langford</td>
<td>Hopf bifurcation with nonsemisimple 1:1 Resonance</td>
</tr>
<tr>
<td>642</td>
<td>R.D. James and D. Kinderlehrer</td>
<td>Frustration in ferromagnetic materials</td>
</tr>
<tr>
<td>643</td>
<td>Carlos Rocha</td>
<td>Properties of the attractor of a scalar parabolic P.D.E.</td>
</tr>
<tr>
<td>644</td>
<td>Debra Lewis</td>
<td>Lagrangian block diagonalization</td>
</tr>
<tr>
<td>645</td>
<td>Richard C. Churchill and David L. Rod</td>
<td>On the determination of Ziglin monodromy groups</td>
</tr>
<tr>
<td>646</td>
<td>Xinfu Chen and Avner Friedman</td>
<td>A nonlocal diffusion equation arising in terminally attached polymer chains</td>
</tr>
<tr>
<td>647</td>
<td>Peter Grittmann and Victor Klee</td>
<td>Inner and outer j- Radii of convex bodies in finite-dimensional normed spaces</td>
</tr>
<tr>
<td>648</td>
<td>P. Smolyn</td>
<td>Analysis of a singularly perturbed traveling wave problem</td>
</tr>
<tr>
<td>649</td>
<td>Stanley Reiter and Carl P. Simon</td>
<td>Decentralized dynamic processes for finding equilibrium</td>
</tr>
<tr>
<td>650</td>
<td>Fernando Reitich</td>
<td>Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions</td>
</tr>
<tr>
<td>651</td>
<td>Russell A. Johnson</td>
<td>Cantor spectrum for the quasi-periodic Schrödinger equation</td>
</tr>
<tr>
<td>652</td>
<td>Wenxiong Liu</td>
<td>Singular solutions for a convection diffusion equation with absorption</td>
</tr>
<tr>
<td>653</td>
<td>Deborah Brandon and William J. Hrusa</td>
<td>Global existence of smooth shearing motions of a nonlinear viscoelastic fluid</td>
</tr>
<tr>
<td>654</td>
<td>James F. Reineck</td>
<td>The connection matrix in Morse–Smale flows II</td>
</tr>
<tr>
<td>655</td>
<td>Claude Baesens, John Guckenheimer, Seunghwan Kim and Robert Mackay</td>
<td>Simple resonance regions of torus diffeomorphisms</td>
</tr>
<tr>
<td>656</td>
<td>Willard Miller, Jr.</td>
<td>Lecture notes in radar/sonar: Topics in Harmonic analysis with applications to radar and sonar</td>
</tr>
<tr>
<td>657</td>
<td>Calvin H. Wilcox</td>
<td>Lecture notes in radar/sonar: Sonar and Radar Echo Structure</td>
</tr>
<tr>
<td>658</td>
<td>Richard E. Blahut</td>
<td>Lecture notes in radar/sonar: Theory of remote surveillance algorithms</td>
</tr>
<tr>
<td>659</td>
<td>D.V. Anosov</td>
<td>Hilbert’s 21st problem (according to Bolibruch)</td>
</tr>
<tr>
<td>660</td>
<td>Stepanache Laederich</td>
<td>Ray–Singer torsion for complex manifolds and the adiabatic limit</td>
</tr>
<tr>
<td>661</td>
<td>Geneviève Raugel and George R. Sell</td>
<td>Navier-Stokes equations in thin 3d domains: Global regularity of solutions I</td>
</tr>
<tr>
<td>662</td>
<td>Emanuel Parzen</td>
<td>Time series, statistics, and information</td>
</tr>
<tr>
<td>663</td>
<td>Andrew Majda and Kevin Lamb</td>
<td>Simplified equations for low Mach number combustion with strong heat release</td>
</tr>
<tr>
<td>664</td>
<td>Ju. S. Il'yashenko</td>
<td>Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation</td>
</tr>
<tr>
<td>665</td>
<td>James F. Reineck</td>
<td>Continuation to gradient flows</td>
</tr>
<tr>
<td>666</td>
<td>Mohamed Sami Elbialy</td>
<td>Simultaneous binary collisions in the collinear N–body problem</td>
</tr>
<tr>
<td>667</td>
<td>John A. Jacquez and Carl P. Simon</td>
<td>Aids: The epidemiological significance of two different mean rates of partner-change</td>
</tr>
<tr>
<td>668</td>
<td>Carl P. Simon and John A. Jacquez</td>
<td>Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations</td>
</tr>
<tr>
<td>669</td>
<td>Matthew Stafford</td>
<td>Markov partitions for expanding maps of the circle</td>
</tr>
<tr>
<td>670</td>
<td>Ciprian Foias and Edriss S. Titi</td>
<td>Determining nodes, finite difference schemes and inertial manifolds</td>
</tr>
<tr>
<td>671</td>
<td>M.W. Smiley</td>
<td>Global attractors and approximate inertial manifolds for abstract dissipative equations</td>
</tr>
<tr>
<td>672</td>
<td>M.W. Smiley</td>
<td>On the existence of smooth breathers for nonlinear wave equations</td>
</tr>
<tr>
<td>673</td>
<td>Hitay Özbay and Janos Turi</td>
<td>Robust stabilization of systems governed by singular integro-differential equations</td>
</tr>
<tr>
<td>674</td>
<td>Mary Silber and Edgar Knobloch</td>
<td>Hopf bifurcation on a square lattice</td>
</tr>
<tr>
<td>675</td>
<td>Christophe Golé</td>
<td>Ghost circles for twist maps</td>
</tr>
<tr>
<td>676</td>
<td>Christophe Golé</td>
<td>Ghost tori for monotone maps</td>
</tr>
<tr>
<td>677</td>
<td>Christophe Golé and Titi</td>
<td>Monotone maps of $T^n \times R^n$ and their periodic orbits</td>
</tr>
<tr>
<td>678</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>Hypergeometric expansions of Heun polynomials</td>
</tr>
<tr>
<td>679</td>
<td>Victor A. Pliss and George R. Sell</td>
<td>Perturbations of attractors of differential equations</td>
</tr>
<tr>
<td>680</td>
<td>Avner Friedman and Peter Knabner</td>
<td>A transport model with micro- and macro-structure</td>
</tr>
<tr>
<td>681</td>
<td>E.G. Kalnins and W. Miller, Jr.</td>
<td>A note on group contractions and radar ambiguity functions</td>
</tr>
<tr>
<td>682</td>
<td>George R. Sell</td>
<td>References on dynamical systems</td>
</tr>
<tr>
<td>683</td>
<td>Shui-Nee Chow, Kening Lu and George R. Sell</td>
<td>Smoothness of inertial manifolds</td>
</tr>
<tr>
<td>684</td>
<td>Shui-Nee Chow, Xiao-Biao Lin and Kening Lu</td>
<td>Smooth invariant foliations in infinite dimensional spaces</td>
</tr>
</tbody>
</table>
Kening Lu, A Hartman–Grobman theorem for scalar reaction-diffusion equations
Christophe Golé and Glen R. Hall, Poincaré’s proof of Poincaré’s last geometric theorem
Mario Taboada, Approximate inertial manifolds for parabolic evolutionary equations via Yosida approximations
Peter Rejto and Mario Taboada, Weighted resolvent estimates for Volterra operators on unbounded intervals
Joel D. Avrin, Some examples of temperature bounds and concentration decay for a model of solid fuel combustion
Susan Friedlander and Misha M. Vishik, Lax pair formulation for the Euler equation
H. Scott Dumas, Ergodization rates for linear flow on the torus
A. Eden, A.J. Milani and B. Nicolaenko, Finite dimensional exponential attractors for semilinear wave equations with damping
A. Eden, C. Foias, B. Nicolaenko & R. Temam, Inertial sets for dissipative evolution equations
A. Eden, C. Foias, B. Nicolaenko & R. Temam, Hölder continuity for the inverse of Mañé’s projection
Michel Chipot and Charles Collins, Numerical approximations in variational problems with potential wells
Huanan Yang, Nonlinear wave analysis and convergence of MUSCL schemes
László Gerencsér and Zsuzsanna Vágó, A strong approximation theorem for estimator processes in continuous time
László Gerencsér, Multiple integrals with respect to L-mixing processes
David Kinderlehrer and Pablo Pedregal, Weak convergence of integrands and the Young measure representation
Bo Deng, Symbolic dynamics for chaotic systems
Charles Collins and Mitchell Luskin, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem
Peter Gritzmann and Victor Klee, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces
A. Ronald Gallant and George Tauchen, A nonparametric approach to nonlinear time series analysis: estimation and simulation
H.S. Dumas, J.A. Ellison and A.W. Sáenz, Axial channeling in perfect crystals, the continuum model and the method of averaging
M.A. Kaashoek and S.M. Verduyn Lunel, Characteristic matrices and spectral properties of evolutionary systems
Xinfu Chen, Generation and Propagation of interfaces in reaction diffusion systems
Avner Friedman and Bei Hu, Homogenization approach to light scattering from polymer-dispersed liquid crystal films
Yoshihisa Morita and Shuichi Jimbo, ODEs on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels
Wenxiong Liu, Blow-up behavior for semilinear heat equations: multi-dimensional case
Hi Jun Choe, Hölder continuity for solutions of certain degenerate parabolic systems
Hi Jun Choe, Regularity for certain degenerate elliptic double obstacle problems
Fernando Reitich, On the slow motion of the interface of layered solutions to the scalar Ginzburg–Landau equation
Xinfu Chen and Fernando Reitich, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling
C.C. Lim, J.M. Pimbley, C. Schmeiser and D.W. Schwendeman, Rotating waves for semiconductor inverter
W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya, Multisummability of formal power series solutions of linear ordinary differential equations
Peter J. Olver and Chehrzad Shakiban, Dissipative decomposition of partial differential equations
Clark Robinson, Homoclinic bifurcation to a transitive attractor of Lorenz type, II
Michelle Schatzman, A simple proof of convergence of the QR algorithm for normal matrices without shifts
Ian M. Anderson, Niky Kamran and Peter J. Olver, Internal, external and generalized symmetries
C. Foias and J.C. Saut, Asymptotic integration of Navier–Stokes equations with potential forces. I
Ling Ma, The convergence of semidiscrete methods for a system of reaction-diffusion equations
Adelina Georgescu, Models of asymptotic approximation
A. Makagon and H.Salehi, On bounded and harmonizable solutions on infinite order arma systems
San-Yih Lin and Yan-Shin Chin, An upwind finite-volume scheme with a triangular mesh for conservation laws
J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart, On the dynamics of fine structure
KangPing Chen and Daniel D. Joseph, Lubrication theory and long waves