THE RECIPROCALS OF SOLUTIONS OF LINEAR
ORDINARY DIFFERENTIAL EQUATIONS

BY

WILLIAM A. HARRIS, JR.

AND

YASUTAKA SIBUYA

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street S.E.
Minneapolis, Minnesota 55455
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THE RECIPROCALS OF SOLUTIONS OF
LINEAR ORDINARY DIFFERENTIAL
EQUATIONS

by

William A. Harris, Jr.*
University of Southern California
Los Angeles, CA  90089-1113

and

Yasutaka Sibuya**
University of Minnesota
Minneapolis, MN  55455

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1. INTRODUCTION. There has been recent interest in linear ordinary differential equations with polynomial coefficients since algebraic functions, generating functions with combinatorial significance, and the special functions of mathematical physics satisfy such equations. Formal power series solutions of linear ordinary differential equations with polynomial coefficients - termed D-finite - are useful since, for example, their coefficients satisfy a linear recurrence relation of fixed order and consequently can be computed fast. Efficient criteria for determining whether a power series is algebraic or D-finite is needed [10].

One useful method for determining whether a power series is algebraic is the rate of growth of the coefficients, thus the reciprocal of an algebraic function is algebraic. For D-finite power series, there is also a growth restriction on the coefficients. Indeed, if \( \sum_{n=0}^{\infty} c_n x^n \) is a formal power series with complex coefficients which satisfies any algebraic differential equation, then there exist two positive constants \( \gamma_1, \gamma_2 \) such that \( |c_n| \leq \gamma_1 n! \gamma_2 \) [3,4]. This estimate is best possible since \( \sum_{n=0}^{\infty} (n!)^k x^n \) satisfies a linear ordinary differential equation with polynomial coefficients; e.g.

\[
\sum_{n=0}^{\infty} n! x^n \quad \text{satisfies} \quad x^2 y'' + (3x - 1) y' + y = 0.
\]

Such functions are of Gevrey type and have been shown recently to have fundamental importance in the theory of linear ordinary differential equations with an irregular singular point [5,6]. Unfortunately, even though the reciprocal of a function of Gevrey type is of Gevrey type, such functions (as we shall show) are not necessarily D-finite, or not necessarily solutions of linear ordinary differential equations with analytic coefficients!

In this note we characterize the class of power series which together with their reciprocals satisfy linear ordinary differential equations.
Let $K$ be a differential field of characteristic zero. We denote by $D_K$ the ring of linear ordinary differential operators with coefficients in $K$, that is

$$D_K = \{ \sum_{k=0}^{m} a_k D^k ; a_k \in K, \ m \in \mathbb{N} \},$$

where $D$ denotes differentiation in any differential field extension of $K$, and $\mathbb{N}$ is the set of all non-negative integers.

**Theorem.** Let $\phi$ be an element of a differential field extension of $K$ such that

(i) $\phi \neq 0$;

(ii) $P(\phi) = 0$ for some $P \in D_K$ - \{0\};

(iii) $Q(1/\phi) = 0$ for some $Q \in D_K$ - \{0\}.

Then, the logarithmic derivative of $\phi$ ($= \phi'/\phi$) is algebraic over $K$.

**Corollary 1.** In case $K$ is the field of rational functions of $x$ with coefficients in $\mathbb{C}$ (the field of complex numbers), if $\phi$ satisfies the hypothesis of the Theorem, then $\phi'/\phi$ is an algebraic function of $x$.

This Corollary contains the result established by L. Carlitz [1] that reciprocal of the D-finite Bessel function

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n! \Gamma(v+n+1)},$$

a solution of $xy'' + (v + 1)y' + y = 0$ is not D-finite; i.e. the reciprocal does not satisfy any linear ordinary differential equation with polynomial coefficients.

Let $\mathcal{C}(x)$ be the ring of convergent power series in $x$ with coefficients in $\mathcal{C}$.

**Corollary 2.** In case $K$ is the quotient field of $\mathcal{C}(x)$, if $\psi$ is a formal power series in $x$ with coefficients in $\mathcal{C}$, and if $\psi$ satisfies the hypothesis of the Theorem, then $\psi \in \mathcal{C}(x)$ (i.e. $\psi$ is convergent).
The divergent power series \( \sum_{n=0}^{\infty} n!x^n \) is D-finite since it is a formal solution of \( x^2y'' + (3x - 1)y' + y = 0 \). Hence Corollary 2 implies that its reciprocal \( 1/\sum_{n=0}^{\infty} n!x^n \) is not only not D-finite, but does not satisfy any linear ordinary differential equation with analytic coefficients (i.e. with coefficients in \( \mathbb{C}(x) \)).

Clearly, if \( q(x) \in \mathbb{C}(x) \) is a convergent power series with coefficients in \( \mathbb{C} \), then \( q(x) \) satisfies a linear ordinary differential equation with analytic coefficients: \( qy' - q'y = 0 \). Also, if \( q(x) \) is a formal power series in \( x \) with coefficients in \( \mathbb{C} \) and satisfies a linear ordinary differential equation with analytic coefficients, then \( q(-x) \) also satisfies a linear ordinary differential equation with analytic coefficients - obtained by replacing \( x \) by \( -x \) in the coefficients and \( \frac{d}{dx} \) by \( -\frac{d}{dx} \).

For a formal power series \( p \) in \( x \) with coefficients in \( \mathbb{C} \) let us set

\[
\psi(x) = p(-x)/p(x).
\]

Thus, \( \psi(x)\psi(-x) = 1 \), and if \( \psi(x) \) satisfies a linear ordinary differential equation with analytic coefficients then so does \( \psi(-x) \). Thus, the divergent power series in \( x \)

\[
f(x) = \sum_{n=0}^{\infty} \frac{n!(-x)^n}{\sum_{n=0}^{\infty} n!x^n}
\]

is such that (from Corollary 2) neither \( f(x) \) nor its reciprocal \( 1/f(x) \) (= \( f(-x) \)) satisfies any linear ordinary differential equation with analytic coefficients.

Corollary 2 can be generalized as follows. Let \( \mathbb{C}[[x]] \) be the ring of formal power series in \( x \) with coefficients in \( \mathbb{C} \), and let \( \mathbb{C}(x) \) be the quotient field of \( \mathbb{C}[[x]] \).
COROLLARY 2'. In case $K$ is the quotient field of $\mathbb{C}(x)$, if

(i) $\phi$ and $\psi \in \mathbb{C}(x)$ - \{0\},

(ii) $P(\phi) = 0$ for some $P \in D_K$ - \{0\}, and

$Q(\psi) = 0$ for some $Q \in D_K$ - \{0\},

(iii) $\phi, \psi \in K$ ,

then, $\phi$ and $\psi \in K$ .

If we set $v = \phi'/\phi$, we can derive two algebraic (non-linear) ordinary differential equations

$$\begin{align*}
F(v,v', ..., v(n)) &= 0 , \\
G(v,v', ..., v(m)) &= 0 ,
\end{align*}$$

(1.1)

from $P(\phi) = 0$ and $Q(1/\phi) = 0$ respectively where $F$ and $G$ are polynomials in $v,v', ..., v^{(n)}$ with coefficients in $K$. If we could derive a non-trivial algebraic equation $H(v) = 0$ with coefficients in $K$ by eliminating all derivatives in $v$ from equations (1.1), then the proof of the Theorem would be completed. L. Carlitz used this idea implicitly in his case where $F(v,v')$ is a simple Riccati equation. In the general case, the complexity of the computations as the degrees of $F$ and $G$ increased led us to use other ideas in our proof. (For differential algebra and elimination theory see, e.g. J. Denef and L. Lipshitz [2], J.F. Ritt [7], and A. Seidenberg [9].)

The authors wish to thank Steven Sperber for stimulating discussions, William Lang for valuable suggestions regarding Section 2, and Dennis Stanton for bringing to our attention a very valuable paper of R.P. Stanley [10].

2. LEMMAS ON ALGEBRAIC EQUATIONS. Let $k$ be a field. We denote by $k[y_1, ..., y_p]$ the ring of polynomials in $p$ indeterminates $y_1, ..., y_p$ with coefficients in $k$. For $F = \sum_{r_1 ... r_p} \alpha_{r_1 ... r_p} y_1^{r_1} ... y_p^{r_p} \in k[y_1, ..., y_p]$, where $\alpha_{r_1 ... r_p} \in k$, we set
\begin{align}
(2.1) \quad w(F) &= \max \left\{ \sum_{j=1}^{p} r_j r_j ; \alpha_1 \ldots \alpha_p \neq 0 \right\}.
\end{align}

Then, for \( F_1 \) and \( F_2 \in k[y_1, \ldots, y_p] \), we have
\[
w(F_1 + F_2) < \max(w(F_1), w(F_2)),
\]
and
\[
w(F_1 F_2) = w(F_1) + w(F_2).
\]

In case \( w(F_1) < w(F_2) \), we have
\[
w(F_1 + F_2) = w(F_2).
\]

If we regard \( F(\alpha_1 t, \alpha_2 t^2, \ldots, \alpha_p t^p) \) as a polynomial in \( t \) whose coefficients are polynomials in \( \alpha_1, \ldots, \alpha_p \), then
\[
w(F) = \deg_t F(\alpha_1 t, \ldots, \alpha_p t^p).
\]

**Lemma 2.1.** Let \( n \) and \( p \) be positive integers, and let \( F \) and \( G \) be polynomials in \( y_1, \ldots, y_p \) with coefficients in \( k \) (i.e. \( F, G \in k[y_1, \ldots, y_p] \)).

Assume that
\begin{enumerate}
  \item \( p > 2 \);
  \item \( w(F - y_p^n) < np \);
  \item \( w(G) = nq \), where \( q \) is a positive integer such that \( 1 < q < p-1 \);
  \item \( w(G(y_1, \ldots, y_q, 0, \ldots, 0) - y_q^n) < nq \).
\end{enumerate}

Let \( H(y_1, \ldots, y_{p-1}) \) be the resultant of \( F \) and \( G \) with respect to \( y_p \).

Then
\begin{enumerate}
  \item \( w(H) = n^2q \);
  \item \( w(H(y_1, \ldots, y_q, 0, \ldots, 0) - y_q^{n^2}) < n^2q. \)
\end{enumerate}
Proof. Set

\[
\begin{align*}
F &= y^n_p + \sum_{\ell=0}^{n-1} F_{\ell}(y_1, \ldots, y_{p-1})y_p^\ell, \\
G &= \sum_{\ell=0}^{m} G_{\ell}(y_1, \ldots, y_{p-1})y_p^\ell,
\end{align*}
\]

(2.2)

where \( F_{\ell}, G_{\ell} \in k[y_1, \ldots, y_{p-1}] \) and \( G_m \neq 0 \). Note that assumption (ii) implies \( \deg_y (F - y^n_p) < n \) and that assumption (iii) implies \( m = \deg_y G < \frac{nq}{p} \). We also have

(2.3) \( G_0(y_1, \ldots, y_{p-1}) = G(y_1, \ldots, y_{p-1}, 0) \).

Furthermore, assumptions (ii) and (iii) imply that

\[
\begin{align*}
w(F_{\ell}) &< np - \ell p = (n-\ell)p \quad (\ell = 0, 1, \ldots, n-1), \\
w(G_{\ell}) &< nq - \ell p \quad (\ell = 0, 1, \ldots, m).
\end{align*}
\]

(2.4)

It also follows from assumption (iv) that

(2.5) \( w(G_0) = nq \).

Let \( A_{jk} \in k[y_1, \ldots, y_{p-1}] \) \( (j, k = 1, \ldots, m+n) \) be given by

(2.6.1) \( A_{jk} = \begin{cases} 1 & (k = j), \\
F_{n-\ell} & (k = j + \ell), \quad \ell = 1, \ldots, n. \\
end{cases} 
\)

for \( j = 1, \ldots, m \);

(2.6.2) \( A_{jk} = G_{m-\ell} \) \( (k = j - (m-\ell)), \quad \ell = 0, 1, \ldots, m, 
\)

for \( j = m + 1, \ldots, m+n \);

(2.6.3) \( A_{jk} = 0 \) otherwise.

Then

(2.7) \( H(y_1, \ldots, y_{p-1}) = \det(A_{jk} ; j, k = 1, \ldots, m+n) \).
Note that

\[ A_{jj} = \begin{cases} 1 & j = 1, \ldots, m, \\ G_0 & j = m+1, \ldots, m+n. \end{cases} \]

If \( A_{jk} \neq 0 \) and \( j \neq k \), then

\[ A_{jk} = \begin{cases} F_{n-(k-j)} & \text{if } j < k, \\ G_{j-k} & \text{if } j > k. \end{cases} \]

Hence, if \( A_{jk} \neq 0 \) and \( j \neq k \), we have

\[ w(A_{jk}) \begin{cases} < (k-j)p & \text{if } j < k, \\ < nq + (k-j)p & \text{if } j > k. \end{cases} \]

(2.8)

Let \((k_1, \ldots, k_{m+n})\) be a permutation of \((1, \ldots, m+n)\) such that

\[ A_{1k_1} A_{2k_2} \cdots A_{m+n,k_{m+n}} \neq 0. \]

Then, it follows from (2.8) that, since at most \( n \) factors in this product can satisfy \( j > k \), we have

(2.9)

\[ w(A_{1k_1} A_{2k_2} \cdots A_{m+n,k_{m+n}}) < n^2q \]

if \((k_1, \ldots, k_{m+n}) \neq (1, \ldots, m+n)\). Since \( A_{11} A_{22} \cdots A_{m+n,m+n} = G_0^n \), we have

(2.10)

\[ w(H - G_0^n) < n^2q. \]

Hence the estimates (2.5) and (2.10) imply \( w(H) = n^2q \). Furthermore, (2.10) implies

\[ w(H(y_1, \ldots, y_q, 0, \ldots, 0) - G_0(y_1, \ldots, y_q, 0, \ldots, 0)^n) < n^2q. \]

We also derive

\[ w(G_0(y_1, \ldots, y_q, 0, \ldots, 0)^n - y_q^{n^2}) < n^2q. \]
from assumption (iv). Hence, we have

\[ w(H(y_1,\ldots, y_q, 0,\ldots, 0) - y_q^{n^2}) < n^2 q. \]

q.e.d.

**LEMMA 2.2.** Let \( n \) and \( p \) be positive integers, and let \( F_1,\ldots, F_p \in k[y_1,\ldots, y_p] \).
Assume that

(i) \[ w(F_p - y_p^n) < np ; \]

(ii) \[ w(F_j) = nj \quad (j=1,\ldots, p-1); \]

(iii) \[ w(F_j(y_1,\ldots, y_j, 0,\ldots, 0) - y_j^{n^2}) < nj \quad (j=1,\ldots, p-1). \]

Then, the system of algebraic equations

\[ (2.11) \quad F_j(y_1,\ldots, y_p) = 0 \quad (j = 1,\ldots, p) \]

admits only a finite number of solutions in any extension field of \( k \), and these solutions are algebraic over \( k \).

Proof by induction on \( p \).

If \( p = 1 \), we have a non-trivial algebraic equation in one unknown. Hence, equation (2.11) has only a finite number of solutions which are algebraic over \( k \). Therefore, assuming \( p > 2 \), we shall eliminate \( y_p \) from the system of equations (2.11). To do this, let \( H_j(y_1,\ldots, y_{p-1}) \) be the resultant of \( F_p \) and \( F_j \) with respect to \( y_p \), where \( j = 1,\ldots, p - 1 \). Then, Lemma 2.1 implies that

\[ w(H_{p-1} - y_{p-1}^{n^2}) < n^2(p-1) ; \]

\[ w(H_j) = n^2 j \quad (j=1,\ldots, p-2) ; \]

\[ w(H_j(y_1,\ldots, y_j, 0,\ldots, 0) - y_j^{n^2}) < n^2 j \quad (j=1,\ldots, p-2). \]

By the induction assumption, there are only a finite number of solutions, which
are algebraic over \( k \), to the system of equations:

\[
H_j(y_1, \ldots, y_{p-1}) = 0 \quad (j = 1, \ldots, p-1).
\]

Hence, by utilizing the non-trivial equation in \( y_p \):

\[
F_p(y_1, \ldots, y_p) = 0,
\]

we can complete the proof of Lemma 2.2. \( \text{q.e.d.} \)

3. **A LEMMA ON DIFFERENTIAL EQUATIONS.** Let \( k \) be a field of characteristic zero, and let \( k[[x]] \) be the ring of formal power series in \( x \) with coefficients in \( k \). Then, a linear ordinary differential equation

\[
y^{(m+1)} + \sum_{\ell=0}^{m} a_{\ell}(x)y^{(\ell)} = 0 \quad (a_{\ell} \in k[[x]])
\]

admits a canonical basis of \( m+1 \) solutions of the form:

\[
f_j = \frac{1}{j!} x^j + \sum_{\ell=m+1}^{\infty} f_{j,\ell} x^\ell \quad (j = 0, 1, \ldots, m),
\]

where \( f_{j,\ell} \in k \).

Let \( \bar{k} \) be a field extension of \( k \). If a formal power series

\[
\phi = \sum_{\ell=0}^{\infty} c_{\ell} x^\ell
\]

with coefficients in \( \bar{k} \) satisfies (3.1), then we have

\[
\phi = \sum_{j=0}^{m} (j!)c_j f_j.
\]

Let us consider another linear ordinary differential equation:

\[
u^{(n+1)} + \sum_{\ell=0}^{n} b_{\ell}(x)u^{(\ell)} = 0 \quad (b_{\ell} \in k[[x]])
\]

This equation also has a canonical basis of \( n+1 \) solutions of the form:

\[
g_i = \frac{1}{i!} x^i + \sum_{\ell=n+1}^{\infty} g_{i,\ell} x^\ell \quad (i = 0, 1, \ldots, n),
\]

where \( g_{i,\ell} \in k \). Let us assume that

\[1 < m < n.\]
LEMMA 3.1. There are at most a finite number of solutions of the form

\[ y = f_0 + \sum_{j=1}^m \alpha_j f_j \quad (\alpha_j \in \mathbb{K}) \]  

of the linear differential equation (3.1) such that

\[ u = \frac{1}{y} \]

satisfies the linear differential system (3.5). For these solutions, the coefficients \( \alpha_1, \ldots, \alpha_m \) are algebraic over \( \mathbb{K} \).

**Proof.** Suppose that, for some \( \alpha_1, \ldots, \alpha_m \in \mathbb{K} \), (3.8) satisfies (3.5). Since \( y(0) = 1 \) for (3.7), we have

\[ \frac{1}{y} = g_0 + \sum_{j=1}^n \beta_j g_j \]

for some \( \beta_1, \ldots, \beta_n \in \mathbb{K} \). Set

\[
\begin{aligned}
&f_0 + \sum_{j=1}^m \alpha_j f_j = \sum_{\ell=0}^{\infty} \lambda_\ell x^\ell \quad (\lambda_\ell \in \mathbb{K}) \\
g_0 + \sum_{j=1}^n \beta_j g_j = \sum_{\ell=0}^{\infty} \mu_\ell x^\ell \quad (\mu_\ell \in \mathbb{K}).
\end{aligned}
\]

Since

\[ (f_0 + \sum_{j=1}^m \alpha_j f_j)(g_0 + \sum_{j=1}^n \beta_j g_j) = 1, \]

we have

\[ \sum_{\ell=0}^{i} \lambda_i - \ell \mu_\ell = 0, \quad i > 1. \]

Note that equations (3.2), (3.6) and (3.10) imply

\[ \begin{cases} \\
\lambda_0 = 1, \quad \mu_0 = 1, \\
\lambda_j = \frac{1}{j!} \alpha_j \quad (j = 1, \ldots, m), \\
\mu_j = \frac{1}{j!} \beta_j \quad (j = 1, \ldots, n), \\
\lambda_\ell = f_{0, \ell} + \sum_{j=1}^m (j!) f_{j, \ell} \lambda_j \quad (\ell > m+1), \\
\mu_\ell = g_{0, \ell} + \sum_{j=1}^n (j!) g_{j, \ell} \mu_j \quad (\ell > n+1).
\end{cases} \]
Equation (3.12) represents an infinite set of equations for the desired solutions (3.7). The utilization of equations (3.13) allows us to consider these equations in terms of $\lambda_j$, $j=1,\ldots, m$ and $\mu_i$, $i=1,\ldots, n$. Any finite subset represents necessary conditions. We consider the $n+m$ subset of equation (3.12) for $i = 1, 2, \ldots, m+n$, i.e.

\[
\begin{align*}
\sum_{\ell=0}^{i} \lambda_{i-\ell} \mu_{\ell} &= 0 \quad (1 < i < m), \\
\sum_{\ell=0}^{i-m-1} \lambda_{i-\ell} \mu_{\ell} + \sum_{\ell=i-m}^{i} \lambda_{i-\ell} \mu_{\ell} &= 0 \quad (m+1 < i < n), \\
\sum_{\ell=0}^{i-m-1} \lambda_{i-\ell} \mu_{\ell} + \sum_{\ell=i-m}^{i} \lambda_{i-\ell} \mu_{\ell} + \sum_{h=n+1}^{i} \lambda_{i-h} \mu_{h} &= 0 \quad (n+1 < i < n+m).
\end{align*}
\]

By utilizing (3.13), we write (3.14) as

\[
\sum_{\ell=0}^{n} a_{i, \ell} \mu_{\ell} = 0 \quad (i=1,\ldots, m+n),
\]

where

\[
a_{i, \ell} = \begin{cases} 
\lambda_{i-\ell} & (0 < \ell < i), \\
0 & (\ell > i),
\end{cases}
\]

for $1 < i < m$;

\[
a_{i, \ell} = \begin{cases} 
f_{0, i-\ell} + \sum_{j=1}^{m} (j!)f_{j, i-\ell} \lambda_{j} & (0 < \ell < i - m - 1), \\
\lambda_{i-\ell} & (i-m < \ell < i), \\
0 & (i+1 < \ell < n).
\end{cases}
\]
for \( m+1 < i < n \):

\[
\begin{cases}
  f_{0,i-\ell} + \sum_{j=1}^{m} (j!)f_{j,i-\ell} \lambda_j + \sum_{h=n+1}^{i} (\ell!)g_{\ell,h} \lambda_{i-h} & (0 < \ell < i-m-1),
  \\
  \lambda_{i-\ell} + \sum_{h=n+1}^{i} (\ell!)g_{\ell,h} \lambda_{i-h} & (i-m < \ell < n),
\end{cases}
\]

(3.18) \( a_{i,\ell} \) = \[
\begin{cases}
  f_{0,i-\ell} + \sum_{j=1}^{m} (j!)f_{j,i-\ell} \lambda_j + \sum_{h=n+1}^{i} (\ell!)g_{\ell,h} \lambda_{i-h} & (0 < \ell < i-m-1),
  \\
  \lambda_{i-\ell} + \sum_{h=n+1}^{i} (\ell!)g_{\ell,h} \lambda_{i-h} & (i-m < \ell < n),
\end{cases}
\]

for \( n+1 < i < n+m \).

These formulas (3.16), (3.17) and (3.18) give the quantities \( a_{i,\ell} \) as (linear) polynomials in \( \lambda_1, \ldots, \lambda_m \) with coefficients in \( k \) (i.e. \( a_{i,\ell} \in k[\lambda_1, \ldots, \lambda_m] \)).

Let us define \( w(F) \) for \( F \in k[\lambda_1, \ldots, \lambda_m] \) in the same manner as in Section 2, i.e.

\[
w(\sum a_{r_1, \ldots, r_m} \lambda_1^{r_1} \cdots \lambda_m^{r_m}) = \max \{ \sum_{j=1}^{m} j r_j ; a_{r_1, \ldots, r_m} \neq 0 \}.
\]

Then,

(3.19) \[ w(a_{i,\ell}) = i - \ell \]

in the following three cases:

(3.20-1) \[ 0 < \ell < i, \quad 1 < i < m; \]

(3.20-2) \[ i-m < \ell < i, \quad m+1 < i < n; \]

(3.20-3) \[ i-m < \ell < n, \quad n+1 < i < n+m. \]

Otherwise,

(3.21) \[ w(a_{i,\ell}) < i - \ell. \]

Let us consider the following \( m \) determinants:

(3.22-1) \[ A_i(\lambda_1, \ldots, \lambda_m) = \begin{vmatrix} a_{i,0} & \cdots & a_{i,n} \\ \cdots & \cdots & \cdots \\ a_{i+n,0} & \cdots & a_{i+n,n} \end{vmatrix}, \quad (i=1, \ldots, m). \]
Then, (3.19) and (3.21) imply that

\[(3.23) \quad w(A_i) = (n+1)i \quad (i=1,\ldots, m).\]

Precisely speaking, if we disregard terms in \(a_{i,\ell}\) with \(w < i-\ell\), the determinant \(A_i\) reduces to

\[
\begin{vmatrix}
\lambda_i & \lambda_{i-1} & \ldots & \lambda_1 & 0 \\
\lambda_{i+1} & \lambda_i & \lambda_{i-1} & \ldots & \lambda_1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\lambda_m & 0 & \ldots & \ldots & \lambda_{i+1} \\
0 & \lambda_m & \ldots & \ldots & \lambda_i
\end{vmatrix}
\]

This proves that

\[(3.24) \quad w(A_i(\lambda_1,\ldots, \lambda_i, 0,\ldots, 0) - \lambda_i^{n+1}) < (n+1)i \quad (i=1,\ldots, m).\]

The quantities \(\lambda_1, \ldots, \lambda_m\) satisfy the system of equations with \(\mu_0 = 1\).

Hence, we must have

\[(3.25) \quad A_i(\lambda_1,\ldots, \lambda_m) = 0 \quad (i=1,\ldots, m).\]

Therefore, we can complete the proof of Lemma 3.1 by using Lemma 2.2.

q.e.d.

4. PROOF OF THEOREM. Utilizing the notation of Section 1, set

\[(4.1) \quad P = \sum_{\ell=0}^{m+1} p_{\ell} D^{\ell}, \quad Q = \sum_{\ell=0}^{n+1} q_{\ell} D^{\ell},\]

where \(m\) and \(n \in \mathbb{N}\); \(p_{\ell}\) and \(q_{\ell} \in \mathbb{K}\) ; in particular

\[(4.2) \quad p_{m+1} \neq 0, \quad q_{n+1} \neq 0.\]
For an element \( f \) of \( K \), let us set
\[
\hat{f} = \sum_{\ell=0}^{\infty} \frac{f^{(\ell)}}{\ell!} x^\ell \in K[[x]].
\]

Then, \( T(f) = \hat{f} \) defines an injective homomorphism of rings:
\[
T : K \rightarrow K[[x]]
\]
such that
\[
T(f') = \frac{d}{dx} T(f) \quad ( = \frac{df}{dx}).
\]

(For similar ideas, see, for example, P. Robba [8; pp. 6-7].)

Corresponding to two operators \( P \) and \( Q \) of (4.1), let us consider two operators:
\[
\hat{P} = \sum_{\ell=0}^{m+1} \hat{P}_\ell \left( \frac{d}{dx} \right)^\ell \quad \text{and} \quad \hat{Q} = \sum_{\ell=0}^{n+1} \hat{Q}_\ell \left( \frac{d}{dx} \right)^\ell.
\]

We assumed that \( \phi \) is an element of a differential field extension of \( K \).

Denote this extension by \( \hat{K} \). Then, \( P(\phi) = 0 \) and \( Q(1/\phi) = 0 \) imply, respectively, that the formal power series
\[
\hat{\phi} = \sum_{\ell=0}^{\infty} \frac{\phi^{(\ell)}}{\ell!} x^\ell \in \hat{K}[[x]]
\]
satisfies \( \hat{P}(\hat{\phi}) = 0 \) and \( \hat{Q}(1/\hat{\phi}) = 0 \).

Note that (4.2) implies \( 1/\hat{P}_{m+1} \in K[[x]] \) and \( 1/\hat{Q}_{n+1} \in K[[x]] \). Therefore
\[
\begin{equation}
(4.3) \quad y = \frac{\hat{\phi}}{\phi} = 1 + \sum_{\ell=1}^{\infty} \frac{\phi^{(\ell)}}{\ell! \phi} x^\ell
\end{equation}
\]
satisfies the differential equation
\[
y(m+1) + \sum_{\ell=0}^{m} \frac{\hat{P}_\ell}{\hat{P}_{m+1}} y(\ell) = 0
\]
and \( u = \phi/\hat{\phi} \) satisfies the differential equation
\[ u^{(n+1)} + \sum_{k=0}^{n} \frac{\hat{q}_k}{q_{n+1}} u^{(k)} = 0. \]

Hence, Lemma 3.1 implies that all coefficients of \((4.3)\) are algebraic over \(K\). This completes the proof of the Theorem in case \(m > 1\) and \(n > 1\). If \(m = 0\), we have \(\phi' / \phi = -p_0 / p_1 \in K\). If \(n = 0\), we have \(\phi' / \phi = q_0 / q_1 \in K\). q.e.d.

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