PERIOD 3 BIFURCATION FOR THE LOGISTIC MAPPING

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In the context of continuous mappings of the interval, one of the most striking features may be Sharkovsky's theorem [6] which, among other things, shows that the existence of a period 3 point implies the existence of periodic points of every period (see also [2], [5]). Therefore, for a one-parameter family of interval mappings, the determination of period 3 bifurcation points may be interesting. In recent years, the logistic mapping $f_\alpha(x) = 1 - \alpha x^2$ has been extensively studied ([1], [4]). By using computer simulation for this family $f_\alpha(x)$, as the parameter α is increased from 0, we can observe the Feigenbaum "cascades" [3]. That is, stable periodic points of double periods accumulate in a geometric and universal way. As the parameter α is approximately equal to 1.7498 ([1], p. 129), there seems to be a period 3 bifurcation. In this note, we show that this family $f_\alpha(x)$ does have a period 3 bifurcation exactly at $\alpha = 7/4$.

<u>Theorem</u> Let $f_\alpha(x) = 1 - \alpha x^2$. Then

(a) when $\alpha < 7/4$, $f_\alpha(x)$ has no period 3 points.

(b) when $\alpha = 7/4$, $f_\alpha(x)$ has 3 period 3 points, each with multiplicity 2.

(c) when $\alpha > 7/4$, $f_\alpha(x)$ has 6 period 3 points, each with multiplicity 1.

The proof of this theorem will be based on a series of lemmas. Let $f_\alpha(x) = 1 - \alpha x^2$ and let $g(\alpha, x) = f_\alpha^3(x) - x$. Then, by direct computation, we have $g(\alpha, x) = p_\alpha(x) h(\alpha, x)$, where

$$p_\alpha(x) = 1 - x - \alpha x^2$$

and
$$\begin{aligned} h(\alpha, x) &= -\alpha^3 [p_\alpha(x)]^3 - 4\alpha^3 x [p_\alpha(x)]^2 + (2\alpha^2 - 6\alpha^3 x^2) p_\alpha(x) - 4\alpha^3 x^3 \\ &\quad + \alpha^2 x^2 + (4\alpha^2 - \alpha)x + 1 - \alpha \\ &= \alpha^6 x^6 - \alpha^5 x^5 + (\alpha^4 - 3\alpha^5)x^4 + (2\alpha^4 - \alpha^3)x^3 + (3\alpha^4 - 3\alpha^3 + \alpha^2)x^2 \\ &\quad + (-\alpha^3 + 2\alpha^2 - \alpha)x - \alpha^3 + 2\alpha^2 - \alpha + 1 \ . \end{aligned}$$

Note that $h(0, x) \equiv 1$.

Lemma 1 If α_0, x_0 satisfy $1 - x_0 - \alpha_0 x_0^2 = 0$ or $1 + x_0 - \alpha_0 x_0^2 = 0$, then $h(\alpha_0, x_0) \neq 0$.

Lemma 2 $h(\alpha_0, x_0) = 0$ if and only if x_0 is a periodic point of $f_{\alpha_0}(x)$ with minimal period 3.

Lemma 3 If $h(\alpha_0, x_0) = 0$, then all zeros of $h(\alpha_0, x)$, as a polynomial in x, are real.

Proof. This follows from Lemma 2 and the fact that any odd degree real polynomial must have at least one real zero.

Lemma 4 Let α be a fixed number satisfying $\alpha \geqslant 3/4$ and $-\alpha^3 + 2\alpha^2 - \alpha + 1 < 0$. Then $h(\alpha, x)$, as a polynomial in x, has 3 positive zeros and 3 negative zeros.

Proof. Under the assumptions on α, there are 3 changes in signs of the coefficients of $h(\alpha, x)$ and 3 changes in signs of the coefficients of $h(\alpha, -x)$. By Descarte's law and Lemma 3, we have the desired result.

Lemma 5 $h(\alpha_0, x_0) = 0$ and $\frac{\partial}{\partial x} h(\alpha_0, x_0) = 0$ for some α_0, x_0 if and only if $\alpha_0 = 7/4$ and $343x_0^3 - 98x_0^2 - 252x_0 + 8 = 0$. In particular,
$h(7/4, x) = (1/64)^2[343x^3 - 98x^2 - 252x + 8]^2$.

Proof. Let us fix $\alpha = \alpha_0$ and consider $h(\alpha_0, x)$ as a polynomial in x. Since $h(\alpha_0, x_0) = 0$ and $\frac{\partial}{\partial x} h(\alpha_0, x_0) = 0$, the multiplicity of x_0 is at least 2. By Lemma 2, $h(\alpha_0, x)$ must be a complete square. Let $h(\alpha_0, x) = (ax^3 + bx^2 + cx + d)^2$. We may assume $a > 0$. By comparing the coefficients of x^6, x^5, x^4, and x^3 of both sides respectively, we obtain that $a = \alpha^3$, $b = -\alpha^2/2$, $c = -\frac{3}{2}\alpha^2 + \frac{3}{8}\alpha$, and $d = \frac{\alpha}{4} - \frac{5}{16}$. By comparing the coefficients of x^2, we obtain that $\frac{1}{64}(48\alpha^4 - 104\alpha^3 + 35\alpha^2) = 0$. So $\alpha = 0$, $5/12$, or $7/4$. By checking with the coefficients of x and constant terms of both sides, we obtain that $\alpha = 7/4$ is the only solution and $h(7/4, x) = (1/64)^2[343x^3 - 98x^2 - 252x + 8]^2$. The converse is trivial.

Proof of the theorem. By Lemma 4, $h(2,x) = 0$ has 6 real solutions. By Lemma 5 and Implicit Function Theorem, $h(\alpha,x) = 0$ has 6 real solutions (each with multiplicity 1) for all $\alpha > 7/4$. This proves (c). (b) follows from Lemma 5, and (a) follows from Lemma 5 and Implicit Function Theorem since $h(0,x)$ has no real zeros.

Open Problem. With reasonable assumptions about the coefficients, every period 3 bifurcation point of a one-parameter family of polynomial mappings will satisfy a polynomial equation. Therefore, these points are algebraic numbers. It would be interesting to characterize the class of all one-parameter families of polynomial mappings whose period 3 bifurcation points are rational. However, this class will not contain all polynomial mappings as shown by the family $g_\alpha(x) = \alpha x + x^2$ whose period 3 points bifurcate at $\alpha = \pm 2\sqrt{2} + 1$.

References

[1] P. Collet and J.-P. Eckmann, Iterated maps on the interval as dynamical systems. Progress in Physics, Vol. 1, Birkhauser: Boston, 1980.

[2] W.A. Coppel, The solution of equations by iteration. Proc. Camb. Phil. Soc., 51 (1955), 41–43.

[3] M.J. Feigenbaum, Quantitative universality for a class of nonlinear transformations. J. Stat. Phys. 19 (1978), 25-52; 21 (1979), 669-706.

[4] J. Guckenheimer, On the bifurcation of maps of the interval, Inventiones Mathematicae, 39 (1977), 165-178

[5] T. Li and J.A. Yorke, Period three implies chaos, Amer. Math. Monthly, 82 (1975), 985-992.

[6] A.N. Sharkovsky, Coexistence of cycles of a continuous map of the line into itself, Ukr. Mat. Z., 16 (1964), 61-71.