The Floquet exponent for two-dimensional linear systems with bounded coefficients

by

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1. Introduction

Our purpose in this paper is to define and study the Floquet exponent for the two-dimensional linear differential equation with bounded coefficients

$$(1) \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' \equiv u' = \begin{pmatrix} \tilde{a}(t) & \tilde{b}(t) + \tilde{c}(t) \\ -\tilde{b}(t) + \tilde{c}(t) & -\tilde{a}(t) \end{pmatrix} u \equiv \tilde{A}(t)u \; .$$

The Floquet exponent is defined using ergodic theory. If $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ are real, it can be described as follows: it is a complex number $w = \beta + i\alpha$ such that: (i) $\alpha = -\lim_{t\to\infty} \frac{\arg u(t)}{t}$ measures rotation of solutions $u(t) \neq 0$ in the u-plane; (ii) $\beta = \lim_{t\to\infty} \frac{1}{t} \ln \|u(t)\|$ measures exponential growth of solutions. Thus α is a rotation number (Johnson-Moser [17]), and β is a Lyapunov characteristic number. (Incidentally, this explains why we take $\operatorname{tr} \tilde{A}(t) \equiv 0$. If we added a term $\begin{pmatrix} \tau(t) & 0 \\ 0 & \tau(t) \end{pmatrix}$ to $\tilde{A}(t)$, then α would not change, and β would change in an easily understandable way determined by the asymptotic behaviour of $\frac{1}{t} \int_0^t \tau(s)ds$).

Our treatment of w is motivated by the discussion of w in [17] for the almost periodic Schrödinger equation, in system form

$$u' = \begin{pmatrix} 0 & 1 \\ -\lambda + q(t) & 0 \end{pmatrix} u \; , \quad \lambda \in \mathbb{R} \; , \; q \;\; \text{real}.$$

See also the beautiful Fermi lectures of Moser [19] ; in fact we follow [19] in calling w the Floquet exponent. In [17] , a fundamental property of $w = w(\lambda)$ is that it admits a holomorphic extension (also called $w(\lambda)$) into the upper-half λ-plane. Thus if $\lambda_0 \in \mathbb{R}$, then $w(\lambda_0) = \lim_{\varepsilon\to 0+} w(\lambda_0 + i\varepsilon)$. It turns out that the Floquet exponent for (1) has an analogous property. In fact let us introduce a parameter λ in (1), as follows:

$$(2) \qquad u' = [\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} + \tilde{A}(t)]u \; .$$

We will obtain a function $w(\lambda)$ which, for real $\tilde{A}$, admits a holomorphic extension to $\operatorname{Im} \lambda > 0$. (If $\tilde{A}$ is complex, then w is defined for $|\mathrm{Im}\,\lambda| > \|\tilde{A}\|_\infty = \|a\|_\infty + \|b\|_\infty + \|c\|_\infty)$.

We will give a definition of this holomorphic extension which has the advantage of generalizing in a satisfying way to Hamiltonian systems of dimension > 2 (to be discussed elsewhere). Here is the basic idea. Suppose $\tilde{A}$ is real, and introduce polar coordinates $r = \|u\|$, $\theta = \tan^{-1} \frac{u_2}{u_1} = \arg u$. We obtain

$$(1)_r \quad r'/r = \tilde{a}(t) \cos 2\theta + \tilde{c}(t) \sin 2\theta$$

$$(1)_\theta \quad \theta' = -\tilde{b}(t) - \tilde{a}(t) \sin 2\theta + \tilde{c}(t) \cos 2\theta .$$

For fixed t, view the function $r'/r - i\theta'$ as a function of the line ℓ in $\mathbb{R}^2$ parameterized by θ . Thus $r'/r - i\theta'$ is a function on $\mathbb{P}^1(\mathbb{R})$ = = set of lines through the origin in $\mathbb{R}^2$. Now $\mathbb{P}^1(\mathbb{R})$ is a subset of $\mathbb{P}^1(\mathbb{C})$, the set of complex lines in $\mathbb{C}^2$. In fact $\mathbb{P}^1(\mathbb{C})$ is a 2-sphere (the Riemann sphere), and $\mathbb{P}^1(\mathbb{R})$ bounds a disc D in this sphere. It turns out that $r'/r - i\theta'$ extends to a holomorphic function on D (§ 3). This function, suitably averaged (here some ergodic theory gets brushed under the rug, see § 3) yields w:

$$w = \lim_{t\to\infty} \frac{1}{t} \int_0^t (r'/r(s) - i\theta'(s))ds .$$

The Schrödinger equation is usually written in operator form:

$$S\varphi = (-\frac{d^2}{dt^2} + q(t))\varphi = \lambda\varphi .$$

The equation (2) is also associated to a differential operator:

$$Lu = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} [\frac{d}{dt} - \tilde{A}(t)]u = \lambda u .$$

The function $w(\lambda)$ is closely related to spectral properties of L . In particular, if $\tilde{A}$ is real, then L is a self-adjoint operator on $L^2(\mathbb{R},\mathbb{C}^2)$, and we prove that $\alpha(\lambda) = \mathrm{Im}\, w(\lambda)$ is monotone increasing and increases exactly on the spectrum of L . This is exactly the relation between rotation number and spectrum that holds for the Schrödinger equation. See [17], and the earlier papers Pastur [22] and Ishii [12]. These latter papers use the "integrated density of states", which equals the rotation number (Avron-Simon [2]).

The operator L is of more than academic interest. Let us set $b = 0$ and make the change of variables

$$v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} u .$$

Then (2) becomes

$$(3) \qquad v' = \begin{pmatrix} i\lambda & q \\ r & -i\lambda \end{pmatrix} v , \quad q = a - ic , \quad r = a + ic .$$

This is the spectral problem studied by Ablowitz, Kaup, Newell, and Segur [1]. Important non-linear evolution equations (for example the modified Korteweg-de Vries equation, the sine-Gordon equation, the non-linear Schrödinger equation) define so-called "isospectral deformations" of (3)([1] ; Zakharov-Shabat [29]). The problem (3) was further studied by Flaschka and Newell [8]. When q and $r \to 0$ rapidly as $|t| \to \infty$, they defined an object $\ell n\, a(\rho)$ which is, formally speaking, almost identical with our $w(\lambda)$. In fact, let $\mathcal{G}(x,y,\lambda)$ be the kernel of the operator $(L-\lambda)^{-1}$ for $|\mathrm{Im}\,\lambda| > \|\tilde{A}\|_\infty$; thus $\mathcal{G}$ (the Green's function) is a 2 x 2 matrix function. We prove that

$$(4) \qquad \frac{dw}{d\lambda} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{tr}\, \mathcal{G}(x,x,\lambda)dx .$$

This result is formally analogous to formulas 4.4 and 4.6 of [8]. However, the similarity between (4) and the formulas of [8] is somewhat misleading; this is explained further in § 5 . See also [9].

We note that (4) above is completely analogous to Theorem 6.4 in [17].

The paper is organized as follows. Section 1 ends with some basic definitions and results. In particular we define exponential dichotomy, which is fundamental in our approach to the Floquet exponent. We also state the fundamental "spectral theorem" of Sacker-Sell [24,25] and Selgrade [27], in particular a refinement due to Sacker and Sell [25]. In § 2 we define w for real systems (1). In § 3 we discuss briefly the spectral theory of L . This is mainly to introduce the Weyl-Titchmarsh m-functions, which we do for complex as well as real $\tilde{A}$. We define the m-functions using exponential dichotomy, and show how a perturbation theorem of Sacker-Sell ([26] ; see also Coppel [4]) yields trivially a description of their asymptotic behavior. In § 4 we consider the holomorphic function $w(\lambda)$ for $\operatorname{Im} \lambda > \|\operatorname{Im} \tilde{A}\|_\infty$. Finally, in § 5 we relate $w(\lambda)$ to the spectral theory of L . We compute the Frechet derivative of w as a functional of $(\tilde{a},\tilde{b},\tilde{c})$. An estimate of Coppel [4, p. 34] is needed for this computation.

This paper owes much to [17]. The main novelty here (an important one) is in the definition of $w(\lambda)$. One cannot simply mimic the definition of [17]. Also we believe that our discussion of $w(\lambda)$ for complex $\tilde{A}$ via exponential dichotomy arguments is of interest.

A preliminary version of this work appeared in [10]. That paper treats only the case of real, Bohr almost periodic $\tilde{A}$. On the other hand, in [10] it is shown how w defines an infinite number of conserved quantities for the non-linear evolution equations mentioned earlier. This discussion generalizes immediately to the case of bounded $\tilde{A}$ via the results presented here.

Our Floquet exponent is ergodic-theoretic in nature. This means that, instead of studying single equations (1) and single differential operators L , we must study families of such equations and operators. The families we treat are obtained as follows. Let Y be a compact metric space with a <u>flow</u> τ . Thus $\tau : Y \times \mathbb{R} \to Y$ is a continuous map such that: (i) $\tau(y,0) = y$; (ii) $\tau(y,t+s) = \tau(\tau(y,t),s)$ for all $y \in Y$, $t,s \in \mathbb{R}$. We often write $\tau_t(y)$ instead of $\tau(y,t)$. Let $a,b,c : Y \to \mathbb{C}$ be continuous functions, and consider the family of ordinary differential equations

$$(1)_y \qquad u' = \begin{pmatrix} a(\tau_t(y)) & b(\tau_t(y)) + c(\tau_t(y)) \\ -b(\tau_t(y)) + c(\tau_t(y)) & -a(\tau_t(y)) \end{pmatrix} u \equiv A(\tau_t(y))u \quad (y \in Y)$$

Inserting the complex parameter λ , we obtain

$$(2)_{y,\lambda} \quad u' = \left[\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} + A(\tau_t(y)) \right] u \qquad (y \in Y \ , \ \lambda \in \mathbb{C}) \ ,$$

or in operator form

$$L_y u = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - A(\tau_t(y)) \right] u = \lambda u \ .$$

If the coefficients $\tilde{a},\tilde{b},\tilde{c}$ in (1) are bounded and uniformly continuous, then the equation (1) can always be embedded in a family $(1)_y$. In fact let $Y = \mathrm{cls}\{\tilde{A}_s(t) \mid s \in \mathbb{R}\}$, where $\tilde{A}_s(t) = \tilde{A}(s+t)$ and the closure is in the topology of uniform convergence on compact subsets of $\mathbb{R}$ (thus $Y \subset C(\mathbb{R}, L(\mathbb{R}^2))$ = continuous, 2 x 2 matrix-valued functions). A flow on Y is defined by translation: $\tau_t(y) = y_t$ $(y \in Y \ , \ t \in \mathbb{R})$. If $A(y) \underset{\mathrm{def}}{=} y(0)$, then equation (1) is just equation $(1)_{\tilde{A}}$ in the family $\{(1)_y \mid y \in Y\}$.

We fix for the rest of the paper an <u>ergodic measure</u> μ on Y . Thus μ is a regular Borel measure on Y satisfying $\mu(Y) = 1$. Moreover μ is invariant (i.e., $\mu(\tau_t(B)) = \mu(B)$ for each Borel $B \subset Y$ and each $t \in \mathbb{R}$), and satisfies the following condition: if $\mu(B \,\Delta\, \tau_t(B)) = 0$ for each $t \in \mathbb{R}$, then either $\mu(B) = 0$ or $\mu(B) = 1$. Here $B \subset Y$ is Borel, and Δ means symmetric difference: $B \,\Delta\, C = \{y \in Y \mid y \in B \ , \ y \notin C \text{ or } y \in C \ , \ y \notin B\}$.

Our Floquet exponent will be defined using the ergodic measure μ . We note here that the assumption that a,b,c be continuous can be considerably weakened. In fact suppose only that $(1)_y$ can be written in the form $u' = X_t(y)u$ where (Y,μ,X_t) is a stationary ergodic process such that $\sup_x \int_x^{x+1} \|X_t(y)\| dt < \infty$ for μ-a.a.y. Then all of our theorems still hold. See [16] for details.

Now let $\Phi_y(t)$ be the fundamental matrix solution of $(1)_y$ satisfying $\Phi_y(0) = I$. Then $\det \Phi_y(t) \equiv 1$ because $\mathrm{tr}\, A(y) \equiv 0$. Observe that the family $\{\Phi_y\}$ induces a flow on the product vector bundle $Y \times \mathbb{C}^2$, as follows: $((y,u),t) \to (\tau_t(y) \ , \ \Phi_y(t)u)$. If $A(y)$ is real, then there is also a flow on $Y \times \mathbb{R}^2 \subset Y \times \mathbb{C}^2$. These flows are linear on fibers in the obvious sense.

The following definition is basic.

1.1. *Definition.* Say that equations $(1)_y$ admit exponential dichotomy if there exist constants $K > 0$, $\alpha > 0$ and vector subbundles V^+, $V^- \subset \Omega \times \mathbb{C}^2$ with the following properties. First, $V^\pm$ are invariant; thus $(y,u) \in V^\pm$ and $t \in \mathbb{R}$ imply that $(\tau_t(y), \Phi_y(t)u) \in V^\pm$. Second,

$$(y,u) \in V^+ \Rightarrow \|\Phi_y(t)u\| \le Ke^{-\alpha t} \qquad (t \in \mathbb{R}),$$

$$(y,u) \in V^- \Rightarrow \|\Phi_y(t)u\| \le Ke^{\alpha t} \qquad (t \in \mathbb{R}) .$$

Third, $Y \times \mathbb{C}^2 = V^+ \oplus V^-$ (Whitney sum).

Note that, if equations $(1)_y$ have exponential dichotomy, then $\dim V^\pm = 1$ (because $\det \Phi_y(t) \equiv 1$).

The following theorem is also basic.

1.2. *Theorem.* The following are equivalent.

(1) Equations $(1)_y$ admit exponential dichotomy.

(2) For each $y \in Y$, equation $(1)_y$ admits no non-zero bounded solution.

This result follows from the Sacker-Sell-Selgrade theorem [24,27] as extended in [25]. First, say that $Y_o \subset Y$ is minimal if Y_o is closed and every orbit $\{\tau_t(y) \mid t \in \mathbb{R}\}$ is (contained in and) dense in $Y_o (y \in Y_o)$. By [24,27], exponential dichotomy over Y_o is equivalent to the statement that no equation $(1)_y$ has a non-zero bounded solution $(y \in Y_o)$. Now, according to [25], one has exponential dichotomy over all of Y if and only if no equation $(1)_y$ has a non-zero bounded solution, and in addition: for all minimal sets Y_o, $\dim V^+$ and $\dim V^-$ are independent of Y_o. Since $\det \Phi_y(t) \equiv 1$, this additional condition is satisfied: $\dim V^\pm = 1$.

Finally we introduce another flow, this time on the projective bundle $P_{\mathbb{C}} \equiv Y \times \mathbb{P}^1(\mathbb{C})$. Here $\mathbb{P}^1(\mathbb{C})$ is the set of complex lines in $\mathbb{C}^2$ containing $(0,0)$. This flow, denoted by $\tilde{\tau}$, is given by $\tilde{\tau}_t(y,\ell) = (\tau_t(y), \Phi_y(t)\cdot\ell)$ for each complex line ℓ. If $A(y)$ is real, then this formula defines also a flow on $P_{\mathbb{R}} = Y \times \mathbb{P}^1(\mathbb{R})$. We agree to regard $\mathbb{P}^1(\mathbb{R})$ as a subset of $\mathbb{P}^1(\mathbb{C})$ via the natural embedding i : if $\ell \subset \mathbb{R}^2$ is the line generated by $u \neq 0$, then $i(\ell) = \{cu \mid c \in \mathbb{C}\}$.

2. The Floquet Exponent

In this section we define the Floquet exponent w for the equations $(1)_y$. We use the ergodic measure μ in the definition. We write $w = \beta + i\alpha$, and consider β and α separately.

To define β we use the Oseledec theory ([21] ; for a simple proof in the case we are considering see [18]). According to this theory, we can find a subset $Y_1 \subset Y$ of μ-measure 1 and real numbers $\gamma_1 \geq \gamma_2$ with the following properties. First, $Y_1 \times \mathbb{R}^2$ is a sum of measurable sub-bundles $W_1 + W_2$; in fact $(\{y\} \times \mathbb{R}^2) \cap W_i = \{u \in \mathbb{R}^2 \mid u = 0$ or $\lim_{t\to\pm\infty} \frac{1}{t} \ln \|\Phi_y(t)u\| = \gamma_i\}$ $(i = 1,2)$. Since $\det \Phi_y(t) = 1$, the only possibilities are $\gamma_1 = \gamma_2 = 0$ or $\gamma_1 = -\gamma_2 > 0$. We <u>define</u> $\beta = -\gamma_1 = \gamma_2$; thus $\beta \leq 0$.

To define α, we use the polar coordinates $r = \|u\|$, $\theta = \tan^{-1} \frac{u_2}{u_1}$, and recall

$$\theta' = -b(\tau_t(y)) - a(\tau_t(y)) \sin 2\theta + c(\tau_t(y)) \cos 2\theta \equiv f(\tau_t(y), \theta(t)) .$$

We take θ mod π, and view θ as a coordinate on $\mathbb{P}^1(\mathbb{R})$. We view $f(y,\theta)$ as a function on $P \equiv P_{\mathbb{R}} = Y \times \mathbb{P}^1(\mathbb{R})$. Thus

$$(2.1) \qquad \theta' = f(\tilde{\tau}_t(p)) ,$$

where $p = (y,\theta) \in P$ and $\tilde{\tau}$ is the flow on P.

2.2. *Theorem* (a) There is a set $Y_\mu \subset Y$ of μ-measure 1 such that, if $y \in Y_\mu$ and $\theta_o \in \mathbb{R}$, then the limit

$$(2.3) \qquad \alpha \underset{\text{def}}{=} - \lim_{t\to\pm\infty} \frac{\theta(t)}{t} = \text{rotation number}$$

exists and is independent of $(y,\theta_o) \in Y_\mu \times \mathbb{R}$. Here $\theta(t)$ is the solution of (2.1) with $\theta(0) = \theta_o$.

(b) If μ is the only ergodic measure on Y, then $Y_\mu = Y$ and the limit in (2.3) is uniform in $(y,\theta) \in P$ and $t \in \mathbb{R}$.

(c) If Λ is a topological space and $(\lambda,y) \to A_\lambda(y) : \Lambda \times Y \to \{2 \times 2$ real matrices with trace zero$\}$ is continuous, then $\lambda \to \alpha(\lambda)$ is continuous.

Proof. A repeat of the proof of Theorem 4.5 in [17] ; we sketch the details. First let $\pi : P \to Y : (y,\theta) \to y$ be the projection, and let ν be an invariant measure on P such that $\pi(\nu) = \mu$. By the Birkhoff ergodic theorem,

$$\lim_{t\to\pm\infty} \frac{\theta(t) - \theta_o}{t} = \lim_{t\to\pm\infty} \frac{1}{t} \int_o^t f(\tilde{\tau}_s(p))ds$$

exists for ν - a.a. $p = (y,\theta) \in P$, say for $p \in P_\nu$. Let $Y_\mu = \pi(P_\nu)$; then $\mu(Y_\mu) = 1$. Since f is π-periodic in θ , we see that, if $\theta_1(t)$, $\theta_2(t)$ satisfy (2.1) with the same y and with $0 < \theta_2(0) - \theta_1(0) \le \pi$, then $0 < \theta_2(t) - \theta_1(t) \le \pi$ for all $t \in \mathbb{R}$. Hence $\lim_{t\to\pm\infty} \frac{\theta(t)}{t}$ exists for all $(y,\theta) \in Y_\mu \times \mathbb{R}$, and the limit is an invariant function $\hat{f}(y)$ of y alone. By ergodicity of μ , $\hat{f}(y)$ = const. μ - a.e., and we can assume that $\hat{f}$ is constant on Y_μ . This proves (a). Note that, from the proof just given, $\alpha = - \int_Y \hat{f}\, d\mu = - \int_P f\, d\nu$.

To prove (b), one uses the Krylov-Bogoliubov argument [20] as in [17, Lemma 4.4] . We omit details.

To prove (c), note $\alpha(\lambda) = - \int_P f_\lambda\, d\nu_\lambda$, where f_λ is the function in (2.1) and ν_λ is an arbitrary measure on P which is invariant with respect to the flow $\tilde{\tau}_\lambda$ on P induced by the equations

$$u' = A_\lambda(\tau_t(y))u \qquad (y \in Y \; ; \; \lambda \in \Lambda) \; .$$

Let $\lambda_n \to \lambda$. Then $f_{\lambda_n} \to f_\lambda$ uniformly on P , and $\tilde{\tau}_{\lambda_n} \to \tilde{\tau}_\lambda$ uniformly on compact subsets of $P \times \mathbb{R}$.

If $\alpha(\lambda_n)$ does not converge to $\alpha(\lambda)$, then (choosing a subnet if necessary), we can assume that $\alpha(\lambda_n) \to q \neq \alpha(\lambda)$. Choosing a further subnet, we can assume $\nu_{\lambda_n} \to \tilde{\nu}$ weakly, and clearly $\tilde{\nu}$ is $\tilde{\tau}_\lambda$-invariant. Then $q = \lim_n - \int_P f_{\lambda_n}\, d\nu_{\lambda_n} = - \int_P f_\lambda\, d\tilde{\nu} = \alpha(\lambda)$, a contradiction.

3. Spectral Theory

To develop the spectral theory of the operators L_y, we need the Weyl-Titchmarsh m-functions together with knowledge of their asymptotic behavior, and Green's identity. We show how to construct the m-functions using exponential dichotomy arguments even when A is complex. We also use the Sacker-Sell perturbation theorem [26] to give a quick proof of the required statement about asymptotic behavior. These methods of proof may be viewed as supplementing the usual analytical ones [11].

We begin by coordinatizing $P^1(\mathbb{C})$ as follows. If $\ell \in \mathbb{P}^1(\mathbb{C})$ contains the non-zero vector $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, let $m = m(\ell) = \frac{u_2}{u_1}$. Then $\mathbb{P}^1(\mathbb{R}) \cong \{m \mid m = \infty$ or $m \in \mathbb{R}\}$.

3.1. *Theorem.* Let A be complex, and suppose $|\text{Im}\ \lambda| > \|\text{Im}\ A\|_\infty = \sup_y(|\text{Im}\ a(y)| + |\text{Im}\ b(y)| + |\text{Im}\ c(y)|)$. Then equations $(2)_{y,\lambda}$ have exponential dichotomy. Write $Y \times \mathbb{C}^2 = V_+(\lambda) \oplus V_-(\lambda)$, where $V_\pm(\lambda)$ are the complex 1-dimensional subbundles of 1.1. Let $M_\pm(y,\lambda)$ be the m-coordinate of $V_\pm(\lambda) \cap (\{y\} \times \mathbb{C}^2)$. Then $\text{sgn}(\text{Im}\ M_\pm \cdot \text{Im}\ \lambda) = \pm 1$.

3.2. *Definition.* We call the $M_\pm(y,\lambda)$ the M-functions corresponding to equations $(4)_{y,\lambda}$. The proof of 3.1 will show that they are completely analogous to the usual m-functions for the Schrödinger equation [3].

<u>*Proof of* 3.1.</u> The proof is based entirely on dynamical systems arguments. The first part of the proof duplicates the first half of the proof of Theorem 3.1. in [13].

We start with the differential equation for $m = \frac{u_2}{u_1}$:

$$(3.3)\qquad m' = (-\lambda + b + c) - 2am - (\lambda + b + c)m^2 ;$$

here we write a,b,c for $a(\tau_t(y))$, etc., and $y \in Y$ is fixed. Suppose $m(0) \in \mathbb{R}$. Then

$$\text{Im}\ m'(0) = (-\text{Im}\ \lambda + \text{Im}\ b + \text{Im}\ c) - 2(\text{Im}\ a)\ m(0) - (\text{Im}\ \lambda + \text{Im}\ b + \text{Im}\ c)\ m(0)^2 .$$

Since $|\text{Im}\,\lambda| > \|\text{Im}\,A\|_\infty$, the quadratic expression on the right-hand side has no real roots. That is, if $\text{Im}\,\lambda > \|\text{Im}\,A\|_\infty$, then $m(0) \in \mathbb{R} \Rightarrow \text{Im}\,m'(0) < 0$; if $\text{Im}\,\lambda < -\|\text{Im}\,A\|_\infty$, then $m(0) \in \mathbb{R} \Rightarrow \text{Im}\,m'(0) > 0$.

Similarly, if $m(0) = \infty$, then $\text{sgn Im}\left(\frac{1}{m}\right)'(0) = \text{sgn Im}\,\lambda$. This means that, if $\text{Im}\,\lambda > \|\text{Im}\,A\|_\infty$, then the disc bundle $D^- = Y \times \{m \mid \text{Im}\,m \leq 0 \text{ or } m = \infty\}$ is mapped strictly into its interior by the time-t map $\tilde{\tau}_t$ on $P_{\mathbb{C}}$ if $t > 0$. Similarly $D^+ = Y \times \{m \mid \text{Im}\,m \geq 0 \text{ or } m = \infty\}$ is mapped strictly into its interior if $t < 0$. Analogous conclusions hold if $\text{Im}\,\lambda < -\|\text{Im}\,A\|_\infty$.

It follows from the previous paragraph that there are compact invariant subsets $B^\pm \subset D^\pm$ such that $B^\pm \cap (\{y\} \times \mathbb{P}^1(\mathbb{C})) \neq \emptyset$ for all $y \in Y$, and such that $B^\pm \cap (Y \times \mathbb{P}^1(\mathbb{R}) = \emptyset$. Using this fact, we will show that no equation $(2)_{y,\lambda}$ can have a bounded solution. It will then follow from Theorem 1.2 that equations $(2)_{y,\lambda}$ admit exponential dichotomy.

Suppose for contradiction that some equation $(2)_{\bar{y},\lambda}$ admits a non-zero bounded solution $u(t)$. Let us suppose, e.g., $\text{Im}\,\lambda > \|\text{Im}\,A\|_\infty$, so that $\tilde{\tau}_t(D^-) \subset D^-$ if $t > 0$ and $\tilde{\tau}_t(D^+) \subset D^+$ if $t < 0$. Using the existence of the invariant sets $B^\pm$, the fact that $\det \Phi_y(t) \equiv 1$, and the preceding sentence, one can show that $\|u(t)\|$ is bounded away from zero either for $t > 0$ or for $t < 0$. Suppose this holds for $t > 0$. Then one can show that <u>all</u> non-zero solutions $\bar{u}(t)$ of $(2)_{\bar{y},\lambda}$ are bounded and bounded away from zero for $t > 0$.

Let Ω be the ω-limit set of $\bar{y}$; thus $\Omega = \{y \in Y \mid y = \lim_{n\to\infty} \tau(\bar{y},t_n)$ for some sequence $t_n \to +\infty\}$. Then Ω is compact and invariant. Let $Z \subset \Omega$ be a minimal subset of Ω (such a Z exists; see, e.g., [7]). It follows from the previous paragraph that all solutions of all equations $(2)_{y,\lambda}$ are bounded ($y \in \Omega$). It then follows from the theory in [14, Sec. 2] that each point $p \in Z \times \mathbb{P}^1(\mathbb{C})$ is <u>recurrent</u> [7], and hence in particular the orbit $\{\tilde{\tau}_t(p) \mid t \in \mathbb{R}\}$ returns infinitely often to every neighborhood of p . However this contradicts the third paragraph of this proof, according to which, if $p \in Z \times \mathbb{P}^1(\mathbb{R})$, then the orbit through p leaves some neighborhood of p and never returns (neither as $t \to \infty$ nor as $t \to -\infty$).

This completes the proof that equations $(2)_{y,\lambda}$ admit exponential dichotomy. Now, using the definition of exponential dichotomy, it is easily seen that $B^\pm = \{(y,\ell) \mid \{\ell\} = V^\pm(\lambda) \cap (\{y\} \times \mathbb{C}^2)\}$. This completes the proof of 3.1.

Next we consider the asymptotic behavior of the M-functions. Using the Sacker-Sell perturbation theorem [26], it will be trivial to prove that $M_\pm(y,\lambda) \to \pm i$ as $|\lambda| \to \infty$, λ in a sector omitting the real λ-axis.

3.4. *Theorem.* Let U be a sector in the λ-plane which is contained in $\delta \le |\arg \lambda| \le \pi - \delta$ for some $\delta > 0$. Then $M_\pm(y,\lambda) \to \pm\, i\, \mathrm{sgn}(\mathrm{Im}\,\lambda)$ uniformly in $(y,\lambda) \in Y \times U$ as $|\lambda| \to \infty$.

Proof. We consider only the case $\mathrm{Im}\,\lambda > 0$, $\lambda \in U$. Note first that, by 3.1, $M_\pm(y,\lambda)$ are defined for large $|\lambda|$, $\lambda \in U$.

We write $\tilde{u}(t) = u\left(\frac{t}{|\lambda|}\right)$. Then

$$(3.5)_y \qquad \frac{d\tilde{u}}{dt} = \left[B_y(t) + \begin{pmatrix} 0 & \frac{\lambda}{|\lambda|} \\ \frac{-\lambda}{|\lambda|} & 0 \end{pmatrix} \right] \tilde{u}(t) \qquad (y \in Y),$$

where $B_y(t) \to 0$ uniformly in $y \in Y$ and $t \in \mathbb{R}$ as $|\lambda| \to \infty$.

We consider the set C of all bounded uniformly continuous 2×2 - matrix functions on $\mathbb{R}$ with the compact-open topology. Then $B_y \to 0$ in C as $|\lambda| \to \infty$, and the convergence is uniform in y.

Now, the constant system

$$\frac{d\tilde{u}}{dt} = \begin{pmatrix} 0 & \frac{\lambda}{|\lambda|} \\ \frac{-\lambda}{|\lambda|} & 0 \end{pmatrix} \tilde{u} = C_\lambda \tilde{u}$$

has exponential dichotomy for all $\lambda \in U$ (because $\arg \lambda \ne 0,\pi$). Moreover the "bundles" $V_o^\pm$ are defined by the eigenspaces of C_λ, which are generated by $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ for all $\lambda \in U$. Thus the m-coordinate of $V_o^\pm$ is $\pm\, i$.

We give C the flow defined by translation: $(c,t) \to c_t$, where $c_t(s) = c(t+s)$. Then $J = \{C_\lambda \mid \lambda \in U\}$ is a compact invariant set in C with respect to this flow. By the perturbation theorem in [26], there is a neighborhood N of J in C such that, for each compact invariant subset Y_1 of N, the equations $u' = y(t)u \quad (y \in Y_1)$ admit exponential dichotomy with bundles $V^\pm$ uniformly close to $V_o^\pm$.

Now, for $\lambda_o \in U$ with $|\lambda_o|$ sufficiently large, the functions $B_y(t)+C_{\lambda_o}$ are all in N. Hence equations $(3.5)_y$ admit exponential dichotomy with bundles $V^{\pm}$ whose fibers have m-coordinates uniformly near $\pm i$. Since equations $(3.5)_y$ are obtained from equations $(2)_{y,\lambda_o}$ via the time-change $t \to \frac{t}{|\lambda_o|}$, it follows that equations $(2)_{y,\lambda_o}$ also have exponential dichotomy, and the bundles have fibers with $\hat{m}$-coordinates uniformly near $\pm i$. This completes the proof of 3.4.

3.6. *Remark.* Theorems 3.1 and 3.4 can be used to gain information about the spectra of the operators L_y. First of all, it is trivial to see that, if equations $(2)_{y,\lambda}$ have exponential dichotomy, then λ is in the resolvent of L_y for all $y \in Y$ (just write down a Green's function using solutions $\psi_{\pm}(t)$ of $(2)_{y,\lambda}$ which decay exponentially at $t = \pm\infty$). Then by 3.1, the resolvent of L_y contains $\{\lambda \mid |\text{Im}\,\lambda| > \|\text{Im}\,A\|_\infty\}$. If A is real, then the L_y are self-adjoint, and the above simply says that the spectrum of L_y is contained in R.

Next suppose that, for some $\lambda_o \in \mathbb{C}$, equations $(2)_{y,\lambda_o}$ have exponential dichotomy. Suppose moreover that the function $y \to M_+(y,\lambda_o) : Y \to \mathbb{P}^1(\mathbb{C})$ is not homotopic to a constant map. (One can construct such examples whenever there exists a homotopically non-trivial map from Y to $\mathbb{P}^1(\mathbb{C}) \cong S^2$, using the Daletskii-Krein construction [5]). By 3.4, the map $y \to M_+(y,\lambda)$ is homotopic to a constant if $|\lambda|$ is large enough. We conclude that the resolvent of each L_y is not path-connected, and in particular λ_o is "surrounded" by spectrum of each L_y. We use the fact that $(y,\lambda) \to M_{\pm}(y,\lambda)$ is jointly continuous; this follows directly from the perturbation theorem of [26] which has already been used.

We finish this section by outlining the spectral theory for $L = L_y$ when A is real and $y \in Y$ is fixed. We will need:

3.7. *Green's Identity.* If $f,g : \mathbb{R} \to \mathbb{C}^2$ are continuously differentiable, then

$$\int_a^b (\langle Lf,g\rangle - \langle f,Lg\rangle)dt = [f_1(t)\overline{g_2(t)} - f_2(t)\overline{g_1(t)}]_{t=a}^{t=b}$$

for all $a < b \in \mathbb{R}$.

The spectral theory of L (viewed as an operator on $L^2(\mathbb{R},\mathbb{C}^2)$) is developed just as the theory of the one-dimensional Schrödinger operator [3]. Therefore we will be brief.

Fix a number $N > 0$, and consider the finite boundary value problem

$$(3.8) \qquad Lu = \lambda u , \quad u_1(-N) = u_1(N) = 0 .$$

This problem has a corresponding <u>spectral matrix</u> $P_N(\lambda) = P_N(y,\lambda)$ (a 2 x 2, symmetric matrix-function) with the following properties: (i) $\lambda_1 \le \lambda_2 \Rightarrow P_N(\lambda_2) - P_N(\lambda_1) \ge 0$; (ii) there is a unitary transformation $U_N : L^2([-N,N], \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2 ; dP_N) \underset{\text{def}}{=} \{f : \mathbb{R} \to \mathbb{C}^2 \mid \int_{-\infty}^{\infty} < f , dP_N(\lambda)f> <\infty\}$ such that $U_N L U_N^{-1}$ is multiplication by λ. In fact the operator in (3.8) has compact resolvent, and P_N can be expressed in terms of the eigenvalues and eigenfunctions of (3.8).

For $\operatorname{Im} \lambda \neq 0$, define $M_\pm(N;y,\lambda) = \dfrac{u_2^\pm(0)}{u_1^\pm(0)}$, where $u^\pm(t) = \begin{pmatrix} u_1^\pm(t) \\ u_2^\pm(t) \end{pmatrix}$ satisfy $Lu^\pm = \lambda u^\pm$ and $u_1^+(N) = 0 = u_1^-(-N)$. Using the definition 1.1 of exponential dichotomy and 3.1, it is easy to show that $\lim_{N\to\infty} M_\pm(N;y,\lambda) = M_\pm(y,\lambda)$, where the convergence is uniform on any set $Y \times K$ with $K \subset \{\lambda \mid \operatorname{Im} \lambda \neq 0\}$ compact. Moreover, using Green's identity (3.7), one can easily show that the matrix

$$\mathcal{G}(N;y,\lambda) = \begin{pmatrix} \dfrac{1}{M_- - M_+} & \dfrac{1}{2}\,\dfrac{M_- + M_+}{M_- - M_+} \\[2ex] \dfrac{1}{2}\,\dfrac{M_- + M_+}{M_- - M_+} & \dfrac{M_- \; M_+}{M_- - M_+} \end{pmatrix} (N,\lambda)$$

satisfies

$$(3.9) \qquad \frac{\operatorname{Im} \mathcal{G}(N;y,\lambda)}{\operatorname{Im} \lambda} = \int_{-\infty}^{\infty} \frac{dP_N(t)}{|t-\lambda|^2} \qquad (\operatorname{Im} \lambda > 0) .$$

Now let $N \to \infty$. Using the Helly theorem [3], one shows that $P_y(\lambda) = \lim_{N\to\infty} P_N(y,\lambda)$ exists for all but countably many $\lambda \in \mathbb{R}$. One shows that (using 3.4 and 3.9):

$$(3.10) \qquad \frac{\operatorname{Im} \mathcal{G}(y,\lambda)}{\operatorname{Im} \lambda} = \int_{-\infty}^{\infty} \frac{dP_y(t)}{|t-\lambda|^2} \quad (\operatorname{Im} \lambda > 0) ,$$

where $\mathcal{G}(y,\lambda) = \begin{pmatrix} \frac{1}{M_- - M_+} & \frac{1}{2} \frac{M_- + M_+}{M_- - M_+} \\ \frac{1}{2} \frac{M_- + M_+}{M_- - M_+} & \frac{M_- \; M_+}{M_- - M_+} \end{pmatrix} (y,\lambda)$. It follows that L is unitarily equivalent to multiplication by λ in the space $L^2(\mathbb{R},\mathbb{C}^2 \; ; dP_y(\lambda))$. As a consequence:

3.11. *Corollary.* $\lambda \to \operatorname{tr} P_y(\lambda)$ increases exactly on the spectrum of L_y as an operator on $L^2(\mathbb{R},\mathbb{C}^2)$.

We note that, if $\mathcal{G}_y(t,s;\lambda)$ is the Green's function, i.e., the integral kernel of the resolvent $(\lambda - L_y)^{-1}$ for $\operatorname{Im} \lambda \neq 0$, then

$$(3.12) \qquad \mathcal{G}(y,\lambda) = \mathcal{G}_y(0,0,\lambda) .$$

This completes the discussion of the operator L_y on $L^2(\mathbb{R},\mathbb{C}^2)$.

At one point we will also need the half-line operator $L^+ = L_y^+$ on $L^2([0,\infty) \; , \mathbb{C}^2)$, defined as follows:

$$(3.13) \qquad L^+ u = \lambda u \; , \quad \frac{u_2(0)}{u_1(0)} = \tan \theta \; , \quad -\frac{\pi}{2} < \theta \le \frac{\pi}{2} .$$

One studies these operators just as one studies the half-line operators for the Schrödinger equation [3,9]. In particular, there are spectral functions $p^+(\theta,y;\lambda)$ such that L_y^+ is unitarily equivalent to multiplication by λ on $L^2([0,\infty) \; , d\rho^+(\theta,y;\lambda))$. One can show that $\lambda \to p^+(\theta,y;\lambda)$ has a jump discontinuity at λ_o if and only if $L_y^+ = L_y^+(\theta)$ has eigenvalue λ_o , i.e. if and only if (3.13) has a square-integrable solution. Hence using Green's identity and a standard argument (e.g. [28]):

3.14. *Proposition.* Suppose $L_y^+(\theta)$ has eigenvalue λ_o . Then $\lim_{\lambda \to \lambda_o n.t.} M_+(y,\lambda)$ exists and equals $\tan \theta$, where the notation means that $\lambda \to \lambda_o$ non-tangentially and $\operatorname{Im} \lambda > 0$.

3.15. *Question.* It is proved in [13] that, if A is real, then for μ - a.a.y, the spectrum of L_y equals the set of $\lambda \in \mathbb{C}$ for which equations $(2)_{y,\lambda}$ have exponential dichotomy. Is this true for complex A ?

4. Complex Rotation

In this section we define and study $w = w(\lambda)$ for complex λ. We begin, however, with equations $(1)_y$, i.e. without λ. Suppose first that $A(y)$ is real.

We embed $\mathbb{P}^1(\mathbb{R})$ in $\mathbb{P}^1(\mathbb{C})$ as in § 1 : if the line ℓ in $\mathbb{R}^2$ contains $u_o \neq 0$, then $i(\ell) = \{c\,u_o \mid c \in \mathbb{C}\} \in \mathbb{P}^1(\mathbb{C})$. However instead of the m-coordinate on $\mathbb{P}^1(\mathbb{C})$, we now use the coordinate

$$\rho = \frac{1 + im}{1 - im}$$

In the ρ-coordinate, one sees easily that $\mathbb{P}^1(\mathbb{R}) = \{\rho \mid |\rho| = 1\}$, i.e. the unit circle. The interior $\{\rho \mid |\rho| < 1\}$ of $\mathbb{P}^1(\mathbb{R})$ corresponds to the upper half-plane $\{m \mid \operatorname{Im} m > 0\}$ in the m-coordinate.

Now fix $y \in Y$, and consider the function $h_y(\ell) = {r'}/{r} - i\theta'$ as a function of $\ell \in \mathbb{P}^1(\mathbb{R})$. In the ρ-coordinate, we obtain using equations $(1)_{r,\theta}$ in § 1 :

$$h_y(\rho) = q_y + \eta_y\rho ,$$

where $q_y = ib(y)$, $\eta_y = a(y) - ic(y)$. This function extends holomorphically to the unit disc $\{\rho \mid |\rho| \le 1\}$, and in fact to the entire complex ρ-plane $\mathbb{P}^1(\mathbb{C}) \setminus \{\rho = \infty\}$.

Next we introduce $\lambda \in \mathbb{R}$. Then $q_y = q_y(\lambda) = i(\lambda + b(y))$, and η_y remains unchanged. We can clearly extend q_y and η_y holomorphically to the entire <u>complex</u> λ-plane via the same formulas. Thus we define

$$h_{y,\lambda}(\rho) = q_y(\lambda) + \eta_y\rho ,$$

a function holomorphic in the complex variables λ and ρ which is an extension of ${r'}/{r} - i\theta'$.

We want to define $w(\lambda)$ as a time average of ${r'}/{r} - i\theta' = h_{y,\lambda}$, or (via the Birkhoff ergodic theorem) as a space average. Here we must be careful. We cannot define the time average along just any orbit in $P_{\mathbb{C}} = Y \times \mathbb{P}^1(\mathbb{C})$. Indeed according to § 2 we must have $\operatorname{Re} w(\lambda) \le 0$ for real λ, and (assuming $w(\lambda)$ extends holomorphically to $\operatorname{Im} \lambda > 0$) this implies that Re $w(\lambda) < 0$ for $\mathrm{Im}\,\lambda > 0$. At this point we recall from 3.1 that, for each $y \in Y$, the orbit $t \to (\tau_t(y)\,,\, M_+(\tau_t(y),\lambda))$ is defined by an exponentially decreasing solution of $(2)_{y,\lambda}$ ($\mathrm{Im}\,\lambda > 0$). We are led to define (noting that $\mathrm{Im}\, M_+ > 0 \Rightarrow |\rho(M_+)| < 1$) :

$$(4.1) \qquad w(\lambda) = \int_Y \left[\, q_y(\lambda) + \eta_y \frac{1 + iM_+(y,\lambda)}{1 - iM_+(y,\lambda)} \right] d\mu(y) \qquad (\mathrm{Im}\,\lambda > 0)\ ;$$

note that, by the Birkhoff theorem, one has for μ - a.a.$y \in Y$:

$$(4.2) \qquad w(\lambda) = \lim_{T\to\infty} \frac{1}{T} \int_0^T h_{\tau_s(y),\lambda}(\rho(M_+(\tau_s(y),\lambda)))ds\ .$$

It is possible to arrive at the definition (4.1) of $w(\lambda)$ via more fundamental arguments. However these arguments seem to require some elementary Lie algebraic ideas. We defer a discussion to another paper, where higher-dimensional coefficients $A(y)$ will be considered.

4.3. *Definition.* If $A(y)$ is complex and $\mathrm{Im}\,\lambda > \|\mathrm{Im}\, A\|_\infty$, then (4.1) still makes sense and we define $w(\lambda)$ via 4.1.

We have defined $w(\lambda)$ for $\mathrm{Im}\,\lambda > 0$ (or, if A is complex, for $\mathrm{Im}\,\lambda > \|\mathrm{Im}\, A\|_\infty$). If A is real, we must still show this (obviously holomorphic) function $w(\lambda)$ has a boundary value $\hat{w}(\lambda)$ (λ real) given by the definition in § 2 . We will do this. First, however, it is convenient to derive another formula for $w(\lambda)$, which holds also if A is complex.

4.4. *Proposition.* If $\mathrm{Im}\,\lambda > \|\mathrm{Im}\, A\|_\infty$, then

$$w(\lambda) = \int_Y [a(y) + (\lambda + b(y) + c(y))\, M_+(y,\lambda)]d\mu(y)\ .$$

Proof. Using the m-equation (3.3), we obtain

$$q_y(\lambda) + \eta_y \frac{1 + iM_+}{1 - iM_+} = a + (\lambda + b + c)M_+ - \frac{iM_+'}{1 - iM_+}\ .$$

By the Birkhoff ergodic theorem, we have for μ - a.a.y :

$$w(\lambda) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [a(\tau_s(y)) + (\lambda + b(\tau_s(y)) + c(\tau_s(y)))\, M_+(\tau_s(y),\lambda)]ds$$

$$+ \lim_{T\to\infty} \frac{1}{T} \ln(1 - iM_+(\tau_s(y),\lambda))\Big|_{s=0}^{s=T}$$

. Here we choose a branch of the logarithm which is holomorphic in the right half-plane. The logarithm term $\to 0$ as $T \to \infty$, so 4.4 is proved with another application of the Birkhoff theorem.

Now we consider the boundary behavior of the holomorphic function w .

4.5. *Theorem.* Suppose A is real, and let $\lambda_o \in \mathbb{R}$. Then if $\operatorname{Im} \lambda > 0$ and $\lambda \to \lambda_o$ non-tangentially, one has $w(\lambda) \to w(\lambda_o)$.

We remark that the convergence holds for <u>all</u> λ_o . This is mildly interesting for $\operatorname{Re} w = \beta$, in view of the fact that $\beta(\lambda)$ need not be continuous for real λ [15]. We also remark that the proof that $\operatorname{Im} w(\lambda) \to \alpha(\lambda_o)$ differs in certain details from that of [17].

Proof of 4.5. We consider separately the real part β and the imaginary part α of w . We remind the reader that, for real λ , $\beta(\lambda)$ and $\alpha(\lambda)$ are given by the definitions in § 2 .

To prove that $\beta(\lambda) \to \beta(\lambda_o)$ as $\lambda \to \lambda_o$ non-tangentially ($\operatorname{Im} \lambda > 0$) , we generalize the proof of [15]. We sketch details. First of all, let $\operatorname{Im} \lambda > 0$, and let $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ be a solution of $(2)_{y,\lambda}$ such that $\frac{u_2(0)}{u_1(0)} = M_+(y,\lambda)$. Note that $u_1' = [a + (\lambda + b + c)M_+]u_1$, and that M_+ , $\frac{1}{M_+}$ are uniformly bounded. It follows from 4.4 and the Birkhoff theorem that, for μ - a.a.y :

$$(4.6) \qquad \beta(\lambda) = \lim_{t\to\infty} \frac{1}{t} \ln\|u(t)\| .$$

Next, for each λ with $\operatorname{Im} \lambda > 0$, define an ergodic measure ν_λ on $P_{\mathbb{C}} = Y \times P^1(\mathbb{C})$, as follows:

$$\int_{P_{\mathbb{C}}} e(y,\ell)d\nu_\lambda = \int_Y e(y,M_+(y,\lambda))d\mu(y)$$

for each continuous $e : P_{\mathbb{C}} \to \mathbb{R}$. The measure ν_λ is indeed ergodic. Moreover, for <u>any</u> $\lambda \in \mathbb{C}$, define a continuous function

In particular, for μ - a.a.y , there is a solution of $(2)_{y,\lambda_o}$ in $L^2[0,\infty)$. Fix such a y , and let $\ell_y = \frac{u_2(0)}{u_1(0)}$ for a square-integrable solution u(t). By 3.14 , we obtain $M_+(y,\lambda) \to \ell_y$ as $\lambda \to \lambda_o$ non-tangentially. Thus we have: $g_\lambda(y,M_+(y,\lambda)) \to g_{\lambda_o}(y,\ell_y)$ pointwise μ - a.e. as $\lambda \to \lambda_o$ n.t. Now using 4.8 and bounded convergence, we get $\beta(\lambda) = \int_Y g_\lambda(y,M_+(y,\lambda))d\mu(y) \to \int_Y g_{\lambda_o}(y,\ell_y)d\mu(y)$. By 4.7 , the Birkhoff theorem, and the definition of $\beta(\lambda_o)$ (see § 2), we get $\beta(\lambda) \to \beta(\lambda_o)$.

Now we turn to $\alpha = \text{Im } w$. We first use a standard theorem [6] to conclude that the non-tangential limit $\lim_{\tilde\lambda\to\lambda} M_+(y,\tilde\lambda)$ exists for Lebesgue - a.a. $\lambda \in \mathbb{R}$ $(y \in Y)$.

By Fubini's theorem, there is a set $R_o \subset \mathbb{R}$, whose complement has zero Lebesgue measure, such that, if $\lambda \in R_o$, then

$$\lim_{\tilde\lambda\to\lambda} M_+(y,\tilde\lambda) \underset{\text{def}}{=} p_\lambda(y)$$

exists for μ - a.a.$y \in Y$. It is not hard to see that $p_\lambda(y)$ is defined if and only if $p_\lambda(\tau_t(y))$ is defined $(t \in \mathbb{R})$, and that, relative to the flow $\tilde\tau^\lambda$ on $P_{\mathbb{C}}$ induced by equations $(2)_{y,\lambda}$, one has

$$\tilde\tau_t^\lambda(y,p_\lambda(y)) = (\tau_t(y) \; , \; p_\lambda(\tau_t(y))) \; . \tag{4.9}$$

That is, p_λ defines a measurable, invariant section of $P_{\mathbb{C}}$ over Y .

Next recall that, if $\text{Im } \tilde\lambda > 0$, then the ρ-coordinate satisfies $|\rho(M_+(y,\tilde\lambda))| < 1$. Hence for $\lambda \in R_o$: $|\rho(p_\lambda(y))| \le 1$. Using (4.1) and bounded convergence, we get for $\lambda \in R_o$:

$$\lim_{\tilde\lambda\to\lambda} w(\tilde\lambda) = \lim_{\lambda\to\tilde\lambda} \int_Y [q_y(\tilde\lambda) + n_y \frac{1 + iM_+(y,\tilde\lambda)}{1 - iM_+(y,\tilde\lambda)}]d\mu(y)$$

(4.10)

$$= \int_Y [q_y(\lambda) + n_y \frac{1 + ip_\lambda(y)}{1 - ip_\lambda(y)}]d\mu(y) \; .$$

$$g_\lambda : P_{\mathbb{C}} \to \mathbb{R} : (y,\ell) \to \mathrm{Re}\ \frac{\langle [\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} + A(y)]u,u\rangle}{\langle u,u\rangle}$$

Here u is any non-zero vector in ℓ . If $\bar{u}(t)$ is any non-zero solution of $(2)_{y,\lambda}$, then

$$(4.7) \qquad \frac{1}{t}\ \ell n\ \frac{\|\bar{u}(t)\|}{\|\bar{u}(0)\|} = \frac{1}{t}\int_0^t g_\lambda(\tilde{\tau}_s(y,\bar{\ell}))ds \quad ,$$

where $\bar{\ell}$ is the line containing $\bar{u}(0)$. By 4.6, 4.7, and the Birkhoff ergodic theorem, we get

$$(4.8) \qquad \beta(\lambda) = \int_{P_{\mathbb{C}}} g_\lambda\ d\nu_\lambda = \int_Y g_\lambda(y,M_+(y,\lambda))d\mu(y) \qquad (\mathrm{Im}\ \lambda > 0)\ .$$

Now recall that $\beta(\lambda_o) \leq 0$ (§ 2). Suppose first $\beta(\lambda_o) = 0$. Then using the Birkhoff theorem, the Oseledec theorem [21], and the Choquet theorem (which gives a representation of an invariant measure in terms of ergodic measures, see Phelps [23]), we get the following. If ν is an invariant measure on $P_{\mathbb{C}}$ such that $\pi(\nu) = \mu$ (where $\pi : P_{\mathbb{C}} \to Y$ is the projection), then $\int_{P_{\mathbb{C}}} g_{\lambda_o}\ d\nu = 0$. The detailed proof (which we omit) uses also 4.7.

Next note that, if λ_n is any sequence in $\mathbb{C}$ such that $\lambda_n \to \lambda_o$, then $g_{\lambda_n} \to g_{\lambda_o}$ <u>uniformly</u> on $P_{\mathbb{C}}$. We claim that, if Im $\lambda > 0$ and $\lambda \to \lambda_o$ (nontangentially or not), then $\beta(\lambda) \to 0 = \beta(\lambda_o)$. To see this, suppose for contradiction that, for some sequence $\lambda_n \to \lambda_o$, one has $\beta(\lambda_n) \to \beta_o < 0$. Using weak compactness of measures, we can suppose that $\nu_{\lambda_n} \to \nu_o$, where clearly ν_o is invariant with respect to the flow $\tilde{\tau}_{\lambda_o}$ on $P_{\mathbb{C}}$ generated by equations $(2)_{y,\lambda_o}$. So we have (4.8) $\beta(\lambda_n) = \int_{P_{\mathbb{C}}} g_{\lambda_n}\ d\nu_{\lambda_n} \to \int g_{\lambda_o} d\nu_o = 0$, a contradiction. Hence $\beta(\lambda) \to 0 = \beta(\lambda_o)$ as claimed.

The remaining possibility is $\beta(\lambda_o) < 0$. In this case, the Oseledec theorem [21] gives us a μ-measurable, 1-dimensional subbundle $V \subset Y \times \mathbb{R}^2$, invariant under solutions of equations $(2)_{y,\lambda_o}$ (for μ - a.a.y , of course), such that, if $(y,u_o) \in V$ and $u_o \neq 0$, then

$$\lim_{t\to\infty} \frac{1}{t}\ \ell n\ \|\Phi_y(t)u_o\| = \beta(\lambda_o) < 0\ .$$

We note that $p_\lambda(y)$ may very well equal ∞, in which case $\dfrac{1+p_\lambda(y)}{1-p_\lambda(y)} = -1$.

The set $P_{\mathbb{R}} = Y \times \mathbb{P}^1(\mathbb{R})$ is a subset of $P_{\mathbb{C}}$ which is invariant under the flow $\tilde{\tau}^\lambda (\lambda \in R_o)$. For fixed $\lambda \in R_o$, we distinguish two cases: (i) $p_\lambda(y) \in \mathbb{P}^1(\mathbb{R})$ for μ - a.a.$y \in Y$, or (ii) $p_\lambda(y) \notin \mathbb{P}^1(\mathbb{R})$ for μ - a.a.$y \in Y$. By ergodicity of μ, (i) and (ii) exhaust the possibilities.

Suppose (i) holds. Recalling that we view $({}^{r'}/_{r} - i\theta')(y,\rho) = h_{y,\lambda}(\rho)$ as a function of $(y,\rho) \in Y \times \{\rho \mid |\rho| \le 1\}$, we can rewrite (4.10) as follows:

$$\lim_{\tilde{\lambda}\to\lambda} w(\tilde{\lambda}) = \int_Y ({}^{r'}/_{r} - i\theta')(y,\rho(p_\lambda(y)))d\mu(y) .$$

Using the Birkhoff ergodic theorem and Theorem 2.2 (a), we get for μ - a.a.y :

$$\lim_{\tilde{\lambda}\to\lambda} \operatorname{Im} w(\tilde{\lambda}) = \lim_{t\to\infty} \frac{1}{t} [\theta(t) - \theta(0)] = \alpha(\lambda) ,$$

where $\theta(0) = \arg \rho(p_\lambda(y))$ and $\theta(t)$ satisfies the θ-equation corresponding to $(2)_{y,\lambda}$. This shows that $\operatorname{Im} w(\tilde{\lambda}) \to \alpha(\lambda)$ for $\lambda \in R_o$.

Now we consider case (ii). For fixed $\lambda \in R_o$, one has for μ - a.a.y : $\operatorname{Im} p_\lambda(\tau_t(y)) > 0$ for all $t \in \mathbb{R}$. For any such y, the function $t \to p_\lambda(\tau_t(y))$ satisfies the m-equation 3.3.

Using the Birkhoff ergodic theorem in (4.10), we get for μ - a.a.y :

$$(4.11) \qquad \lim_{\tilde{\lambda}\to\lambda} w(\tilde{\lambda}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [q_{\tau_s(y)}(\lambda) + n_{\tau_s(y)} \frac{1+p_\lambda(\tau_s(y))}{1-p_\lambda(\tau_s(y))}]ds$$

The quantity $\int_0^T \operatorname{Im} \dfrac{-ip_\lambda'(\tau_s(y))}{1-ip_\lambda(\tau_s(y))} ds$ is uniformly bounded in T (the prime means $\frac{d}{ds}$). We add $\frac{1}{T}$ times this integral in (4.11). Using the m-equation (3.3) as in the proof of (4.4), we get

$$(4.12) \qquad \lim_{\tilde{\lambda}\to\lambda} \operatorname{Im} w(\tilde{\lambda}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [a(\tau_s(y)) + (\lambda + b(\tau_s(y)) + c(\tau_s(y))) \, p_\lambda(\tau_s(y))] ds$$

for μ - a.a.$y \in Y$.

Fix $y \in Y$ for which (4.12) holds. Let $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$ be the solution of $(2)_{y,\lambda}$ with $u_1(0) = 1$, $u_2(0) = p_\lambda(y)$. Then

$$\frac{u_1'}{u_1}(t) = a(\tau_t(y)) + (\lambda + b(\tau_t(y)) + c(\tau_t(y))p_\lambda(\tau_t(y)) \equiv \mathcal{f}(t) + i\sigma(t) ,$$

where $\mathcal{f}$ and σ are real. It is clear from (4.12) what we must do: namely we must show that $\lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma(s)ds = \alpha(\lambda)$.

To do so, we recall 2.2(a): for μ - a.a.y , every real solution $v(t)$ of $(2)_{y,\lambda}$ satisfies $-\lim_{T\to\infty} \frac{1}{T} \arg v(T) = \alpha(\lambda)$.

We fix y for which this property together with (4.12) hold, and choose $v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \text{Re } u(t)$. Let $\theta(t) = \arg v(t)$ (i.e., $\theta(t)$ = argument of the complex number $v_1(t) + iv_2(t)$). We also write

(4.13)

a) $v_1(t) = \delta(t) \cos [\int_0^t \sigma(s)ds + \sigma_o]$

b) $v_2(t) = v_1(t)\cdot\text{Re } p_\lambda(\tau_t(y)) - \delta(t)\cdot\text{Im } p_\lambda(\tau_t(y))\cdot\sin [\int_0^t \sigma(s)ds+\sigma_o]$,

where $\delta(t) = \exp \mathcal{f}(t) > 0$.

Assume for simplicity that $-\frac{\pi}{2} < \theta(0) < \frac{\pi}{2}$ (the other possibilities can be treated similarly). By 4.13 (a), we can also assume $-\frac{\pi}{2} < \sigma_o < \frac{\pi}{2}$. Let $t_1 > 0$ be the least positive zero of $v_1(t)$. Then $\theta(t_1) = \frac{\pi}{2}$ iff $v_2(t_1) > 0$, and $\theta(t_1) = -\frac{\pi}{2}$ iff $v_2(t_1) < 0$. By 4.13 (b), $\int_0^{t_1} \sigma(s)ds + \sigma_o = \pm\frac{\pi}{2}$ iff $\theta(t_1) = \mp\frac{\pi}{2}$. (If $t_1 = \infty$, then both $\theta(t)$ and $\int_0^t \sigma(s)ds + \sigma_o$ remain always between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$; in this case $\alpha(\lambda) = 0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma(s)ds$).

Note further that, if $t > t_1$ is near t_1 , then: (i) $\theta(t_1) = \frac{\pi}{2}$, $\theta(t) > \frac{\pi}{2} \Rightarrow \int_0^t \sigma(s)ds + \sigma_o < -\frac{\pi}{2}$; (ii) $\theta(t_1) = -\frac{\pi}{2}$, $\theta(t) < -\frac{\pi}{2} \Rightarrow \int_0^t \sigma(s)ds+\sigma_o > \frac{\pi}{2}$. Thus $\theta(t)$ leaves $[-\frac{\pi}{2} , \frac{\pi}{2}]$ iff $\int_0^t \sigma(s)ds + \sigma_o$ does so, and they leave in opposite directions.

Clearly we can continue this analysis for all positive time. The conclusion we draw is that $\int_0^t \sigma(s)ds + \sigma_o$ and $-\theta(t)$ are always within π radians of each other:

$$\left| \int_0^t \sigma(s)ds + \sigma_o + \theta(t) \right| < \pi \qquad (t > 0) .$$

It follows immediately that $\alpha(\lambda) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \sigma(s)ds$ for each $\lambda \in R_o$.

Using (4.12), we see that $\lim_{\tilde{\lambda}\to\lambda} \operatorname{Im} w(\tilde{\lambda}) = \alpha(\lambda)$ for all $\lambda \in R_o$. We finish the proof of Theorem 4.5 in the following way. By 4.6 , $\operatorname{Re} w(\tilde{\lambda}) \leq 0$ for $\operatorname{Im} \tilde{\lambda} > 0$, and by the maximum principle for harmonic functions, $\operatorname{Re} w(\tilde{\lambda}) < 0$. Thus w is holomorphic in the upper half-plane with negative real part there.

Furthermore the boundary value α of Im w is continuous (2.2 (c)).Under these circumstances, we have $\lim_{\tilde{\lambda}\to\lambda} \operatorname{Im} w(\tilde{\lambda}) = \alpha(\lambda)$, and in fact the convergence is unrestricted, not just non-tangential. See, e.g., [6]. This completes the proof of 4.5.

5. w(λ) and the Green's function

Our final project is to relate $w(\lambda)$ to the Green's matrix (i.e., kernel) $\mathcal{G}_y(t,s;\lambda)$ of the operator $(\lambda - L_y)^{-1}$ $(y \in Y)$. The fundamental relation is

$$(5.1) \qquad \frac{dw}{d\lambda} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \operatorname{tr} \mathcal{G}_y(s,s;\lambda)ds \qquad (\mu - \text{a.a.}\, y \in Y) .$$

Using the elementary equality $\mathcal{G}_y(t,t;\lambda) = \mathcal{G}_{\tau_t(y)}(0,0;\lambda)$ and the Birkhoff ergodic theorem, we see that (5.1) is equivalent to

$$(5.2) \qquad \frac{dw}{d\lambda} = \int_Y \operatorname{tr} \mathcal{G}_y(0,0;\lambda)d\mu(y) .$$

The result (5.1 - 5.2) is true for complex A if $|\mathrm{Im}\,\lambda| > \|\mathrm{Im}\,A\|_\infty$. The proof involves a computation very similar to that in [17, Section 7]. Our computation is also formally similar to that of [8, Section 4]. We use a fundamental estimate of Coppel [4, p. 34].

Before proving 5.1 - 5.2, however, we will use it to derive the following important corollary.

5.3. *Theorem.* Suppose A is real, and let $\alpha(\lambda)$ be the rotation number for equations $(2)_{y,\lambda}$. Let $dP_y(\lambda)$ be the spectral matrix of § 3 $(y \in Y)$. Then

$$d\alpha = \pi \int_Y (\operatorname{tr} dP_y)\, d\mu(y) ,$$

where the equality is to be interpreted as follows: if $f : \mathbb{R} \to \mathbb{R}$ is continuous with compact support, then $\int_{-\infty}^{\infty} f(t)d\alpha(t) = \pi \int_Y (\operatorname{tr} \int_{-\infty}^{\infty} f(t)dP_y(t))d\mu(y)$.

Note that, from the θ-equation $(1)_\theta$ and the definition of α , $\lambda \to \alpha(\lambda)$ is monotone increasing,hence $d\alpha$ is defined.

It is shown in [13] that, if A is real, then the spectrum of L_y is independent of y for μ - a.a.y . (For the one-dimensional Schrödinger operator, this is a well-known result of Ishii [12] and Pastur [22]). From 5.3 and 3.11 we obtain the important

5.4. *Corollary.* For μ - a.a.y , the rotation number α increases exactly on the spectrum of L_y .

Proof of 5.3. First we recall from 3.12 that, if $\mathcal{G}(y,\lambda) = \mathcal{G}_y(0,0;\lambda)$, then

$$\operatorname{tr}\mathcal{G}(y,\lambda) = \frac{1 + M_-(y,\lambda)\, M_+(y,\lambda)}{M_-(y,\lambda) - M_+(y,\lambda)} \qquad (\operatorname{Im}\lambda > 0) .$$

It follows from 5.2 that $\operatorname{Im}\frac{dw}{d\lambda} = \operatorname{Im}\int_Y \operatorname{tr}\mathcal{G}(y,\lambda)d\mu(y) > 0$ for $\operatorname{Im}\lambda > 0$. Hence we can write [6] :

$$\operatorname{Im}\frac{dw}{d\lambda} = \operatorname{Im}\lambda \cdot \left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\gamma(t)}{|t-\lambda|^2} + \delta\right] ,$$

where $\delta \geq 0$ and $\gamma(t)$ is a monotone increasing function of $t \in \mathbb{R}$.

By the formula in 3.4, $\delta = 0$. By a standard argument [6], we get

$$\lim_{\varepsilon\to 0^+}\int_{\lambda_1}^{\lambda_2}\operatorname{Im}\frac{dw}{d\lambda}(t + i\varepsilon)dt = \gamma(\lambda_2) - \gamma(\lambda_1)$$

at all continuity points λ_1, λ_2 of γ. However by Theorem 4.5 and continuity of α, $\operatorname{Im} w(\lambda_2 + i\varepsilon) - \operatorname{Im} w(\lambda_1 + i\varepsilon) \to \alpha(\lambda_2) - \alpha(\lambda_1)$ for <u>all</u> $\lambda_1, \lambda_2 \in \mathbb{R}$. This implies that

$$\operatorname{Im}\frac{dw}{d\lambda} = \operatorname{Im}\lambda \cdot \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\alpha(t)}{|t-\lambda|^2} \qquad (\operatorname{Im}\lambda > 0) ,$$

and that

$$(5.5) \qquad \lim_{\varepsilon\to 0^+}\int_{\lambda_1}^{\lambda_2}\int_Y \operatorname{Im}\mathcal{G}(y,t + i\varepsilon)d\mu(y)dt = \alpha(\lambda_2) - \alpha(\lambda_1)$$

for all λ_1, $\lambda_2 \in \mathbb{R}$.

It follows from 3.10 that $\operatorname{Im}\mathcal{G}(y,\lambda) = \operatorname{Im}\lambda \cdot \int_{-\infty}^{\infty}\operatorname{tr}\frac{dP_y(t)}{|t-\lambda|^2}$ for $\operatorname{Im}\lambda > 0$. The function $(y,\lambda) \to \mathcal{G}(y,\lambda)$ is jointly continuous. Combining these two sentences, we can find a constant K such that (set $\lambda = i$) :

$$(5.6) \qquad \left\| \int_{-\infty}^{\infty}\frac{dP_y(t)}{1 + t^2}\right\| \leq K \qquad (y \in Y) .$$

Let $\lambda_1 < \lambda_2 \in \mathbb{R}$ and $\varepsilon > 0$ be given. Then

$$\int_{\lambda_1}^{\lambda_2} \operatorname{Im} \mathcal{G}(y,s+i\varepsilon)ds = \int_{-\infty}^{\infty} \operatorname{tr} dP_y(t) \int_{\lambda_1}^{\lambda_2} \frac{\varepsilon\, ds}{(t-s)^2+\varepsilon^2} =$$

$$\int_{-\infty}^{\infty} \left[\tan^{-1} \frac{\lambda_2 - t}{\varepsilon} - \tan^{-1} \frac{\lambda_1 - t}{\varepsilon} \right] \operatorname{tr} dP_y(t) .$$

Letting $\varepsilon \to 0$ and using (5.6), we get

$$\text{(5.7)} \quad \lim_{\varepsilon\to 0^+} \int_{\lambda_1}^{\lambda_2} \operatorname{Im} \mathcal{G}(y,s+i\varepsilon)ds = \pi \int_{(\lambda_1,\lambda_2)} \operatorname{tr} dP_y(t) + \frac{\pi}{2} \operatorname{tr}[dP_y\{\lambda_2\} + dP_y\{\lambda_1\}]$$

and the convergence is bounded in y . The first integral on the right is over the <u>open</u> interval (λ_1,λ_2).

Now combine (5.5) , (5.7), and bounded convergence to get

$$\text{(5.8)} \quad \alpha(\lambda_2) - \alpha(\lambda_1) = \int_Y \left\{ \pi \int_{(\lambda_1,\lambda_2)} \operatorname{tr} dP_y(t) + \frac{\pi}{2} \operatorname{tr}[dP_y\{\lambda_2\} + dP_y\{\lambda_1\}]\right\} d\mu(y)$$

Letting $\lambda_2 \to \lambda_1$ and using bounded convergence again, we see that

$$0 = \int_Y (\operatorname{tr} dP_y\{\lambda_1\}) d\mu(y) \qquad (\lambda_1 \in \mathbb{R}) ;$$

thus for μ - a.a.y , λ_1 is not an eigenvalue of L_y (this is well-known for the Schrödinger equation; e.g., Pastur [22]). Hence using (5.8) again:

$$\text{(5.9)} \quad \alpha(\lambda_2) - \alpha(\lambda_1) = \pi \int_Y \left(\int_{\lambda_1}^{\lambda_2} \operatorname{tr} dP_y(t)\right) d\mu(y) .$$

Now, the formula $f \to \int_Y \int_{-\infty}^{\infty} f(t) \operatorname{tr} dP_y(t) d\mu(y)$ ($f : \mathbb{R} \to \mathbb{R}$ continuous of compact support) defines a Radon measure $d\sigma$ on $\mathbb{R}$, and (5.9) shows that $d\sigma$ and $d\alpha$ agree on closed intervals $[\lambda_1,\lambda_2]$. Hence they agree on Borel sets, and Theorem 5.3 is proved.

Now we turn to the proof of 5.1. We emphasize that we will prove 5.1 for complex A as well as real A , so long as $\operatorname{Im} \lambda > \|\operatorname{Im} A\|_\infty$.

Let us first assume that the closed support of the ergodic measure μ is all of Y . There is no loss of generality in doing so, since $\mu(\operatorname{Supp}(\mu)) = 1$.

Then in particular $\mu(O) > 0$ for every open $O \subset Y$. Using metrizability of Y, we see that a set of points y in Y of full μ-measure has the property that the orbit $\{\tau_t(y) \mid t \in \mathbb{R}\}$ is dense in Y. Since $C(Y)$ is separable, there is also a set of points y in Y of full μ-measure such that

$$(5.10) \qquad \int_Y f(\bar{y})d\mu(\bar{y}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\tau_s(y))ds$$

for <u>all</u> $f \in C(Y)$.

Fix a point $y_o \in Y$ with dense orbit for which (5.10) holds. Let $B = \{g : \mathbb{R} \to \mathbb{C} \mid g(t) = f(\tau_t(y_o))$ for some $f \in C(Y)\}$. Give B the uniform norm; then B is isomorphic as a complex Banach space to $C(Y)$ via $i : C(Y) \to B : i(f)(t) = f(\tau_t(y_o))$. We will abuse notation and write $a(t) = a(\tau_t(y_o))$, $b(t) = b(\tau_t(y_o))$, $c(t) = c(\tau_t(y_o))$. Further, we write $m_\pm(t,\lambda) = M_\pm(\tau_t(y_o),\lambda)$.

We fix λ with $\mathrm{Im}\,\lambda > \|\mathrm{Im}\,A\|_\infty$, and study $w = w(\lambda)$ as a functional of $p = (a,b,c) \in B^3 = B \times B \times B$. We will do more than is necessary, and compute the complex Frechet derivative $Dw_p : B^3 \to \mathbb{C}$. Thus $|w(p + \delta p) - w(p) - Dw_p(\delta p)| = O(\delta p)$ for small variations $\delta p = (\delta a, \delta b, \delta c) \in B^3$, where $\|\delta p\| = \|\delta a\|_\infty + \|\delta b\|_\infty + \|\delta c\|_\infty$. In particular, $\frac{dw}{d\lambda} = Dw_p(0,1,0)$. The Frechet derivative is useful in the context of a certain Hamiltonian structure on B^3([10]).

Let $\varphi(t) = \begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\end{pmatrix}$, $\psi(t) = \begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix}$ be the solutions of $(2)_{y_o,\lambda}$ such that $\varphi(0) = \begin{pmatrix}1\\ m_-(0,\lambda)\end{pmatrix}$, $\psi(0) = \begin{pmatrix}1\\ m_+(0,\lambda)\end{pmatrix}$. Then

$$(5.11) \qquad \begin{aligned} &a) \quad \varphi_1' = [a + (\lambda + b + c)m_-]\varphi_1 \\ &b) \quad \psi_1' = [a + (\lambda + b + c)m_+]\psi_1 . \end{aligned}$$

From (4.4) and (5.10) we get

$$w = \lim_{T\to\infty} \frac{1}{T}\int_0^T [a + (\lambda + b + c)m_+]ds = \lim_{T\to\infty} \frac{1}{T} \ln \psi_1(T) .$$

We also have

$$-w = \lim_{T\to\infty} \frac{1}{T}\int_0^T [a + (\lambda + b + c)m_-]ds = \lim_{T\to\infty} \frac{1}{T} \ln \varphi_1(T) .$$

Indeed, the Wronskian

$$W = \varphi_1\psi_2 - \varphi_2\psi_1 = \varphi_1\psi_1[m_+ - m_-]$$

is constant, and since $\operatorname{Im} m_+ > 0 > \operatorname{Im} m_-$ we have $\ell n\, W = \ell n\, \varphi_1 + \ell n\, \psi_1 - \ell n(m_+ - m_-) \Rightarrow \lim_{T\to\infty} \frac{1}{T} \ell n\, \psi_1(T) = - \lim_{T\to\infty} \frac{1}{T} \ell n\, \varphi_1(T)$. Thus

$$(5.12) \qquad w = \frac{1}{2} \lim_{T\to\infty} \frac{1}{T} \int_0^T (\lambda + b + c)(m_+ - m_-)ds \; .$$

The functional $(f,g) \to \lim_{T\to\infty} \frac{1}{T} \int_0^T f(s)\, g(s)ds : B^2 \to \mathbb{C}$ is complex bilinear and bounded. Let U be a neighborhood of p in B^3 on which w, m_-, m_+ are defined; in particular m_+ and m_- are then maps from U to B . Assuming $m_\pm$ are Frechet differentiable at p , we have

$$(5.13) \qquad \delta w \underset{\text{def}}{=} Dw_p(\delta p) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [(\delta b + \delta c)(m_+(p) - m_-(p)) + (\lambda + b + c)(Dm_+(p) - Dm_-(p))(\delta p)]ds \; .$$

We compute e.g. $Dm_-(p)(\delta p) \equiv \widetilde{\delta m}_-$. From the m-equation (3.3), we see that $\widetilde{\delta m}_-$ ought to satisfy:

$$(5.14) \qquad \begin{aligned} &(\widetilde{\delta m}_-)' + 2Q_-(t)\, \widetilde{\delta m}_- = R_-(t) \; ; \\ &Q_-(t) = a(t) + (\lambda + b(t) + c(t))m_-(t) \\ &R_-(t) = -2m_-(t)\, \delta a(t) - (1 + m_-^2(t))\, \delta b(t) + (1 - m_-^2(t))\, \delta c(t) \; . \end{aligned}$$

Here and below $m_\pm$ denote $m_\pm(p)$. Observe that, from the fact that equations $(2)_{y,\lambda}$ have exponential dichotomy, the estimates (1.1), and 5.11 (a), we get

$$(5.15) \qquad \lim_{|t-s|\to\infty} \frac{1}{t-s} \int_s^t Q_-(r)dr = \sigma > 0 \; .$$

Hence (5.14) has a unique bounded solution which, with an eye to later developments, we denote by δm_- :

$$(5.16) \qquad \delta m_-(t) = \int_{-\infty}^t R_-(s) \exp(2 \int_t^s Q_-(r)dr)ds \; .$$

To prove that $Dm_-(p)(\delta p)$ really exists and equals the δm_- of (5.16), we proceed as follows. Write $\Delta m_- = m_-(p + \delta p) - m_-(p)$. Then

$$(5.17) \qquad (\Delta m_-)' + [2Q_- + 2m_-(\delta b + \delta c) + \Delta m_-(\lambda + b + \delta b + c + \delta c)]\Delta m_- = R_-(t) \ .$$

By the estimate of Coppel [4, p. 34], we have $\|\Delta m_-\|_\infty = O(\delta p)$ for small δp (this uses the fact that equations $(2)_{y,\lambda}$ have exponential dichotomy). Using 5.15, we see that, for δp small, (5.17) has a unique bounded solution (which must be Δm_-):

$$\Delta m_-(t) = \int_{-\infty}^{t} R_-(s)\cdot\exp\int_t^s 2Q_-(u)du\cdot\exp[O(\delta p)(s-t)]ds \ .$$

Using (5.15) and the fact that $\|R_-\|_\infty = O(\delta p)$, we see that $\|\Delta m_- - \delta m_-\|_\infty = O((\delta p)^2)$. This shows that $Dm_-(p)$ exists, and that $Dm_-(p)(\delta p) = \delta m_-$.

We now use 5.11 (a) to rewrite δm_- :

$$\delta m_-(t) = \int_{-\infty}^{t} \frac{\varphi_1^2(s)}{\varphi_1^2(t)} R_-(s)ds \ .$$

One can also compute $Dm_+(p)(\delta p) \underset{\text{def}}{=} \delta m_+$:

$$\delta m_+(t) = -\int_t^\infty \frac{\psi_1^2(s)}{\psi_1^2(t)} R_+(s)ds \ ,$$

where R_+ is obtained from R_- by replacing m_- by m_+ .

Now plug these expressions for $\delta m_\pm$ in (5.13). To obtain a more convenient formula for δw , we add $\frac{dh}{dt}$ in the integrand in (5.13), where h(t) is the <u>bounded</u> function

$$h(t) = \frac{1}{W}\varphi_1(t)\psi_1(t)\,[\delta m_-(t) - \delta m_+(t)] \ .$$

Then δw is unchanged. After a computation:

(5.18) $$\delta w = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left[\frac{m_- + m_+}{m_- - m_+} \delta a + \frac{1 + m_- m_+}{m_- - m_+} \delta b - \frac{1 - m_- m_+}{m_- - m_+} \delta c \right] ds .$$

We can also express δw in terms of μ :

(5.19) $$\delta w = \int_Y \left[\frac{M_- + M_+}{M_- - M_+} \delta a + \frac{1 + M_- M_+}{M_- - M_+} \delta b - \frac{1 - M_- M_+}{M_- - M_+} \delta c \right] d\mu(y) .$$

In particular, setting $\delta a = 0 = \delta c$ and $\delta b = 1$, we get (5.1) and (5.2).

5.20. *Remarks* (1) The functions $\delta m_\pm$ are not just bounded and continuous, but are actually in B .

(2) Though formula (5.18) resembles the Flaschka-Newell formula for $\delta \ln a$ [8], it differs from that formula in an important respect. To explain this, suppose Y is the hull of a matrix function $\tilde{A}$ which decreases rapidly at $\pm\infty$. Then Y is a circle, and the flow on Y has a unique fixed point y_0 which attracts all other orbits. Thus there is only one ergodic measure μ on Y ; it is the Dirac measure at y_0 . Using this measure μ in (5.19), we obtain δw for the identically zero matrix $\tilde{A} \equiv 0$. This is clearly not the Flaschka-Newell formula.

(3) If $\operatorname{Im} \lambda < - \|\operatorname{Im} A\|_\infty$, one can compute δw and $\frac{dw}{d\lambda}$. The formulas are the same as those in (5.18 - 19) and (5.1 - 2).

References

[1] ABLOWITZ, M., KAUP, D., NEWELL, A., and SEGUR, H., The inverse scattering transform: Fourier analysis for non-linear systems, Studies in Applied Math. 53 (1974), 249-315.

[2] AVRON, J., SIMON, B., Almost periodic Schrödinger operators II. The integrated density of states, Duke Math.J. 50 (1983), 369-391.

[3] CODDINGTON, E., and LEVINSON, N., Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[4] COPPEL, A., Dichotomies in Stability Theory, Lecture Notes in Math. # 629, Springer-Verlag, New York-Berlin, 1978.

[5] DALETSKII, J., KREIN, M., Stability of Solutions of Differential Equations in Banach Space, AMS Translations # 43, Providence, R.I., 1974.

[6] DUREN, P., Theory of H^p Spaces, Academic Press, New York, 1970.

[7] ELLIS, R., Lectures on Topological Dynamics, Benjamin, Reading, Mass., 1969.

[8] FLASCHKA, H., and NEWELL, A., Integrable systems of non-linear evolution equations, in Lecture Notes in Phys. 38, Springer-Verlag, New York-Berlin, 1974.

[9] GIACHETTI, R., JOHNSON, R., A Hamiltonian system from gange transformations of the Zakharov-Shabat system, Phys. Letters A, 1984.

[10] GIACHETTI, R., JOHNSON, R., Spectral theory of two-dimensional almost periodic differential operators and its relation to certain non-linear evolution equations, to appear in Nuovo Cimento B .

[11] HILLE, E., Lectures on Ordinary Differential Equations, Addison-Wesley, Reading, Mass., 1969.

[12] ISHII, K., Localization of eigenstates and transport phenomena in the one-dimensional disordered system, Supp.Theor.Phys. 53 (1973), 77-138.

[13] JOHNSON, R., Exponential dichotomy, rotation number, and linear differential operators with bounded coefficients, Heidelberg preprint, 1984.

[14] JOHNSON, R., On a Floquet theory for two dimensional, almost periodic differential equations, J.of Diff.Eqns. 37 (1980), 184-205.

[15] JOHNSON, R., Lyapunov exponents for the almost periodic Schrödinger equation, Illinois J.of Math., 1984.

[16] JOHNSON, R., Remarks on linear differential equations with measurable coefficients, in preparation.

[17] JOHNSON, R., MOSER, J., The rotation number for almost periodic potentials, Comm.Math.Phys. 84 (1982), 403-438.

[18] JOHNSON, R., PALMER, K., SELL, G., Ergodic theory of linear dynamical systems, IMA preprint 1984, University of Minnesota.

[19] MOSER, J., Integrable Hamiltonian Systems and Spectral Theory, Lezioni Fermiane, Pisa, 1981.

[20] NEMYTSKII, V., and STEPANOV, V., Qualitative Theory of Differential Equations, Princeton Univ.Press, Princeton, N.J., 1960.

[21] OSELEDEC, V., A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, Trans.Moscow Math.Soc. 19 (1968), 197-231.

[22] PASTUR, L., Spectral properties of disordered systems in the one-body approximation, Comm.Math.Phys. 75 (1980), 179-196.

[23] PHELPS, R., Lectures on Choquet Theory, American Book Co., New York, 1966.

[24] SACKER, R., SELL, G., Dichotomies and invariant splittings for linear differential systems I , J.of Diff.Eqns. 15 (1974), 429-458.

[25] SACKER, R., SELL, G., Dichotomies and invariant splittings for linear differential systems II , J.of Diff.Eqns. 22 (1976), 478-496.

[26] SACKER, R., SELL, G., A spectral theory for linear differential systems, J.of Diff.Eqns. 27 (1978), 320-358.

[27] SELGRADE, J., Isolated invariant sets for flows on vector bundles, Trans. AMS 203 (1975), 359-390.

[28] TITCHMARSH, E., Eigenfunction Expansions Associated with Second-Order Differential Equations, Part I, 2nd Ed., Clarendon Press, Oxford, 1962.

[29] ZAKHAROV, V., SHABAT, A., Exact theory of two-dimensional self-focussing and one-dimensional self-modulating waves in non-linear media, Sov.Phys. JETP 34 (1972), 62-69.