THE CONSTRUCTION OF INERTIAL MANIFOLDS FOR REACTION-DIFFUSION EQUATIONS BY ELLIPTIC REGULARIZATION

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THE CONSTRUCTION OF INERTIAL MANIFOLDS FOR REACTION-DIFFUSION EQUATIONS BY ELLIPTIC REGULARIZATION*

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Abstract. We demonstrate that the method of elliptic regularization developed in Fabes, Luskin, and Sell (1988) can be used to construct invariant manifolds for reaction diffusion equations.

Key words. reaction diffusion equations, dissipative dynamical systems, elliptic regularization, inertial manifolds.

AMS(MOS) subject classifications. 34C30, 35J60, 35K15,58F99.

1. Introduction. We show in this paper that the existence of inertial manifolds for reaction-diffusion equations can be demonstrated by the method of elliptic regularization (See Fabes, Luskin, and Sell (1988)). An inertial manifold $\mathfrak{M}$ is a Lipschitz continuous, positively invariant, finite dimensional manifold in the ambient infinite dimensional phase space which includes the global attractor $\mathfrak{A}$. We shall analyze the scalar reaction-diffusion partial differential equation for $u : \Omega \equiv [0,1]^n \to \mathbb{R}$ given by

$$\frac{\partial u}{\partial t} = \Delta u - u - g(u), \qquad (x,t) \in \Omega \times \mathbb{R}^+, \tag{1.1}$$

with periodic boundary conditions where $g(s) \in C^3$ and $n \leq 2$. Our analysis can be extended to reaction-diffusion systems with Dirichlet or Neumann boundary conditions under appropriate hypotheses on g.

2. The Evolution Equation. We define $L^p(\Omega)$ for $1 \leq p \leq \infty$ to be the usual class of real-valued measurable functions on Ω with norm for $1 \leq p < \infty$ given by

$$\| v \|_{L^p(\Omega)} = \left(\int_\Omega |v(x)|^p dx \right)^{1/p}, \qquad v \in L^p(\Omega),$$

and for $p = \infty$ given by

$$\| v \|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|, \qquad v \in L^\infty(\Omega).$$

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We define the Sobolev space (see Adams (1975)) $H^m(\Omega)$ for m a non-negative integer to be the restriction to Ω of functions $v(x) \in H^m(\mathbb{R}^n)$ which are Ω-periodic, i.e.,

$$v(x + \sum_{j=1}^{n} L_j e_j) = v(x)$$

for integers, L_j , where e_j is the canonical basis for $\mathbb{R}^n$. We use the norm

$$\| v \|_{H^m(\Omega)} = \left[\sum_{|\alpha| \le m} \int_\Omega |D^\alpha v|^2 dx \right]^{1/2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, the α_i are non-negative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$D^\alpha v \equiv \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}} v.$$

We define the positive definite, self-adjoint operator $Av \equiv -\Delta v + v$ with domain $\mathcal{D}(A) = H^2(\Omega)$. For $v \in \mathcal{D}(A)$, we denote

$$\| v \|_A \equiv \| -\Delta v + v \|_{L^2(\Omega)} . \tag{2.1}$$

It is easily proven by Fourier analysis that the norm $\| v \|_A$ is equivalent to the norm $\| v \|_{H^2(\Omega)}$. Since A is positive defnite and self-adjoint, we can define the fractional powers of A (see Dunford and Schwartz (1958)). It is also easily proven by Fourier analysis that $\mathcal{D}(A^{3/2}) = H^3(\Omega)$ and that the norm $\| v \|_{A^{3/2}} \equiv \| A^{3/2} v \|_{L^2(\Omega)}$ is equivalent to $\| v \|_{H^3(\Omega)}$.

We also recall the continuous Sobolev embeddings (see Adams (1975)) for $n \le 2$

$$H^2(\Omega) \subset L^\infty(\Omega) \qquad \text{and} \qquad L^4(\Omega) \subset H^1(\Omega)$$

with positive constants c_1 and c_2 such that

$$\begin{aligned} \| v \|_{L^\infty(\Omega)} &\le c_1 \| v \|_{H^2(\Omega)}, \qquad v \in H^2(\Omega), \\ \| v \|_{L^4(\Omega)} &\le c_2 \| v \|_{H^1(\Omega)}, \qquad v \in H^1(\Omega). \end{aligned} \tag{2.2}$$

We note that it follows from (2.2) that

$$\| \frac{\partial v}{\partial x_i} \|_{L^4(\Omega)} \le c_2 \| v \|_{H^2(\Omega)}, \qquad i = 1, \ldots, n; \quad v \in H^2(\Omega).$$

We assume for initial conditions

$$u(x, 0) = u_0(x), \qquad x \in \Omega, \tag{2.3}$$

with $u_0 \in L^2(\Omega)$ that (1.1) has a unique solution $u(t)$ which satisfies

$$u \in L^2(0,T;H^1(\Omega)) \cap C([0,T);L^2(\Omega)) \tag{2.4}$$

for all $T > 0$. We further assume that

$$u(t) \in C((0,T);H^2(\Omega)) \tag{2.5}$$

and that (1.1) is absorbing in $H^2(\Omega)$; i.e., there exists $\rho > 0$ and $T^* = T^*(u_0)$ such that for $t \geq T^*(u_0)$

$$u(t) \in \mathcal{B}_{\rho/2} \equiv \{v \in H^2(\Omega) : \| v \|_{H^2(\Omega)} \leq \rho/2\} \tag{2.6}$$

for all $u_0 \in L^2(\Omega)$. For example, the techniques in Temam (1988) can be used to show that if

$$g(s) = \sum_{j=0}^{3} b_j s^j, \qquad b_3 > 0, \tag{2.7}$$

then the above existence and regularity results and the absorbing condition holds.

Next, we construct $\theta \in C_c^\infty([0,\infty))$ so that $0 \leq \theta(s) \leq 1$ and

$$\theta(s) = \begin{cases} 1 \text{ if } 0 \leq s \leq \rho/2 \\ 0 \text{ if } s \geq \rho \end{cases} \tag{2.8}$$

and

$$|\theta'(s)| \leq 4/\rho. \tag{2.9}$$

We then define the operator

$$F(v) = \theta(\| Av \|_{L^2(\Omega)}) g(v).$$

We shall show in the next section that $F : \mathcal{D}(A) \to \mathcal{D}(A)$ is Lipschitz continuous and we shall construct an inertial manifold for the evolution equation for $u(t) : \mathbb{R}^+ \to H \equiv L^2(\Omega)$

$$\begin{aligned} &u' + Au = F(u), \qquad t > 0, \\ &u(0) = u_0. \end{aligned} \tag{2.10}$$

We note that the global attractor of (1.1) is contained in the global attractor of (2.10). In fact, it can be shown that if ρ is sufficiently large, then the global attractor of (1.1) is equal to the global attractor of (2.10) under appropriate conditions on g.

3. The Elliptic Regularization Theorem. In this section, we review the theory for the construction of inertial manifolds by elliptic regularization given in Fabes, Luskin, and Sell (1988). Let $\{e_i\}_{i=1}^{\infty}$ be a basis of eigenvectors of A ,

$$Ae_i = \lambda_i e_i, \qquad i \geq 1,$$

with $0 < \lambda_i < \lambda_2 \leq \dots$ and assume that $\{e_i\}_{i=1}^{\infty}$ is orthonormal in $H \equiv L^2(\Omega)$. Let M and N be positive integers and set

$$\begin{aligned} H^p &= \text{span } \{e_1, \dots, e_M\}, \\ H^q &= \text{Closure span}\{e_{M+1}, \dots\}, \\ H^r &= H^{r(N)} = \text{span}\{e_{M+1}, \dots, e_{M+N}\}. \end{aligned}$$

Let $v \cdot w$ denote the inner product in $H = L^2(\Omega)$.

We shall prove that F satisfies the following properties, (3.1)-(3.9). The Theorem in Fabes, Luskin, and Sell (1988) which is stated at the end of this section will then give the existence of an M-dimensional manifold which is invariant for (2.10). We shall show that

$$F : \mathcal{D}(A) \to \mathcal{D}(A); \tag{3.1}$$

$$|F|_{1,\infty} \equiv \sup_{v \in \mathcal{D}(A)} \| F(v) \|_A < \infty; \tag{3.2}$$

$$\begin{aligned} &e_i \cdot F(p + \Phi(p)), \qquad p \in H^p, \\ &\text{is } C^2 \text{ when } \Phi : H^p \to H^{r(N)} \text{ is } \mathcal{C}^2; \end{aligned} \tag{3.3}$$

$$\begin{aligned} &F : \mathcal{D}(A^{3/2}) \to \mathcal{D}(A^{3/2}) \text{ and for positive constants } c_8 \text{ and } c_9, \\ &\| F(v) \|_{A^{3/2}} \leq c_8 + c_9 \| v \|_{A^{3/2}} . \end{aligned} \tag{3.4}$$

We also shall show that $F : \mathcal{D}(A) \to \mathcal{D}(A)$ is continuous and Gateaux differentiable everywhere and

$$\| DF \|_{1,1,\infty} \equiv \sup_{v \in \mathcal{D}(A)} \| DF(v) \|_{1,1} < \infty \tag{3.5}$$

where DF is the Gateaux derivative of F and

$$\| DF(v) \|_{1,1} = \sup_{\substack{w \in \mathcal{D}(A) \\ \|w\|_A = 1}} \| DF(v)w \|_A .$$

Let us define

$$|F|_{\mathrm{div},\infty} \equiv \sup_{v\in\mathfrak{D}(A)} |\sum_{\alpha=1}^{M} e_\alpha \cdot DF(v)e_\alpha|$$

and set

$$\tilde{\rho}(\xi) = \frac{2 \parallel DF \parallel_{1,1,\infty}}{\xi - 2 \parallel DF \parallel_{1,1,\infty}}.$$

We shall show that for some M sufficiently large

$$\lambda_{M+1} - \lambda_M \geq (2 + 4/\sqrt{3}) \parallel DF \parallel_{1,1,\infty}; \tag{3.6}$$

$$2\lambda_{M+1} + [1 - 2\tilde{\rho}(\lambda_{M+1} - \lambda_M) \parallel DF \parallel_{1,1,\infty} /\lambda_1] \sum_{\alpha=1}^{M} \lambda_\alpha > |F|_{\mathrm{div},\infty}; \tag{3.7}$$

$$\lambda_{M+1} > [\tilde{\rho}(\lambda_{M+1} - \lambda_M) + 1] \parallel DF \parallel_{1,1,\infty}; \tag{3.8}$$

and

$$\lambda_{M+1} \geq c_9. \tag{3.9}$$

The following result is given in Fabes, Luskin, and Sell (1988).

THEOREM 1. *If A and F satisfy (3.1)-(3.9) for the positive integer M, then there exists a globally Lipschitz continuous function $\Phi : H^p \to H^q \cap \mathfrak{D}(A)$ such that $\mathfrak{M} =$ Graph Φ is an M-dimensional manifold which is invariant for (2.10).*

In the next section, we shall prove the following result.

THEOREM 2. *There exists a positive integer M such that (3.1)-(3.9) are satisfied. Thus, there exists a globally Lipschitz continuous function $\Phi : H^p \to H^q \cap \mathfrak{D}(A)$ such that $\mathfrak{M} =$ Graph Φ is an M-dimensional manifold which is invariant for (2.10).*

4. Proof of Theorem 2.

To show that $F : \mathfrak{D}(A) \to \mathfrak{D}(A)$, we note that

$$\frac{\partial}{\partial x_i} g(v) = g'(v)\frac{\partial v}{\partial x_i} \tag{4.1}$$

and

$$\frac{\partial^2}{\partial x_i \partial x_j} g(v) = g'(v)\frac{\partial^2 v}{\partial x_i \partial x_j} + g''(v)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}. \tag{4.2}$$

Thus, (3.1) and (3.2) follow from the continuous embeddings $H^2(\Omega) \subset L^\infty(\Omega)$ and $L^4(\Omega) \subset H^1(\Omega)$ given by (2.2).

Next, (3.3) follows by Lebesque's Dominated Convergence Theorem since $g \in C^2$. Now

$$
\begin{aligned}
\frac{\partial^3}{\partial x_i \partial x_j \partial x_k} g(v) =& g'(v)\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_k} \\
&+g''(v)\left[\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial v}{\partial x_k} + \frac{\partial^2 v}{\partial x_i \partial x_k}\frac{\partial v}{\partial x_j} + \frac{\partial^2 v}{\partial x_j \partial x_k}\frac{\partial v}{\partial x_i}\right] \\
&+g'''(v)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\frac{\partial v}{\partial x_k}.
\end{aligned} \tag{4.3}
$$

So, by the Sobolev embeddings (2.2) there exist constants $c_{10} = c_{10}(\rho)$ and $c_{11} = c_{11}(\rho)$ such that

$$
\|g(v)\|_{H^3(\Omega)} \leq c_{10} + c_{11}\|v\|_{H^3(\Omega)} \qquad \text{for } v \in H^3(\Omega) \text{ and } \|v\|_{H^2(\Omega)} \leq \rho. \tag{4.4}
$$

Thus, (3.4) holds since

$$
\| F(v) \|_{H^3(\Omega)} \leq c_{10} + c_{11} \| v \|_{H^3(\Omega)}, \qquad v \in H^3(\Omega). \tag{4.5}
$$

For $v \in \mathcal{D}(A)$ the Gateaux derivative $DF(v)$ exists and for $w \in \mathcal{D}(A)$,

$$
DF(v)w = \theta'(\| v \|_A)\frac{Av \cdot Aw}{\| Av \|_{L^2(\Omega)}} g(v) + \theta(\| v \|_A) g'(v) w. \tag{4.6}
$$

Now from (4.1) and (4.2) we know there exists a positive constant $c_{12} = c_{12}(\rho)$ such that

$$
\| Ag(v) \|_{L^2(\Omega)} \leq c_{12} \qquad \text{for } v \in H^2(\Omega), \quad \| Av \|_{L^2(\Omega)} \leq \rho. \tag{4.7}
$$

Further,

$$
\frac{\partial}{\partial x_i}\{g'(v)w\} = g''(v)\frac{\partial v}{\partial x_i} w + g'(v)\frac{\partial w}{\partial x_i}
$$

and

$$
\frac{\partial^2}{\partial x_i \partial x_j}\{g'(v)w\} = g'''(v)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} w + g''(v)\left[\frac{\partial^2 v}{\partial x_i \partial x_j} w + \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + \frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_i}\right] + g'(v)\frac{\partial^2 w}{\partial x_i \partial x_j}.
$$

Thus, it follows from the Sobolev embeddings (2.2) that there exists a positive constant $c_{13} = c_{13}(\rho)$ such that

$$
\| Ag'(v)w \|_{L^2(\Omega)} \leq c_{13} \| Aw \|_{L^2(\Omega)}, \qquad \text{for } v, w \in H^2(\Omega), \quad \| Av \|_{L^2(\Omega)} \leq \rho. \tag{4.8}
$$

The result (3.5) now follows from (4.7) and (4.8).

Next, to estimate $|F|_{\mathrm{div},\infty}$ we observe that by Plancherel's Theorem, for $v \in \mathcal{D}(A)$

$$
\begin{aligned}
&|\sum_{\alpha=1}^{M} \left(\frac{Av \cdot Ae_\alpha}{\|Av\|_{L^2(\Omega)}} \right) (g(v) \cdot e_\alpha)| \\
(4.9) \qquad &\leq \left[\sum_{\alpha=1}^{M} \lambda_\alpha^2 \left(\frac{Av \cdot e_\alpha}{\| Av \|_{L^2(\Omega)}} \right)^2 \right]^{\frac{1}{2}} \left[\sum_{\alpha=1}^{M} (g(v) \cdot e_\alpha)^2 \right]^{\frac{1}{2}} \\
&\leq \lambda_M \|g(v)\|_{L^2(\Omega)}
\end{aligned}
$$

and

$$
(4.10) \qquad |\sum_{\alpha=1}^{M} g'(v) e_\alpha \cdot e_\alpha| \leq M \parallel g'(v) \parallel_{L^\infty(\Omega)} \parallel e_\alpha \parallel_{L^\infty(\Omega)}^2 .
$$

Now

$$
\parallel e_\alpha \parallel_{L^\infty(\Omega)}^2 = 2^n,
$$

so it follows from (4.9) and (4.10) that there exists positive constants $c_{14} = c_{14}(\rho)$ and $c_{15} = c_{15}(\rho)$ such that

$$
(4.11) \qquad |F|_{\mathrm{div},\infty} \leq c_{14}\lambda_M + c_{15} M.
$$

Now the eigenvalues of A are

$$
(4.12) \qquad (2\pi k)^2 + (2\pi l)^2 + 1
$$

for integers k and l, and it is known that there exist positive constants c_{16} and c_{17} such that

$$
(4.13) \qquad c_{16} j \leq \lambda_j \leq c_{17} j.
$$

Thus, (3.7) holds if

$$
(4.14) \qquad \tilde{\rho}(\lambda_{M+1} - \lambda_M) \parallel DF \parallel_{1,1,\infty} \leq \lambda_1/4
$$

and

$$
(4.15) \qquad \lambda_{M+1} + \frac{(M+1)M}{8} \geq (c_{14}c_{17} + c_{15})M/2.
$$

Hence, we see that (3.6)-(3.9) hold if λ_{M+1} and $\lambda_{M+1} - \lambda_M$ are sufficiently large. Now it follows from the results in Hardy and Wright (1962) and Richards (1982) that there are arbitrarily large gaps in the spectrum of A. This concludes the proof of the theorem.

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