BOUNDENESS OF SOLUTIONS OF DUFFING'S EQUATION

By

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BOUNDDEDNESS OF SOLUTIONS OF DUFFING'S EQUATION

by

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BOUNDDEDNESS OF SOLUTIONS OF DUFFING'S EQUATION

Abstract. J. Littlewood, L. Markus and J. Moser proposed independently the boundedness problem for solutions of Duffing's equation: \( \ddot{x} + g(x) = p(t) \), where \( p(t) \) is continuous and periodic and \( g(x) \) is superlinear at infinity. The purpose of this paper is to prove that all solutions of the above-mentioned Duffing's equation are bounded for \( t \in \mathbb{R} \) when \( p(t) \) is even (or when \( p(t) \) is odd and \( g(x) \) is odd).

I. Introduction

The study of the bounded solutions of Hamiltonian systems is closely related to the problem for the stability of motions, for example, the problem of collisions for \( n \) bodies [1]. It challenges one's attention since estimating the bounds for solutions of Hamiltonian systems is really not easy.


\[ \ddot{x} + g(x) = p(t) , \]

where \( p(t) \) is continuous and \( 2\pi \)-periodic in \( t \in \mathbb{R} \), and \( g(x) \) is continuously differentiable in \( x \in \mathbb{R} \) and satisfies

\[ \lim_{x \to \infty} \frac{1}{x} g(x) = \infty \], as \( |x| \to \infty \).

Up to now, the only contribution to the above problem is the work of G. Morris [5], in which he proves that each solution of the special Duffing's equation: \( \ddot{x} + 2x^3 = p(t) \), is bounded for \( t \in \mathbb{R} \). The purpose of this paper is to prove the following theorems.

Theorem 1. All solutions of the Duffing equation (1) are bounded for \( t \in \mathbb{R} \) whenever \( p(t) \) is even.
Theorem 2. All solutions of the Duffing equation (1) are bounded for $t \in \mathbb{R}$ whenever $p(t)$ is odd and $g(x)$ is odd.

Our method is based on a careful study of some Lagrange stable sets together with the twist property of the Poincaré map. The method of stable sets developed in this paper can be generalized further to measure-preserving homeomorphisms in $n$-dimensional space, and is useful in studying the bounded solutions for Hamiltonian systems [4].

II. Preliminaries

In the subsequent, we will need some basic facts stated in the following Propositions.

Proposition 1. Let $\{A_k\}$ be a sequence of nonempty connected compact sets in $\mathbb{R}^2$, and let

$$A_1 \supset A_2 \supset \ldots \supset A_k \supset \ldots .$$

Then the intersection $\bigcap_{k=1}^{\infty} A_k$ is nonempty, connected and compact (see [6, p. 163]).

Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism, and let $G \subset \mathbb{R}^2$ be a bounded connected open set. Then the image $\Phi(G)$ is also a bounded connected open set in $\mathbb{R}^2$, and $\partial \Phi(G) = \Phi(\partial G)$, where $\partial$ is the boundary operator for sets in $\mathbb{R}^2$.

Let $z_0 \in \mathbb{R}^2$ be a fixed point of $\Phi$. For any $r > 0$, denote by $B_r$ the open disk in $\mathbb{R}^2$ centered at the origin 0 with radius $r$, and denote the boundary $\partial B_r$ by $S_r$. Let $G \subset \mathbb{R}^2$ be a connected open set with $z_0 \in G$. Assume $r > |z_0|$, where $|z_0|$ is the distance from 0 to $z_0$. Then $\Phi(G) \cap B_r$ is a nonempty open set with $z_0$ as an interior point. Set

$$G_r = \sigma_{z_0} [\Phi(G) \cap B_r] ,$$

where $\sigma_{z_0} [\cdot]$ denotes the largest connected subset of $[\cdot]$ containing $z_0$. 

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Proposition 2. Let $G \subset \mathbb{R}^2$ be a connected open set, and let $z_0 \in G$ be a fixed point of $\Phi$. Let $r > |z_0|$ and assume that

$$\Phi(\overline{G}) \cap S_r \neq \emptyset,$$

(2)

where $\overline{G}$ denotes the closure of $G$. Then we have

$$\partial G_r^1 \cap S_r \neq \emptyset.$$

Proof. First, we assume that

$$\Phi(G) \subset B_r.$$

(3)

Then we have

$$G_r^1 = \sigma_{z_0} \Phi(G) \cap B_r = \sigma_{z_0} \Phi(G) = \Phi(G).$$

It follows from (3) that $\Phi(G) \cap S_r = \emptyset$. Then (2) yields the desired conclusion

$$\partial G_r^1 \cap S_r = \partial \Phi(G) \cap S_r = \Phi(\partial G) \cap S_r \neq \emptyset.$$

Next, assume (3) is not true. Then, since $\Phi(G)$ is open, there is at least one point $p \in G$ such that

$$\Phi(p) \in \mathbb{R}^2 \setminus B_r.$$

Since $z_0$ and $\Phi(p)$ belong to the connected open set $\Phi(G)$, there is a continuous curve

$$\Gamma: z = z(t), \quad (0 \leq t \leq 1),$$

in $\Phi(G)$, with $z(0) = z_0$ and $z(1) = \Phi(p)$. Hence, there is a unique $t_0 \in (0, 1)$ such that

$$z(t) \in \Phi(G) \cap B_r, \quad \text{for } t \in [0, t_0).$$
and $z(t_0) \in \Omega(G) \cap S_r$. This means that

$$z(t_0) \in \partial G^1_r \cap S_r,$$

and thus $\partial G^1_r \cap S_r \neq \emptyset$.

The proof of Proposition 2 is thus completed.

Let $x_1 < x_2$ and define

$$W = \left\{(x, y) \in \mathbb{R}^2 \mid x_1 < x < x_2, \ -\infty < y < \infty \right\},$$

and

$$L(x_k) = \left\{(x, y) \in \mathbb{R}^2 \mid x = x_k, \ -\infty < y < \infty \right\}, \ (k = 1, 2).$$

**Proposition 3.** If $E \subset \mathbb{R}^2$ is a bounded, connected open set and satisfies

$$E \cap L(x_k) \neq \emptyset, \text{ for } k = 1, 2,$$

then there is a connected closed set $\Lambda$ in $\partial E \cap \overline{W}$ with

$$\Lambda \cap L(x_k) \neq \emptyset, \text{ for } k = 1, 2.$$

(We will prove this proposition in the Appendix to this paper.)

**Proposition 4.** Let

$$\Gamma : z = z(t), \quad (0 \leq t \leq 1),$$

be a continuous curve in $\overline{W}$ with $z(0) \in L(x_1)$ and $z(1) \in L(x_2)$. Let $F \subset \overline{W}$ be a connected set. If there are two points $p_1$ and $p_2$ of $F$ such that $p_1$ is located above $\Gamma$ and $p_2$ is located below $\Gamma$, then we have

$$\Gamma \cap F \neq \emptyset.$$

(The proof is easy.)
III. **Lagrange Stable Sets**

A connected closed set \( K \subset \mathbb{R}^2 \) is said to be simply connected if for any bounded open set \( E \), "\( \partial E \subset K \)" implies "\( E \subset K \)."

Let \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \) be a homeomorphism. A set \( F \subset \mathbb{R}^2 \) is said to be positively (or negatively) Lagrange stable for \( \Phi \) if, for any point \( p \in F \), the sequence

\[
\{ \Phi^k(p) : k = 1, 2, \ldots, \text{(or } k = -1, -2, \ldots) \}
\]

is bounded.

**Lemma 1.** If \( \Phi : \mathbb{R}^2 \to \mathbb{R}^2 \) is an area-preserving homeomorphism with a fixed point \( z_0 \), then for any constant \( r > |z_0| \), \( \Phi \) has a positively Lagrange stable set \( K^+_r \) and a negatively Lagrange stable set \( K^-_r \) satisfying

i) \( K^+_r \) and \( K^-_r \) are simply connected closed sets passing through \( z_0 \);

ii) \( K^+_r \cap S_r \neq \emptyset \), \( K^-_r \cap S_r \neq \emptyset \), \( K^+_r \cup K^-_r \subset B_r \);

iii) \( \Phi(K^+_r) \subset K^+_r \), \( \Phi^{-1}(K^-_r) \subset K^-_r \).

**Proof.** We will break the proof up into the following steps.

**Step 1:** Let \( r > |z_0| \), and set \( H^1_r = B_r \). Then we have

\[
z_0 \in H^1_r \cap \Phi(H^1_r) ; \quad \bar{H}^1_r \cap S_r \neq \emptyset .
\]

If \( \Phi(\bar{H}^1_r) \cap S_r = \emptyset \), then we have

\[
\Phi(\bar{H}^1_r) \subset H^1_r .
\]

It follows that \( H^1_r \setminus \Phi(\bar{H}^1_r) \) is a nonempty set. Then we have

\[
\text{mes} \left[ H^1_r \right] - \text{mes} \left[ \Phi(\bar{H}^1_r) \right] = \text{mes} \left[ H^1_r \setminus \Phi(\bar{H}^1_r) \right] > 0 ,
\]

which is in conflict with the fact that \( \Phi \) is an area-preserving homeomorphism.
and $H^1_r$ is an open set in $\mathbb{R}^2$. Hence, we have

$$\Phi(H^1_r) \cap S_r \neq \emptyset.$$ 

Then set

$$H^2_r = \sigma^1_{z_0} [\Phi(H^1_r) \cap B_r].$$

It follows from Proposition 2 that

$$H^2_r \cap S_r \neq \emptyset.$$ 

Since $H^2_r \subset \Phi(H^1_r) \cap B_r \subset B_r = H^1_r$, we have

$$\Phi(H^2_r) \subset \Phi(H^1_r).$$

(4)

Note that $\Phi(H^2_r)$ is a bounded connected open set with $z_0 \in \Phi(H^2_r)$.

Assume $\Phi(H^2_r) \cap S_r = \emptyset$. Then we have

$$\Phi(H^2_r) \subset B_r.$$ 

It follows from (4) that

$$\Phi(H^2_r) \subset \Phi(H^1_r) \cap B_r,$$ 

and consequently

$$\Phi(H^2_r) \subset \sigma^1_{z_0} [\Phi(H^1_r) \cap B_r] = H^2_r.$$ 

Hence, the assumption $\Phi(H^2_r) \cap S_r = \emptyset$ implies that $H^2_r \setminus \Phi(H^2_r)$ is a nonempty set, and thus

$$\text{mes} [H^2_r] - \text{mes} [\Phi(H^2_r)] = \text{mes} [H^2_r \setminus \Phi(H^2_r)] > 0,$$ 

which is in conflict with
\[
\text{mes} \left[ H_r^2 \right] = \text{mes} \left[ \Phi(H_r^2) \right] = \text{mes} \left[ \Phi(H_r^2) \right].
\]

Then we have
\[
\Phi(H_r^2) \cap S_r \neq \emptyset.
\]

Now, set
\[
H_r^3 = \sigma_{z_0} \left[ \Phi(H_r^2) \cap B_r \right].
\]

It follows from Proposition 2 that
\[
H_r^3 \cap S_r \neq \emptyset.
\]

Since \( H_r^3 \subset \Phi(H_r^2) \cap B_r \subset \Phi(H_r^1) \cap B_r \), we have
\[
H_r^3 \subset \sigma_{z_0} \left[ \Phi(H_r^1) \cap B_r \right] = H_r^2
\]
and
\[
\Phi(H_r^3) \subset \Phi(H_r^2).
\]

Note that \( \Phi(H_r^3) \) is a bounded connected open set in \( \mathbb{R}^2 \) with \( z_0 \in \Phi(H_r^3) \).

Using the same technique as mentioned above, we can prove by induction that
\[
H_r^{k+1} = \sigma_{z_0} \left[ \Phi(H_r^k) \cap B_r \right], \quad (k = 1, 2, \ldots),
\]
are connected bounded open sets with
\[
H_r^1 \supset H_r^2 \supset \ldots \supset H_r^k \supset \ldots,
\]
\[
z_0 \in H_r^k \text{ and } H_r^k \cap S_r \neq \emptyset, \quad (k = 1, 2, \ldots).
\]

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Step 2: Set

\[ K_r^- = \bigcap_{k=1}^{\infty} \overline{H_r^k} , \quad (r > |z_0|) . \]

It follows from Proposition 1 that \( K_r^- \) is a connected compact set. Moreover, we have

\[ z_0 \in K_r^- , \quad K_r^- \subset \overline{B_r} , \quad K_r^- \cap S_r \neq \emptyset . \]

Since \( H_r^{k+1} \subset \Phi(H_r^k) \), we have

\[ \Phi^{-1}(H_r^k) \subset \overline{H_r^k} . \]

Then, from (5), we get

\[ K_r^- = \bigcap_{k=1}^{\infty} \overline{H_r^k} = \bigcap_{k=1}^{\infty} \overline{H_r^{k+1}} \subset \overline{B_r} . \]

Hence,

\[ \Phi^{-1}(K_r^-) = \bigcap_{k=1}^{\infty} \Phi^{-1}(H_r^{k+1}) \subset \bigcap_{k=1}^{\infty} \overline{H_r^k} = K_r^- , \]

which also implies that \( K_r^- \) is negatively Lagrange stable for \( \Phi \).

To indicate the related homeomorphism \( \Phi \) and the fixed point \( z_0 \), we denote \( K_r^- \) by \( K_r^{-}(z_0, \Phi) \).

Step 3: Since \( \Phi^{-1} \) is also an area-preserving homeomorphism with fixed point \( z_0 \), we get a connected compact set \( K_r^{-}(z_0, \Phi^{-1}) \). Define

\[ K_r^+ = K_r^+(z_0, \Phi) = K_r^{-}(z_0, \Phi^{-1}) . \]

Then \( K_r^+ \) is a connected compact set with

\[ z_0 \in K_r^+ , \quad K_r^+ \cap S_r \neq \emptyset , \quad K_r^+ \subset \overline{B_r} . \]
It can be seen by definition that

\[ \Phi(K_+^r) \subset K_+^r, \]

which also implies that \( K_+^r \) is positively Lagrange stable for \( \Phi \).

Step 4. It is not hard to verify that all \( \text{H}^{-k}_r \) (\( k = 1, 2, \ldots \)) are simply connected sets. Hence, \( K_r^- \) is a simply connected closed set, and so is \( K_r^+ \).

The proof of Lemma 1 is complete.

With an observation on the proof of Lemma 1, we can easily verify the following conclusion.

**Lemma 2.** For any constants \( r \) and \( s \) with \( s > r \), we have

\[ K_+^r \subset K_+^s, \quad K_r^- \subset K_s^- \]

IV. **Twist Property of the Poincaré Map**

The Duffing equation (1) is equivalent to a Hamiltonian system

\[
\dot{x} = H_y'(t, x, y), \quad \dot{y} = -H_x'(t, x, y), \tag{6}
\]

where

\[ H(t, x, y) = \frac{1}{2} y^2 + \int_0^x g(x) dx - p(t) x, \quad \text{for} \ (t, x, y) \in \mathbb{R}^3. \]

Let \((u, v) \in \mathbb{R}^2\). Denote by

\[ x = x(t, u, v), \quad y = y(t, u, v), \tag{7} \]

the solution of (6) satisfying the initial condition

\[ x(0, u, v) = u, \quad y(0, u, v) = v. \]

It is known that the solution (7) exists for all \( t \in \mathbb{R} \) and is differentiable in
\((t, u, v) \in \mathbb{R}^3\). Moreover, the Poincaré map

\[
\Phi : \mathbb{R}^2 \to \mathbb{R}^2; \quad (u, v) \mapsto (x(2\pi, u, v), y(2\pi, u, v))
\]

is an area-preserving diffeomorphism. It can be proved that the Poincaré map \(\Phi\) above admits an infinite number of fixed points \(z_i (i = 0, 1, 2, \ldots)\) (see [7]).

Let \(u = r \cos \theta, \ v = r \sin \theta \ (r \geq 0)\), and let

\[
x = R \cos \Theta, \quad y = R \sin \Theta,
\]

where \(R = R(t, r, \theta) = \left[x^2(t, u, v) + y^2(t, u, v)\right]^{1/2}\) is continuous in \((t, r, \theta) \in \mathbb{R} \times [0, \infty) \times \mathbb{R}\), and \(2\pi\)-periodic in \(\theta\). It can be seen that there is a constant \(a_0 > 0\) such that

\[
R(t, r, \theta) > 0, \quad \text{for } t \in [0, 2\pi] \text{ and } \theta \in \mathbb{R},
\]

whenever \(r \geq a_0\). In this case, (6) is equivalent to the following system

\[
\begin{cases}
\dot{R} = [R \cos \Theta - g(R \cos \Theta) + p(t)] \sin \Theta, \\
\dot{\Theta} = \frac{1}{R} g(R \cos \Theta) \cos \Theta - \sin^2 \Theta + \frac{1}{R} p(t) \cos \Theta.
\end{cases}
\]

It follows that \(R = R(t, r, \theta) > 0\) and \(\Theta = \Theta(t, r, \theta)\) are continuous in \((t, r, \theta) \in [0, 2\pi] \times [a_0, \infty) \times \mathbb{R}\), where \(R(0, r, \theta) = r\) and \(\Theta(0, r, \theta) = \theta\). Moreover,

\[
\Theta(t, r, \theta + 2\pi) = \Theta(t, r, \theta) + 2\pi.
\]

Hence, for \(r \geq a_0\), the Poincaré map \(\Phi\) can be put into the following form:

\[
\rho = R(2\pi, r, \theta), \quad \phi = \Theta(2\pi, r, \theta) + 2\pi \quad (8)
\]

where \(R(2\pi, r, \theta)\) and \(h(r, \theta) = \Theta(2\pi, r, \theta) - \theta\) are continuous in \((r, \theta) \in [a_0, \infty) \times \mathbb{R}\) and \(2\pi\)-periodic in \(\theta \in \mathbb{R}\).

The following Lemma is just the Lemma 4.3 in [7], which expresses the twist property of the Poincaré map \(\Phi\) for the Duffing equation (1).
Lemma 3. For any given integer \( m > 0 \), there is a constant \( a_1 > a_0 \) such that

\[
h(r, \theta) < -2m \pi, \quad \text{for} \quad r \geq a_1 \quad \text{and} \quad \theta \in \mathbb{R}.
\]

V. Property A

Let \( z_0 \) and \( z_0' \) be fixed points of the Poincaré map \( \Phi \) for the Duffing equation (1), and let

\[
K^+_r = K^+_r(z_0, \Phi), \quad K^-_r = K^-_r(z_0', \Phi)
\]

for \( r > |z_0| + |z_0'| \) (see Lemma 1).

Definition. \( K^+_r \) (or \( K^-_r \)) is said to possess Property A if and only if for any constant \( d > |z_0| \) (or \( d > |z_0'| \)) there is a constant \( s > d \) and a fixed point \( z_d \) of \( \Phi \) such that

\[
z_d \in K^+_s \setminus B_d \quad \text{(or,} \quad z_d \in K^-_s \setminus B_d).\]

Now, we prove the following

Lemma 4. If \( K^+_r \) and \( K^-_r \) possess Property A, then all solutions of (1) are bounded for \( t \in \mathbb{R} \).

Proof. First, we claim that for any constant \( a > 0 \) there is a constant \( b > a \) with

\[
B_a \subset K^+_b.
\]

Assume the contrary. Then there is a constant \( r_0 > |z_0| \) such that

\[
(B_r \setminus K^+_r) \cap B \neq \emptyset, \quad \text{for} \quad r > r_0.
\]

Without loss of generality, assume \( r_0 > a_0 > |z_0| \), where \( a_0 \) is the constant chosen in Section IV. Now, choose constants \( r_1 \) and \( r_2 \) (\( r_2 > r_1 > r_0 \)) with

\[
B_{r_0} \subset \Phi(B_{r_1}) \subset B_{r_2}
\]
Since $K^+_r$ has Property A, we can take $r_2$ such that $\Phi$ has a fixed point $z_1$ with
\[ z_1 \in K^+_r \cap S_{r_2}, \quad \text{for some } r_3 > r_2. \]

Then there is a connected closed set $F_1 \subset K^+_r$ with the property that
\[ z_1 \in F_1, \quad F_1 \subset \mathbb{R}[r_1, r_3], \]
and $F_1$ connects $z_1$ to $S_{r_1}$, where $\mathbb{R}[r_1, r_3]$ is the closed annulus bounded by $S_{r_1}$ and $S_{r_3}$. Let
\[ h_0 = \max|h(r, \theta)|, \quad \text{for } r \in [r_0, r_2] \text{ and } \theta \in \mathbb{R}, \]
where $h(r, \theta)$ is defined in (8). Then take two integers $\ell$ and $m$ satisfying
\[ \ell > h_0/2\pi, \quad m > \ell + 1. \quad (12) \]

Using Lemma 3, we choose a constant $d_0 > r_3$ such that
\[ h(r, \theta) < -2m\pi, \quad \text{for } r > d_0 \text{ and } \theta \in \mathbb{R}. \quad (13) \]

Similarly, choose constants $d_1, d_2, (d_2 > d_1 > d_0 > r_3)$ satisfying
\[ B_{d_0} \subset \Phi(B_{d_1}) \subset B_{d_2}, \quad (14) \]
such that there is a fixed point
\[ z_2 \in K^+_{d_1} \cap S_{d_0}. \]

Then we can take a connected closed set $F_2 \subset K^+_{d_1}$ in $\mathbb{R}[r_1, d_1]$ containing $z_2$ and connecting $z_2$ to $S_{r_1}$.

From (10), we have
\[ B_{d_2} \setminus K^+_{d_2} = \bigcup_{i=1}^w E_i. \]
where \( w \) is a positive integer or infinity, and each \( E_i \subset B_{d_2} \) is a nonempty connected open set with

\[
\partial E_i \subset K_{d_2}^+ \cup S_{d_2}, \quad (i = 1, \ldots, w). \]

Since \( K_{d_2}^+ \) is a simply connected compact set with \( K_{d_2}^+ \cap S_{d_2} \neq \emptyset \), \( A_i = \partial E_i \cap S_{d_2} \) is a nondegenerate closed arc of \( S_{d_2} \), \((i = 1, \ldots, w)\). It follows from (10) that there is at least one set, say \( E_1 \) for definiteness, such that

\[
E_1 \cap S_{r_0} \neq \emptyset, \quad (A_1 = \partial E_1 \cap S_{d_2} \neq \emptyset). \]

Now, we are going to prove the following inequality:

\[
K_{d_2}^+ \cap E_1 \neq \emptyset. \tag{15} \]

To this aim, take a continuous and simple curve

\[
C : z = z(t) = (\eta(t)\cos \alpha(t), \eta(t)\sin \alpha(t)), \quad (0 \leq t \leq 1),
\]

in \((E_1 \cup A_1) \cap \mathbb{R} \left[ r_0, d_2 \right] \) with \( z(0) \in S_{r_0} \) and \( z(1) \in S_{d_2} \). Since \( \theta = \alpha(t) \) is determined up to \( 2\pi \), we have

\[
\theta = \alpha(t) = \alpha_j(t) = \alpha_0(t) + 2j\pi, \quad (j \in \mathbb{Z}),
\]

where \( \alpha_0(t) \) is a single-valued continuous function \((0 \leq t \leq 1)\) with \( \alpha_0(0) = \theta_0 \in [0, 2\pi) \), and \( \alpha_0(1) = \theta_1 \) uniquely determined.

Now, consider \((r, \theta)\) as the Cartesian coordinates of the \((r, \theta)\)-plane. Set

\[
T_a = \{(r, \theta) \mid r > a \text{ and } \theta \in \mathbb{R}\}.
\]

Let \( \Phi^*: T_a \rightarrow T_0 \) be a mapping defined by (8); i.e., \( \Phi^* \) is a lifting of \( \Phi \) in \( \mathbb{R}^2 \setminus \mathbb{B}_{a_0} \).
Then we have an infinite number of continuous curves

\[ C^{(i)} : r = \rho(t), \quad \theta = \alpha_i(t), \quad (0 \leq t \leq 1), \]

in \( T_0 \) \( (i \in \mathbb{Z}) \), which are liftings of \( C \). Since \( C \cap K^+_{d_2} = \emptyset \), and \( K^+_{d_2} \) connects \( S_{d_2} \) and \( S_{r_0} \), then

\[ C^{(i)} \cap C^{(j)} = \emptyset, \quad \text{whenever } i \neq j. \]

Denote by \( L_a \) the vertical line \( r = a \) in \( T_0 \); i.e., \( L_a \) is the lifting of \( S_a \). Then \( C^{(i)}, C^{(i+1)}, L_{r_0} \), and \( L_{d_2} \) bound a simply connected open set \( V^{(i)} \) in \( T_0 \) for \( i \in \mathbb{Z} \).

Note that \( F_2 \subset K^+_{d_2} \) and \( F_1 \subset K^+_{r_3} \subset K^+_{d_2} \). Let \( z_1^{(i)}, z_2^{(i)}, F_1^{(i)} \) and \( F_2^{(i)} \) be the liftings of \( z_1, z_2, F_1 \) and \( F_2 \), respectively, with

\[ z_1^{(i)} \in F_1^{(i)} \subset V^{(i)}, \quad z_2^{(i)} \in F_2^{(i)} \subset V^{(i)}, \quad (i \in \mathbb{Z}). \]

There are two cases to be considered; i.e.,

Case I: \( F_1 \cap F_2 \neq \emptyset \);

Case II: \( F_1 \cap F_2 = \emptyset \).

For Case I, set

\[ F = F_1 \cup F_2. \]

Then \( F \subset R[r_1, d_1] \cap K^+_{d_2} \) is a connected closed set containing two fixed points \( z_1 \) and \( z_2 \). Let

\[ F^{(i)} = F_1^{(i)} \cup F_2^{(i)}. \]

Then \( F^{(i)} \subset V^{(i)} \) \( (i \in \mathbb{Z}) \) are liftings of \( F \).

Because of (11) and (14), the connected closed set \( \Phi^*(F^{(0)}) \) is located between \( L_{r_0} \) and \( L_{d_2} \). Since \( z_1 \) and \( z_2 \) are fixed points of \( \Phi \) with \( z_1^{(0)}, z_2^{(0)} \in F^{(0)} \), we have
\[ \Phi^*(z_1^{(0)}) = z_1^{(k)} , \quad \Phi^*(z_2^{(0)}) = z_2^{(n)} \]

for some integers \( k \) and \( n \). Hence,

\[ \Phi^*(F^{(0)}) \cap V^{(k)} \neq \emptyset , \quad \Phi^*(F^{(0)}) \cap V^{(n)} \neq \emptyset . \]

By (12) and (13), we get

\[ k \geq -\ell > -m+1 \geq n+1 , \]

which implies that \( V^{(k)} \cap V^{(n)} = \emptyset \). It follows that \( z_1^{(k)} \) is above \( C^{(-m)} \) and \( z_2^{(n)} \) is below \( C^{(-m)} \). Using Proposition 4, we have

\[ \Phi^*(F^{(0)}) \cap C^{(-m)} \neq \emptyset , \]

which yields

\[ \Phi(F) \cap C \neq \emptyset . \]

Note that

\[ \Phi(F) \subset \Phi(K_1^+) \subset K_2^+ , \quad C \subset E_1 \cup A_1 , \quad \Phi(F) \cap A_1 = \emptyset . \]

Then we conclude that (15) holds in Case I.

Now, we turn to consider case II. Let \( W \) be the open strip bounded by \( L_{r_1} \) and \( L_{d_1} \) in \( T_0 \), where \( d_1 \) is some constant \( (d_0 < d_1 \leq d_1) \) such that \( F_2^{(0)} \subset W \) connects \( L_{r_1} \) and \( L_{d_1} \). Then we have two simply connected open sets \( W_a \) and \( W_b \) in \( W \) separated by \( F_2^{(0)} \) such that \( W_a \) extends to infinity in the positive direction of the \( \theta \)-axis and \( W_b \) extends to infinity in the negative direction of the \( \theta \)-axis.

Since \( F_1 \cap F_2 = \emptyset \), we have

\[ F_1^{(i)} \cap F_2^{(j)} = \emptyset , \quad \text{for any } i \text{ and } j . \]

It follows that \( F_1^{(-1)} \) is below \( F_2^{(0)} \); i.e.,
Note that $z_{1}^{(-1)} \in F_{1}^{(-1)} \cap W_{b}$. Now, we have

$$
\Phi^{*}(z_{1}^{(-1)}) = z_{1}^{(k-1)} , \quad \Phi^{*}(z_{2}^{(0)}) = z_{2}^{(n)} ,
$$

for some integers $k$ and $n$. By (12) and (13), we have

$$
k \geq -\ell > -m+1 \geq n+1 . \tag{16}
$$

Note that $\Phi^{*}(W_{b})$ is a simply connected open set extending to infinity in the negative direction of the $\theta$-axis, and its boundary

$$
\partial \Phi^{*}(W_{b}) \subset \Phi^{*}(F_{2}^{(0)}) \cup \Phi^{*}(L_{r_{1}}) \cup \Phi^{*}(L_{d_{1}}) .
$$

Without loss of generality, we assume that $B_{2} \subset \Phi(B_{d_{1}})$. It follows from (11) that

$$
\Phi^{*}(F_{2}^{(0)}) \cap L_{r_{2}} \neq \emptyset .
$$

Moreover, $\Phi^{*}(L_{r_{1}}) \subset \partial \Phi^{*}(W_{b})$, and $\Phi^{*}(L_{d_{1}})$ is at the right of $L_{r_{2}}$. Hence, there is at least one point $q \in \Phi^{*}(F_{2}^{(0)}) \cap L_{r_{2}}$, which is above $z_{1}^{(k-1)}$,

$$
(z_{1}^{(k-1)} \in \Phi^{*}(L_{r_{2}}) \cap W_{b}) .
$$

It follows that $q \in \Phi^{*}(F_{2}^{(0)})$ is above $C^{(k-1)}$. Since $\Phi^{*}(z_{2}^{(0)}) = z_{2}^{(n)} \in \Phi^{*}(F_{2}^{(0)})$ is below $C^{(n+1)}$. Because of (16), $C^{(k-1)}$ is above $C^{(n+1)}$ or $C^{(k-1)} = C^{(n+1)}$. Then $\Phi^{*}(F_{2}^{(0)})$ has a point $q$ above $C^{(n+1)}$ and a point $z_{2}^{(n)}$ below $C^{(n+1)}$. Using Proposition 4, we have

$$
\Phi^{*}(F_{2}^{(0)}) \cap C^{(n+1)} \neq \emptyset ,
$$

which yields

$$
\Phi(F_{2}) \cap C \neq \emptyset .
$$

It follows that (15) is also valid in Case II.
However, (15) is in conflict with the definition of $E_1$. This contradiction proves the desired conclusion, (9).

In a similar way, it can be proved that for any constant $a > |z_0'|$ there is a constant $b > a$ such that

$$B_a \subset K_b^{-}. \quad (17)$$

By (9) and (17), for any constant $a > 0$, there is a constant $d > a$ such that $B_a \subset K_d^+ \cap K_d^-$. It follows that $B_a$ is a (positively and negatively) Lagrange stable set for the Poincaré map $\Phi$ of (1). Then, for any initial point $(u, v) \in B_a$, the solution (7) is bounded for $t \in \mathbb{R}$. Since $a$ is arbitrary, Lemma 4 is thus proved.

VI. Property $A_1^*$

Now, assume that $K_r^+$ (or $K_r^-$) does not possess Property A. Then there is a constant $b_0 > 0$ such that $K_r^+ \setminus B_{b_0}$ (or $K_r^- \setminus B_{b_0}$) contains no fixed point of $\Phi$ for any $r > b_0$. Let

$$B_r \setminus K_r^+ = \bigcup_{i=1}^{\infty} E^+_{r,i} \quad \text{(or, } B_r \setminus K_r^- = \bigcup_{i=1}^{\infty} E^-_{r,i})$$

where $E^+_{r,i}$ (or $E^-_{r,i}$) is a simply connected open set $(i = 1, 2, \ldots)$. Since $\Phi$ has infinitely many fixed points $\{z_j\}$ with $|z_j| \to \infty$ as $j \to \infty$, we have

$$z_k \in E^+_{r,i_k} \quad \text{(or } z_k \in E^-_{r,i_k}) \quad \text{for } r > |z_k| > b_0.$$

Then we have to consider the following properties.

Property $A_1^*$. There is a constant $b_1 > b_0$ such that for any $z_k \in E^+_{r,i_k}$ (or, $z_k \in E^-_{r,i_k}$) with $b_1 < z_k < r$, we have

$$\partial E^+_{r,i_k} \cap S_{b_1} \neq \emptyset \quad \text{(or, } \partial E^-_{r,i_k} \cap S_{b_1} \neq \emptyset).$$

Property $A_2^*$. For any constant $b > b_0$ there is a fixed point $z_k \in E^+_{r,i_k}$ (or, $z_k \in E^-_{r,i_k}$) such that
\[ \forall E^+_{r,i_k} \cap S_b = \emptyset \quad \text{(or, } \forall E^-_{r,i_k} \cap S_b = \emptyset) \]

with \( r > |z_k| > b \).

It follows that if \( K^+_r \) does not possess Property \( A \), then it possesses Property \( A^*_1 \) or Property \( A^*_2 \). Similarly, if \( K^-_r \) does not possess Property \( A \), then it possesses Property \( A^*_1 \) or Property \( A^*_2 \).

**Lemma 5.** \( K^+_r \) and \( K^-_r \) do not have Property \( A^*_1 \).

**Proof.** Since the proof is very similar to that of Lemma 4, we only give some key points and omit the details for the proof below.

Assume that \( K^+_r \) has Property \( A^*_1 \).

Then, choose constants \( r_0', r_1', r_2', d_0', d_1 \) and \( d_2 \)
\( (b_1 < r_0 < r_1 < r_2 < d_0 < d_1 < d_2) \) such that (11) and (14) are satisfied. Moreover, because of Property \( A^*_1 \), we can choose two fixed points
\[ z_1 \in E^+_{d_2',i_1} \cap S_{r_2}, \quad z_2 \in E^+_{d_2',i_2} \cap S_d \]
with
\[ \forall E^+_{d_2',i_1} \cap S_{r_1} \neq \emptyset, \quad \forall E^+_{d_2',i_2} \cap S_{d_1} \neq \emptyset. \]

Note that
\[ \forall E^+_{d_2',i_1} \cap S_{d_2} \neq \emptyset, \quad \forall E^+_{d_2',i_2} \cap S_{d_2} \neq \emptyset. \]

Then there are continuous and simple curves
\[ C_1 \subseteq E^+_{d_2',i_1} \cap R[r_1,d_2] \] joining \( z_1 \) to a point \( q_1 \in S_{r_1} \)
and
\[ C_2 \subseteq E^+_{d_2',i_2} \cap R[r_1,d_2] \] joining \( z_2 \) to a point \( q_2 \in S_{r_1} \).

Moreover, there is a continuous and simple curve
\[ C \subset (E_{d_2}^+, U A_{i_1}) \cap R[0, d_2] \]

connecting \( S_{r_0} \) and \( S_{d_2} \), where \( A_{i_1} = \partial E_{d_2}^+ \cap S_{d_2} \) is a nondegenerate closed arc of \( S_{d_2} \).

Since \( E_{d_2}^+, i_1 \) is open, we can assume that \( C_1 \) and \( C_2 \) do not intersect \( C \).

Using Proposition 3, we get a connected closed set

\[ F \subset (\partial E_{d_2}^+, i_1) \cap K_{d_2}^+ \cap R[1, d_1] \]

connecting \( S_{r_1} \) and \( S_{d_1} \).

Denote the liftings of \( z_1, z_2, C_1, C_2, C \) and \( F \) by \( z_1^{(i)}, z_2^{(i)}, C_1^{(i)}, C_2^{(i)}, C^{(i)} \) and \( F^{(i)} \), respectively (\( i \in \mathbb{Z} \)). Let \( V^{(i)} \) be the open set bounded by \( C^{(i)} \), \( C^{(i+1)} \), \( L_{r_0} \) and \( L_{d_2} \). Note that

\[ F^{(0)} \cap C_1^{(i)} = \emptyset, \]

and \( C_1^{(i)} \) joins \( z_1^{(i)} \) to \( q_1^{(i)} \in V^{(i)} \cap L_{r_1}, (i \in \mathbb{Z}) \). It follows that \( z_1^{(-1)} \) is below \( F^{(0)} \) (i.e., \( z_1^{(-1)} \) belongs to \( W_b \)) (see the proof of Lemma 4).

Similarly, it can be seen that \( z_2^{(1)} \) is above \( F^{(0)} \) (i.e., \( z_2^{(1)} \) belongs to \( W_a \)).

Then, employing a similar method used in the proof of Lemma 4 for Case II, we can prove

\[ \phi^*(F^{(0)}) \cap C^{(n+1)} = \emptyset. \]

It follows that

\[ K_d^+ \cap E_{d_2}^+, i_1 = \emptyset, \]

which is a contradiction. This proves that \( K_r^+ \) does not possess Property \( A_1^* \).

In a similar way, it can be proved that \( K_r^- \) does not possess Property \( A_1^* \).

Lemma 5 is then proved.
VII. Mutual Symmetry of $K^+_r$ and $K^-_r$

Now, assume that the $2\pi$-periodic function $p(t)$ is even; i.e., $p(-t) = p(t)$, for $t \in \mathbb{R}$.

Let

$$\xi = x, \quad \eta = -y, \quad \tau = -t. \tag{18}$$

Then, from (6), we get

$$\xi' = \eta, \quad \eta' = -g(\xi) + p(\tau), \tag{19}$$

where $\xi'$ and $\eta'$ are derivatives of $\xi$ and $\eta$ with respect to $\tau$. It follows from (7) and (18) that

$$\xi = x(-\tau, u, v), \quad \eta = -y(-\tau, u, v), \quad (\tau \in \mathbb{R}),$$

is a solution of (19) satisfying the initial condition

$$\xi(0) = u, \quad \eta(0) = -v. \tag{20}$$

On the other hand, since (6) and (19) are equations in the same form, we conclude from (7) that

$$\xi = x(\tau, u, -v), \quad \eta = y(\tau, u, -v)$$

is a solution of (19) satisfying the initial condition (20). Then the uniqueness theorem yields that

$$x(\tau, u, -v) = x(-\tau, u, v), \quad y(\tau, u, -v) = -y(-\tau, u, v)$$

for $\tau \in \mathbb{R}$. Hence, we have

$$x(-2\pi, u, -v) = x(2\pi, u, v), \quad y(-2\pi, u, -v) = -y(2\pi, u, v), \quad (21)$$

for $(u, v) \in \mathbb{R}^2$. 

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Note that
\[ \Phi : (u, v) \mapsto (x(2\pi, u, v), y(2\pi, u, v)) , \quad \text{for } (u, v) \in \mathbb{R}^2 \]
and
\[ \Phi^{-1} : (u, -v) \mapsto (x(-2\pi, u, -v), y(-2\pi, u, -v)) , \quad \text{for } (u, -v) \in \mathbb{R}^2 . \]

It follows from (21) that
\[ \Phi^{-1} : (u, -v) \mapsto (x(2\pi, u, v), -y(2\pi, u, v)) , \quad \text{for } (u, v) \in \mathbb{R}^2 . \]

Therefore, \( \Phi \) and \( \Phi^{-1} \) are mutually symmetric with respect to the x-axis in the sense that \( \Phi(x, y) = (u, v) \) if and only if \( \Phi^{-1}(x, -y) = (u, -v) \). It follows that if \( z_0 = (x_0, y_0) \) is a fixed point of \( \Phi \), then \( z_0' = (x_0, -y_0) \) is a fixed point of \( \Phi^{-1} \) and thus a fixed point of \( \Phi \). Hence, the set of fixed points of \( \Phi \) is symmetric to the x-axis in \( \mathbb{R}^2 \).

Using the symmetric property of \( \Phi \) and \( \Phi^{-1} \) mentioned above, we can verify directly that \( K^+_r(z_0, \Phi) \) and \( K^-_r(z_0', \Phi) \) (\( = K^+_r(z_0', \Phi^{-1}) \)) are mutually symmetric with respect to the x-axis in \( \mathbb{R}^2 \).

In summary, we obtain the following

**Lemma 6.** Assume that \( p(t) \) is even and \( z_0 = (x_0, y_0) \) is a fixed point of the Poincaré map \( \Phi \) of (1). Then \( z_0' = (x_0', -y_0) \) is a fixed point of \( \Phi \), and \( K^+_r = K^+_r(z_0, \Phi) \) and \( K^-_r = K^-_r(z_0, \Phi) \) are mutually symmetric with respect to the x-axis.

In a similar way, we can prove the following

**Lemma 7.** Assume that \( p(t) \) and \( g(x) \) are odd functions. Let \( z_0 = (x_0, y_0) \) be a fixed point of \( \Phi \). Then \( z_0' = (-x_0, y_0) \) is a fixed point of \( \Phi \), and \( K^+_r = K^+_r(z_0, \Phi) \) and \( K^-_r = K^-_r(z_0', \Phi) \) are mutually symmetric with respect to the y-axis.
VIII. Minimum Sets

From now on, we assume that Lemma 6 or Lemma 7 holds. Then $\hat{\Phi}$ and $\hat{\Phi}^{-1}$ are mutually symmetric (with respect to the x-axis or the y-axis). The set of fixed points of $\hat{\Phi}$ is then symmetric to the x-axis or the y-axis.

Let $z_0$ and $z_0'$ be two symmetric fixed points of $\hat{\Phi}$, and let $K^+_r = K^+_r(z_0, \hat{\Phi})$ and $K^-_r = K^-_r(z_0', \hat{\Phi})$. Then $K^+_r$ and $K^-_r$ are mutually symmetric to the x-axis or the y-axis. Let $z_1$ and $z_1'$ be two symmetric fixed points of $\hat{\Phi}$. Then $z_1 \in B_r \backslash K^+_r$ if and only if $z_1' \in B_r \backslash K^-_r$.

Denote the interior set of $K^+_r$ by $\text{int}[K^+_r]$. Then we have

$$K^+_r = \text{int}[K^+_r] \cup \partial K^+_r.$$ 

Since $\hat{\Phi}$ is an area-preserving homeomorphism and $\hat{\Phi}(K^+_r) \subset K^+_r$, we get

$$\hat{\Phi} \left( \text{int}[K^+_r] \right) = \text{int}[K^+_r], \quad \hat{\Phi}(\partial K^+_r) \subset \partial K^+_r.$$  \hspace{1cm} (22)

Set

$$G^+ = \left( \mathbb{R}^2 \cup \{\omega\} \right) \backslash K^+_r.$$ 

Since $K^+_r$ is a simply connected closed set in $\mathbb{R}^2$, $G^+$ is a simply connected open set in the extended plane for $\mathbb{R}^2$ and $\partial G^+ = \partial K^+_r$.

Let $z_1 \in B_r \backslash K^+_r (r > |z_1|)$, and let

$$B_r \backslash K^+_r = \bigcup_{i=1}^{\infty} E^+_i.$$ 

Then, without loss of generality, assume that $z_1 \in E^+_1$, where $E^+_1$ is a simply connected open set in $G^+$, and $A^+_1 = \partial E^+_1 \cap S^-_r$ is a nondegenerate closed arc of $S^-_r$. Let $F^+_1 = \partial E^+_1 \cap K^+_r$. Then $F^+_1$ is a simply connected closed set in $\partial K^+_r$ and $E^+_1$ is bounded by $F^+_1$ and $A^+_1$. 

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Assume that $r$ is a sufficiently large constant. Then there are constants $r_1$ and $r_2$ ($r > r_2 > r_1 > |z_1|$) such that
\[ \mathcal{B}_{r_1} \subset \Phi(B_{r_2}) \subset \mathcal{B}_r, \] (23)
which implies that $\Phi(S_{r_1}^{+}) \cap \mathcal{B}_{r_1}^{-} = \emptyset$. We can choose an open arc $C_i^+$ of $S_i$ in $E_i^+$ and a simply connected closed set $J_i^+ \subset F_1^+$ such that $C_i^+$ and $J_i^+$ bound a simply connected open set $H_i^+ \subset E_i^+$ with $z_i \in H_i^+$, for $i = 1, 2$. Then we have

\[ J_1^+ \subset J_2^+, \quad H_1^+ \subset H_2^+ \subset \mathcal{G}^+. \]

**Lemma 8.** Let $J_1^+$ and $J_2^+$ be defined as above. Then we have
\[ J_1^+ \subset \Phi(J_2^+) \subset F_1^+. \] (24)

**Proof.** First, we claim that
\[ \Phi(J_2^+) \cap J_2^+ \neq \emptyset. \] (25)

Since $\Phi$ is area-preserving, we have
\[ \Phi(H_2^+) \cap \text{int}[K_T^+] = \emptyset, \]
which yields
\[ \Phi(H_2^+) \subset \mathcal{G}^+. \]

Hence, we have
\[ \Phi(H_2^+) \subset \mathcal{G}^+, \quad (H_2^+ \subset \mathcal{G}^+). \]

Now, assume that (25) is false.

Then $\Phi(J_2^+) \cap J_2^+ = \emptyset$. Note that $\Phi(J_2^+) \cap H_2^+ = \emptyset$ and $\Phi(J_2^+) \cap C_2^+ = \emptyset$. It follows that
\[ \Phi(J_2^+) \cap \overline{H_2^+} = \emptyset. \]
Since \( z_1 \in \Phi(H_2^+) \), \( z_1 \) is surrounded by \( \partial \Phi(H_2^+) = \Phi(J_2^+) \cup \Phi(C_2^+) \) in \( G^+ \). It follows that

\[
\Phi(C_2^+) \cap \overline{B}_{r_1}^+ \neq \emptyset,
\]

which is in conflict with (23). This contradiction proves the desired conclusion, (25).

Note that \( J_2^+ \subset F_1^+, \Phi(J_2^+) \subset \partial K_r^+ \) and \( \Phi(J_2^+) \cap G^+ = \emptyset \). Then, (23) and (25) imply (24).

The proof of Lemma 8 is thus completed.

Let

\[
I_i^+ = \Phi(J_i^+) , \quad \text{for } i = 1, 2.
\]

Then we have

\[
I_1^+ \subset I_2^+ \subset F_1^+ \subset \partial K_r^+,
\]

and (24) is equivalent to

\[
\Phi^{-1}(I_1^+) \subset I_2^+ \subset F_1^+.
\] (26)

Similarly, \( z_1' \in B_r \setminus K_r^- \) (\( r > |z_1'| \)), and

\[
B_r \setminus K_r^- = \bigcup_{i=1}^{\infty} E_i^-.
\]

Without loss of generality, assume that \( z_1' \in E_1^- \), where \( E_1^- \) is a simply connected open set. Let

\[
F_1^- = \partial E_1^- \cap K_r^-,
\]

and so on. In a similar way mentioned above, we get

\[
I_1^- \subset I_2^- \subset F_1^- \subset \partial K_r^-.
\]
and
\[ \Phi(I^-_1) \subset I^-_2 \subset F^-_1, \quad \text{(27)} \]

where \( I^-_1, I^-_2 \) and \( F^-_1 \) are symmetric to \( I^+_1, I^+_2 \) and \( F^+_1 \) (with respect to the x-axis or the y-axis), respectively.

Let \( C \) be a continuous and simple curve in
\[ (B_r \setminus K_r^+) \cap R[r_0, r_1] \quad \text{(or, } (B_r \setminus K_r^-) \cap R[r_0, r_1]) \]
joining \( S_{r_0} \) and \( S_{r_1} \). Define a continuous function
\[ \psi(z) = \arg z - \theta_0, \quad \text{for } z \in C, \]
where \( \theta_0 = \arg p_0 \) \( (0 \leq \theta_0 < 2\pi) \) and \( p_0 \) is the end point of \( C \) on \( S_{r_0} \). Then, with an observation of the method employed in the proof of Lemma 4, one can get the following conclusion without difficulty.

For any given constant \( \alpha > 0 \), we have
\[ \sup_{z \in C} [\psi(z)] > \alpha \quad \text{or} \quad \inf_{z \in C} [\psi(z)] < -\alpha, \]
whenever \( r_1 \) is sufficiently large.

This conclusion together with the mutual symmetry of \( I^+_1 \) and \( I^-_1 \) (with respect to the x-axis or y-axis) implies that \( I^+_1 \) and \( I^-_1 \) intersect each other when \( r_1 \) is large enough. Moreover, there are simply connected closed sets
\[ \Lambda^+ \subset I^+_1, \quad \Lambda^- \subset I^-_1 \]

such that \( \Lambda^+ \) and \( \Lambda^- \) bound a simply connected open set \( D \subset F^+_1 \) with \( z_1 \in D \), provided that \( r_1 \) is large enough. It follows from (24) and (27) that
\[ \Phi(\Lambda^+) \subset F^+_1, \quad \Phi(\Lambda^-) \subset F^-_1 \quad \text{(28)} \]
which are basic conditions for the following discussion.
To make the arguments clear, we use the following briefly sketched figures.

Let $p, q \in \Lambda^+ \cap \Lambda^-$ (Figure 1), and let

$$p_1 = \Phi(p), \quad q_1 = \Phi(q).$$

Since $D$ is a simply connected open set with $z_1 \in D$, $\Phi(D)$ is also a simply connected open set containing $z_1$. Because of (28), we have

$$p_1, q_1 \in F_1^+ \cap F_1^-.$$

It follows that if $p_1$ does not belong to $\Lambda^+$, then $\Phi(D)$ is not a simply connected open set (Figure 2). Hence, we have

$$p_1 \in \Lambda^+.$$

In a similar way, we get

$$q_1 \in \Lambda^+.$$
Then we have

\[ \phi(\Lambda^+) \subseteq \Lambda^+ , \]

which implies the existence of a minimum set of \( \phi \) in \( \Lambda^+ \). Hence, we have proved the following

**Lemma 9.** Under the conditions mentioned above, there exists a minimum set \( M \) of \( \phi \) in \( F_1^+ \).

**IX. Property \( A^*_2 \)**

Assume that Lemma 6 or 7 holds, and let \( K^+_r \) and \( K^-_r \) be defined as in Section VIII.

**Lemma 10.** If \( K^+_r \) and \( K^-_r \) possess Property \( A^*_2 \), then all solutions of the Duffing equation (1) are bounded for \( t \in \mathbb{R} \).

**Proof.** It suffices to prove that for any given constant \( a > 0 \) there is a constant \( b > a \) such that

\[ B_a \subseteq K^+_b \quad \left( B_a \subseteq K^-_b \right) . \]

Assume the contrary. Then there is a constant \( r_0 > |z_0| \) such that

\[ (B_r \setminus K^+_r) \cap B_{r_0} \neq \emptyset , \quad \text{for } r > r_0 . \]

Let

\[ B_r \setminus K^+_r = \bigcup_{i=1}^{\infty} E^+_{r, i} \]

where \( E^+_{r, i} \) is a simply connected open set \( (i = 1, 2, \ldots) \). Recalling the proof of Lemma 4, we get at least one set, say \( E^+_{r, 1} \) for definiteness, such that

\[ \partial E^+_{r, 1} \cap S_{r_0} \neq \emptyset , \quad \partial E^+_{r, 1} \cap S_r \neq \emptyset . \]
Since $K_{r}^{+}$ possesses Property $A_{r}^{*}$, then for any constant $r_{1} > r_{0}$ there is a fixed point $z_{1} \in E_{r, i_{1}}^{+}$ such that

$$\partial E_{r, i_{1}}^{+} \cap S_{r_{1}} = \emptyset.$$ 

It follows from Lemma 9 that there is a minimum set $M_{1} \subset \mathbb{R} \cap [r_{1}, r_{2}]$ of $\Phi$ such that

$$M_{1} \subset K_{r_{3}, i_{1}}^{+} \cap \partial E_{r_{3}, i_{1}}^{+} = F_{r_{3}, i_{1}}^{+},$$

provided that $r_{3}$ is large enough. Note that $F_{r_{3}, i_{1}}^{+}$ is simply connected, and

$$M_{1} \subset K_{r_{3}}^{+} \cap \mathbb{R} \cap [r_{1}, r_{2}].$$

Since $F_{r_{3}, i_{1}}^{+} \subset \partial K_{r_{3}}^{+}$ and $K_{r_{3}, i_{1}}^{+} \cap \partial E_{r_{3}, i_{1}}^{+} \subset \partial K_{r_{3}}^{+}$ are simply connected closed sets and $\partial K_{r_{3}}^{+}$ is connected with

$$(K_{r_{3}}^{+} \cap \partial E_{r_{3}, i_{1}}^{+}) \cap S_{r_{0}} \neq \emptyset,$$

there is a connected closed set $F_{1}$ in $\partial K_{r_{3}}^{+} \cap \mathbb{R} \cap [r_{1}, r_{3}]$, which contains $M_{1}$ and intersects $S_{r_{1}}$. Moreover, $F_{1}$ can be chosen to be simply connected since $F_{r_{3}, i_{1}}^{+}$ is simply connected.

Similarly, using Lemma 9, we get a minimum set $M_{2} \subset K_{d_{2}}^{+} \cap \mathbb{R} [d_{0}, d_{1}]$ for $\Phi$ and a connected closed set $F_{2}$ in $\partial K_{d_{2}}^{+} \cap \mathbb{R} \cap [r_{1}, d_{1}]$, which contains $M_{2}$ and intersects $S_{r_{1}}$.

Without loss of generality, assume

$$r_{0} < r_{1} < r_{2} < r_{3} < d_{0} < d_{1} < d_{2}.$$ 

such that (11) and (14) are satisfied. Then,
\[ M_1 \subseteq F_1 \subseteq K^+_d, \quad M_2 \subseteq F_2 \subseteq K^+_d. \]

Since
\[ \partial E^+_{d_2,1} \cap S_{r_0} \neq \emptyset, \quad \partial E^+_{d_2,1} \cap S_{d_2} \neq \emptyset, \]
there is a continuous and simple curve \( C \) in \( (E^+_{d_2,1} \cup S_{d_2}) \cap R[r_0, d_2] \) connecting \( S_{r_0} \) and \( S_{d_2} \).

Then, using a similar technique employed in the proof of Lemma 4 and replacing the fixed points \( z_1 \) and \( z_2 \) by the minimum sets \( M_1 \) and \( M_2 \), we get a contradiction.

Lemma 10 is then proved.

X. Main Theorems

Now we are in a position to prove the following main theorems of this paper.

**Theorem 1.** If \( p(t) \) is even, then all solutions of the Duffing equation (1) are bounded for \( t \in \mathbb{R} \).

**Proof.** It follows from Lemma 6 that \( K^+_r \) and \( K^-_r \) are mutually symmetric with respect to the x-axis. Hence, both \( K^+_r \) and \( K^-_r \) satisfy Property A, or Property \( A^*_1 \), or Property \( A^*_2 \). Then Theorem 1 is an immediate consequence of Lemma 4, Lemma 5 and Lemma 10.

Similarly, we can prove the following

**Theorem 2.** If \( p(t) \) is odd and \( g(x) \) is odd, then all solutions of the Duffing equation (1) are bounded for \( t \in \mathbb{R} \).

Appendix

In this Appendix, we prove the following

**Proposition 3.** If \( E \subseteq \mathbb{R}^2 \) is a bounded, connected open set and satisfies
\[ E \cap L(x_k) \neq \emptyset, \quad \text{for } k = 1, 2, \]

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then there is a connected closed set $\Lambda \subset \partial E \cap \overline{W}$ with $\Lambda \cap L(x_k) \neq \emptyset$, $(k = 1, 2)$, where $L(x_k)$ and $W$ are defined in Section II.

**Proof.** Recall that a set $F \subset \mathbb{R}^2$ is connected if and only if it is impossible to find two disjoint open sets $G_1$ and $G_2$ whose union contains $F$ while neither $G_1$ nor $G_2$ alone contains $F$.

Let $F$ be any set in $\mathbb{R}^2$, and let $x, y \in F$. If it is impossible to find two disjoint open sets $G_1$ and $G_2$ such that

$$x \in G_1, \quad y \in G_2 \quad \text{and} \quad F \subset G_1 \cup G_2,$$

then $x$ is said to be connected with $y$ in $F$, and this relation is denoted by the following simple notation

$$x \sim_F y.$$

Then it follows that

i) $x \sim_F x$, for any $x \in F$;

ii) if $x \sim_F y$, then $y \sim_F x$;

iii) if $x \sim_F y$ and $y \sim_F z$, then $x \sim_F z$.

This equivalence relationship induces the following partition of $F$:

$$F = \bigcup_{\alpha \in \Delta} F_{\alpha}$$

where $\Delta$ is some index set with

1) $F_{\alpha} \cap F_{\beta} = \emptyset$ whenever $\alpha \neq \beta$ and $\alpha, \beta \in \Delta$;

2) $x, y \in F_{\alpha}$ if and only if $x \sim_F y$.

(see [6]).

**Lemma A.** If $F$ is closed, then $F_{\alpha}$ is closed for any $\alpha \in \Delta$. 

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In fact, if $F_\alpha$ is not closed for some $\alpha \in \Delta$, then there is a sequence
$\{x_k\} \subset F$ such that $x_n \to x_0$ as $n \to \infty$, and $x_0 \notin F_\alpha$. Since $\{x_k\} \subset F_\alpha \subset F$ and
$F$ is closed, we have

$$x_0 \in F_\beta \quad \text{for some } \beta \in \Delta, \text{ with } \beta \neq \alpha.$$ 

Hence $x_1$ is not connected with $x_0$ in $F$. There are two disjoint sets $G_1$ and
$G_0$ such that

$$x_1 \in G_1, \quad x_0 \in G_0, \quad \text{and } F \subset G_1 \cup G_0.$$ 

Since $x_n \in F_\alpha$ (n > 1) or $x_n \sim x_1$ (n > 1), we have

$$x_n \in G_1 \text{ and } x_n \notin G_0.$$ 

On the other hand, since $x_n \to x_0$ and $x_0 \notin G_\alpha$, there is a positive integer
$n_0$ such that

$$x_n \in G_0, \text{ whenever } n > n_0.$$ 

This contradiction proves Lemma A.

**Lemma B.** If $F$ is closed and bounded, then $F_\alpha$ is connected for any $\alpha \in \Delta$.

Assume on the contrary that $F_\alpha$ is not connected for some $\alpha \in \Delta$. Then
there are two disjoint open sets $G_1$ and $G_2$ such that

$$G_1 \cap F_\alpha \neq \emptyset, \quad G_2 \cap F_\alpha \neq \emptyset, \quad \text{and } F_\alpha \subset G_1 \cup G_2.$$ 

Note that $F_\alpha$ is bounded and closed. Then we can assume that the above open sets
$G_1$ and $G_2$ are bounded. Hence, we have

$$\text{dist}(F_\alpha, \partial G_1) > 0, \quad \text{dist}(F_\alpha, \partial G_2) > 0.$$ 

Then we can assume further that
\[ \overline{G}_1 \cap \overline{G}_2 = \emptyset. \]

It follows that, for any point \( p \in \mathcal{F} \), we can choose a neighborhood \( N_p \) of \( p \) satisfying

\[ N_p \cap G_1 = \emptyset \quad \text{or} \quad N_p \cap G_2 = \emptyset. \]

Set

\[ G_1^* = G_1 \cup \left[ \bigcup_{N_p \cap G_1 \neq \emptyset} N_p \right] \cup \left[ \bigcup_{N_p \cap (G_1 \cup G_2) = \emptyset} N_p \right], \]

\[ G_2^* = G_2 \cup \left[ \bigcup_{N_p \cap G_2 \neq \emptyset} N_p \right]. \]

Then \( G_1^* \) and \( G_2^* \) are two disjoint open sets whose union contains \( \mathcal{F} \) while

\[ \mathcal{F} \cap G_1^* \supseteq \mathcal{F}, \quad \mathcal{F} \cap G_2^* \supseteq \mathcal{F}, \quad \mathcal{F} \cap G_1^* \cap G_2^* \neq \emptyset. \]

Let \( x \in \mathcal{F} \cap G_1^* \) and \( y \in \mathcal{F} \cap G_2^* \). Then \( x \in G_1^* \) and \( y \in G_2^* \). It follows that \( x \) is not connected with \( y \) in \( \mathcal{F} \). This contradicts that \( x, y \in \mathcal{F} \) if and only if \( x \) is connected with \( y \) in \( \mathcal{F} \). Lemma B is thus proved.

Now, we are going to prove the above-mentioned Proposition 3.

Let \( E_1 \supseteq \mathcal{E} \) be a simply connected open set with \( \partial E_1 \subseteq \partial \mathcal{E} \), and let

\[ F = \partial E_1 \cap \overline{\mathcal{W}}. \]

Then \( F \) is bounded and closed. From Lemma B, we have

\[ F = \bigcup_{\alpha \in \Delta} F_\alpha, \]

where \( F_\alpha \) is connected and compact for any \( \alpha \in \Delta \). Since \( E_1 \) is simply connected, the relation
does not hold for any \( \alpha \in \Delta \). Then, for any \( \alpha \in \Delta \), we have

\[
F_{\alpha} \cap L(x_1) \neq \emptyset \quad \text{or} \quad F_{\alpha} \cap L(x_2) \neq \emptyset .
\]

We claim that there is at least one \( \alpha \in \Delta \) satisfying

\[
F_{\alpha} \cap L(x_1) \neq \emptyset \quad \text{and} \quad F_{\alpha} \cap L(x_2) \neq \emptyset .
\]

On the contrary, we have the following alternatives:

(I) \[
F_{\alpha} \cap L(x_1) \neq \emptyset \quad \text{and} \quad F_{\alpha} \cap L(x_2) = \emptyset ,
\]
or

(II) \[
F_{\alpha} \cap L(x_2) = \emptyset \quad \text{and} \quad F_{\alpha} \cap L(x_2) \neq \emptyset ,
\]

for any \( \alpha \in \Delta \). Hence, we have

\[
F = H_1 \cup H_2
\]

where

\[
H_1 = \bigcup F_{\alpha_1} \quad \text{(in which } F_{\alpha_1} \text{ are in case (I)),}
\]

and

\[
H_2 = \bigcup F_{\alpha_2} \quad \text{(in which } F_{\alpha_2} \text{ are in case (II)).}
\]

Then \( H_1 \) and \( H_2 \) are disjoint.

We want to prove that \( H_1 \) and \( H_2 \) are closed.

Assume \( H_1 \) is not closed. Then there is a sequence \( \{x_n\} \subset H_1 \) with \( x_n \to x_0 \) as \( n \to \infty \) such that \( x_0 \notin H_1 \). Since \( F \) is closed, we have \( x_0 \in H_2 \).

Then there is an index \( \alpha_2^0 \) in case (II) such that

\[
x_0 \in F_{\alpha_2^0}
\]

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Let \( x_n \in F_{\alpha_n} \), where \( F_{\alpha_n} \) is in Case (I), \((n = 1, 2, \ldots)\), and let \( a_n \in L(x_n) \cap F_{\alpha_n} \), \((n = 1, 2, \ldots)\).

Then \( \{a_n\} \) is bounded. Without loss of generality, we can assume

\[
a_n \to a_0, \quad \text{as } n \to \infty.
\]

Since \( L(x_n) \cap F \) is closed, \( a_0 \in F_{\alpha_1}^0 \) for some \( \alpha_1 \) in case (I).

It follows that \( a_0 \) is not connected with \( x_0 \) in \( F \). Then there are two disjoint open sets \( G_1^0 \) and \( G_2^0 \) with

\[
a_0 \in G_1^0, \quad x_0 \in G_2^0, \quad F \subseteq G_1^0 \cup G_2^0.
\]

Since \( a_n \to a_0 \) and \( x_n \to x_0 \) as \( n \to \infty \), there is a positive integer \( n_0 \) such that

\[
a_n \in G_1^0, \quad x_n \in G_2^0, \quad \text{whenever } n > n_0,
\]

which together with the properties of \( G_1^0 \) and \( G_2^0 \) implies that \( a_n \) is not connected with \( x_n \) in \( F \). This is, however, in conflict with

\[
a_n, x_n \in F_{\alpha_n}^*.
\]

Hence, \( H_1 \) is closed. Similarly, it can be proved that \( H_2 \) is closed.

Hence, \( H_1 \) and \( H_2 \) are two disjoint compact sets with the properties that

\[
H_1 \cup L(x_1) \quad \text{and} \quad H_2 \cup L(x_2)
\]

are connected sets. Therefore, we can construct two open sets (neighborhoods) \( V_1 \) and \( V_2 \) such that

\[
V_1 \supset H_1, \quad V_2 \supset H_2, \quad V_1 \cap V_2 = \emptyset, \quad V_1 \cap L(x_2) = \emptyset, \quad V_2 \cap L(x_1) = \emptyset.
\]
Set
\[ G_1 = V_1 \cup \{(x,y) \mid x < x_1, \ -\infty < y < \infty \} , \]
\[ G_2 = V_2 \cup \{(x,y) \mid x > x_2, \ -\infty < y < \infty \} . \]

Then \( G_1 \) and \( G_2 \) are two disjoint open sets whose union contains \( \partial E_1 \). This proves that \( \partial E_1 \) is not connected. However, this conclusion contradicts that \( E_1 \) is simply connected. Hence, there is at least one \( F_\alpha \) such that \( F_\alpha \) is a connected closed set in \( \partial E_1 \cap \overline{W} \) with
\[ F_\alpha \cap L(x_1) \neq \emptyset , \quad F_\alpha \cap L(x_2) \neq \emptyset . \]

Since \( \partial E_1 \subset \partial E \), then \( F_\alpha \subset \partial E \cap \overline{W} \). Let
\[ \Lambda = F_\alpha . \]

Then we arrive at the desired conclusion of Proposition 3.

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