SHORT WAVE INSTABILITIES RESULTING FROM MEMORY SLIP

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Short wave instabilities resulting from memory slip

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Abstract

It is shown that the equations governing the flow of an upper convected Maxwell fluid can become ill-posed if the no slip condition at the wall is replaced by a law of "memory slip" which relates the slip velocity to the history of the shear stress.

1. Introduction

Recent experiments by Ramamurthy [1] have demonstrated that the onset of melt fracture is associated with the development of slip at the wall. A sequence of several different instabilities can be observed, see e.g. Kalika and Denn [2]. After slip begins, short lengthscale irregularities develop which manifest themselves as loss of gloss and "sharkskin". These irregularities seem to be more or less confined to the boundary, while the overall flow appears to be steady. At higher shear rates, sharkskin regions with slow slip alternate with glossy regions of rapid slip (stick-slip regime). As the shear rate is increased still further, the extrudate again becomes completely glossy, and wavy distortions appear.

The mechanism by which slip occurs and how slip leads to the observed instabilities are not understood. Twenty years ago Pearson and Petrie [3] investigated the stability of shear flows of various fluids when the slip velocity at the wall is a given function of the shear stress: $u = f(\tau)$. As one might expect, they found instabilities when f is decreasing. However, no instabilities are found when f is increasing. Their analysis may be relevant in the stick-slip regime, but does not seem to be capable of explaining what happens in the sharkskin and wavy regimes.

The observed phenomena in the sharkskin regime suggest an instability to short wave disturbances, with eigenmodes that are essentially confined to the boundary. In recent work by the author [4], such instabilities were found for elastic solids sliding under Coulomb friction. The analysis of [4] easily carries over to Maxwell fluids, but its relevance appears doubtful. While significant effects of pressure on slip have been observed in rubber compounds [5], the effect of pressure in polymer melts seems to be controversial and in any case not very large. This makes it unlikely that the instability criterion derived in [4] would apply. In the present paper, we investigate a different possibility, namely that of memory slip. Rather than postulating that the slip velocity is a function of the shear stress, we make it depend on the history of the shear stress; specifically, we postulate a relationship of the form $\frac{du}{dt} + \eta u = f(\tau)$. On a physical level, a relationship of this kind may arise from a competition between high stresses, which cause loss of adhesion, and a relaxation mechanism, by which adhesion is restored. The possibility of memory slip was suggested in [3], but its consequences were not fully explored. We shall see that short wave instabilities occur in this case. These instabilities are an instance of ill-posedness, since the growth rate tends to infinity as the wavelength tends to zero. In that respect, the instability is similar to Hadamard instabilities (see e.g. [6]), which have been proposed as an explanation for melt fracture in the past. The main difference is that, while Hadamard instabilities occur in the interior of the flow domain, the instability discussed here is confined to the boundary.

2. Ill-posedness for a Maxwell fluid with memory slip

We consider two-dimensional flow of an upper convected Maxwell fluid with the constitutive law

$$\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T + \Lambda\mathbf{T} = \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T). \tag{1}$$

In components, we write

$$\mathbf{v} = (u, v),\ \mathbf{T} = \begin{pmatrix} \sigma & \tau \\ \tau & \gamma \end{pmatrix}. \tag{2}$$

We consider perturbations of a basic shear flow in the domain $y > 0$. At the boundary $y = 0$, we have the slip velocity U, and stress components Σ, Υ and $\Gamma = 0$. We linearize the equations of motion at these values. We separate variables and seek solutions proportional to $\exp(i\alpha x + \lambda t)$, where $\alpha > 0$. Derivatives with respect to y will be denoted by a prime. We are interested in the case where α is large, and we look for eigenmodes such that $\lambda + i\alpha U$ is of order $\sqrt{\alpha}$, stresses are of order $\sqrt{\alpha}$ relative to velocities, and the eigenfunction decays away from the boundary at a rate proportional to α. To leading order in α, the resulting linear problem is

$$\begin{gathered}
(\lambda + i\alpha U)\sigma - 2(\Sigma + \mu)i\alpha u - 2\Upsilon u' = 0, \\
(\lambda + i\alpha U)\tau - (\Sigma + \mu)i\alpha v - \mu u' = 0, \\
(\lambda + i\alpha U)\gamma - 2\Upsilon i\alpha v - 2\mu v' = 0, \\
i\alpha p - i\alpha\sigma - \tau' = 0, \\
p' - i\alpha\tau - \gamma' = 0, \\
i\alpha u + v' = 0.
\end{gathered} \tag{3}$$

We seek solutions of (3) which decay exponentially for $y > 0$ and satisfy the boundary conditions

$$\begin{gathered}
v = 0, \\
(\lambda + i\alpha U)u = \nu\tau.
\end{gathered} \tag{4}$$

If we find solutions of this kind for which Re $\lambda > 0$, then we have blow-up of shortwave disturbances at the boundary and the problem is ill-posed. The second equation in (4) is the linearization of the memory slip law discussed in the introduction.

The most economic way to solve (3) and (4) is by introducing the streamfunction and vorticity (see [6]). Let $u = \psi'$, $v = -i\alpha\psi$ and $\zeta = u' - i\alpha v = \psi'' - \alpha^2\psi$. From (3) we obtain

$$\alpha^2(\Sigma + \mu)\zeta - 2i\alpha\Upsilon\zeta' - \mu\zeta'' = 0. \tag{5}$$

Equation (5) admits solutions $\exp(-\beta y)$, where

$$\alpha^2(\Sigma + \mu) + 2i\alpha\beta\Upsilon - \mu\beta^2 = 0, \text{ and hence}$$

$$\beta = \frac{i\alpha\Upsilon \pm \sqrt{-\alpha^2\Upsilon^2 + \alpha^2\mu(\mu+\Sigma)}}{\mu}. \tag{6}$$

We note that for the upper convected Maxwell model, we always have $(\mu+\Sigma)\mu - \Upsilon^2 > 0$. One of the two values given by (6) has positive and one has negative real part. Since we want solutions which decay for $y > 0$, we must choose the value with positive real part. Henceforth β denotes this value.

The vorticity has the form $\zeta = \zeta_0 \exp(-\beta y)$ and the streamfunction is readily obtained as

$$\psi = \frac{\zeta_0}{\beta^2 - \alpha^2} e^{-\beta y} + \psi_0 e^{-\alpha y}. \tag{7}$$

The first equation in (4) yields

$$\psi(0) = 0, \text{ and hence } \psi_0 = -\frac{\zeta_0}{\beta^2 - \alpha^2}. \tag{8}$$

At the boundary $y = 0$, we can now compute the derivatives of the velocities,

$$u = \psi' = (\beta - \alpha)\psi_0,\ v = 0,\ u' = \psi'' = (\alpha^2 - \beta^2)\psi_0,\ v' = -i\alpha\psi' = i\alpha(\alpha - \beta)\psi_0, \tag{9}$$

and the shear stress,

$$\tau = \frac{1}{\lambda + i\alpha U}((\Sigma + \mu)i\alpha v + \mu u') = \frac{\mu}{\lambda + i\alpha U}(\alpha^2 - \beta^2)\psi_0, \tag{10}$$

We insert these values into the second boundary condition (4) and multiply by the factor $(\lambda + i\alpha U)/((\beta - \alpha)\psi_0)$. This yields

$$(\lambda + i\alpha U)^2 = -\nu\mu(\alpha + \beta). \tag{11}$$

Since β as given by equation (6) has non-vanishing imaginary part, one of the roots of (11) has positive real part. Since β is proportional to α, the real part of λ, i.e. the growth rate, turns out proportional to $\sqrt{\alpha}$.

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