A CLASS OF GLOBAL SMOOTH SOLUTIONS OF THE ONE DIMENSIONAL GAS DYNAMIC SYSTEM

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A CLASS OF GLOBAL SMOOTH SOLUTIONS OF THE ONE DIMENSIONAL GAS DYNAMICS SYSTEM

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Abstract. A class of nontrival global smooth solutions to Cauchy problem for the one dimensional gas dynamics system are obtained under certain assumptions on the initial data.

§1. Introduction

In this paper, we study the Cauchy problem for one dimensional gas dynamics system in Lagrange coordinate

$$\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \frac{\partial S}{\partial t} = 0, \tag{1.1}$$

$$t = 0 : v = v_0(x), u = u_0(x), S = S_0(x), -\infty < x < \infty, \tag{1.2}$$

where v, u, p and S are respectively the specific volume, velocity, pressure and entropy of the gas.

It is well-known that the smooth solutions to Cauchy problems for quasilinear hyperbolic systems, generally speaking, exist only locally in time and will occur singularities in finite time, even if the initial data are sufficiently smooth and small. In paper [1], F. John considered the following Cauchy problem:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.3}$$

$$t = 0 : u = u_0(x), -\infty < x < \infty, \tag{1.4}$$

where $u = (u_1, \cdots, u_n)$ are the unknown functions, $A(u)$ the $n \times n$ matrix. It was proved in [1] that if (1.3) is strictly hyperbolic and each characteristic field is genuinely nonlinear, then the C^2 solution of problem (1.3)-(1.4) will blow up in finite time, provided the initial data are compact support and small enough. In paper [2], T.P. Liu weakened the F. John's condition, i.e., it is allowed that some of the characteristic fields are linearly degenerate. Therefore, the interesting problem is that what conditions can ensure the global existence of classical solutions for quasilinear hyperbolic systems. Up to now, the most results on global existence of classical solutions of quasilinear hyperbolic systems are on reduciable

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systems (cf. [3-4]). It is not clear at all, however, for general quasilinear hyperbolic systems which contain $n(n \geq 3)$ unknown functions.

In this paper, we consider the polytropic gases, that is

$$p = k \cdot exp(S/c_v) \cdot \rho^\gamma, \tag{1.5}$$

where $k > 0$ is a constant, $c_v > 0$ the specific heat capacity, $\gamma > 1$ the adiabatic exponent and $\rho = 1/v$ the density of the gas. The sound speed defined by

$$c = (\frac{\partial p}{\partial \rho})^{1/2}. \tag{1.6}$$

If the vacuum state does not occur (i.e. $\rho > 0$), then the system (1.1) is strictly hyperbolic, the first and third characteristic fields are genuinely nonlinear, while the second is linearly degenerate, and (1.1)-(1.2) can be rewritten as the following characteristic form:

$$\begin{aligned} &(\frac{\partial u}{\partial t} - \lambda\frac{\partial u}{\partial x}) - \alpha e^{\beta S}(\frac{\partial \overline{c}}{\partial t} - \lambda\frac{\partial \overline{c}}{\partial x}) = 0, \\ &\frac{\partial S}{\partial t} = 0, \\ &(\frac{\partial u}{\partial t} + \lambda\frac{\partial u}{\partial x}) + \alpha e^{\beta S}(\frac{\partial \overline{c}}{\partial t} + \lambda\frac{\partial \overline{c}}{\partial x}) = 0, \end{aligned} \tag{1.7}$$

$$t = 0 : u = u_0(x), \overline{c} = \overline{c}_0(x), S = S_0(x), -\infty < x < \infty, \tag{1.8}$$

where

$$\begin{aligned} &\lambda = c\rho, \alpha = \frac{2}{\gamma - 1}, \beta = \frac{1}{2\gamma c_v}, \\ &\overline{c} = e^{-\beta S}c, \overline{c}_0(x) = \overline{c}(0, x). \end{aligned} \tag{1.9}$$

In fact, from (1.6) and (1.1), we see that

$$\begin{aligned} \frac{\partial c}{\partial t} &= -\frac{\gamma - 1}{2}c\rho\frac{\partial u}{\partial x}, \\ \frac{\partial c}{\partial x} &= -\frac{\gamma - 1}{2}c\rho\frac{\partial v}{\partial x} + \frac{1}{2c_v}c\frac{\partial S}{\partial x}, \end{aligned} \tag{1.10}$$

and from (1.5)-(1.6), we have

$$p = \frac{c^2}{\gamma v}, \tag{1.11}$$

hence

$$\frac{\partial p}{\partial x} = \alpha c\rho\frac{\partial c}{\partial x} - \alpha\beta c^2\rho\frac{\partial S}{\partial x}, \tag{1.12}$$

then, (1.7) follows directly from (1.1).

It is known from [2] that , for problem (1.1)-(1.2), if the initial data is compact support and its C^1 norm is small enough, then the smooth solution will blow up in finite time. Therefore, some additional conditions on the initial data must be added to guarantee the global existence of smooth solutions. The another difficult point for problem (1.1)-(1.2), on the other hand, is the vacuum problem, that is, if there is no vacuum state at the initial time, then will there never exist any vacuum state in the later time? (the vacuum problem will be met even in the case of isentropic gas, cf. [5-6])

Now we make the following hypotheses on the initial data:

(H1). $u_0(x)$, $c_0(x)$ and $S_0(x)$ are C^1 bounded and there is no vacuum state at the initial time, i.e.

$$\rho_0(x) = 1/v_0(x) > 0, \forall x \in (-\infty, \infty). \tag{1.13}$$

(H2). There exists a function $h(x) > 0$, such that

$$\overline{c}_0(x) \leq h(x), \forall x \in (-\infty, \infty), \tag{1.14}$$

and $h(x)$ satisfies

$$\min_{x_2 \leq x \leq x_1} h(x) = \min(h(x_1), h(x_2)), \forall x_2 < x_1. \tag{1.15}$$

(H3). The following inequality holds:

$$u_0'(x) - \alpha e^{\beta S_0(x)} |\overline{c}_0'(x)| \geq M h(x) e^{2\beta S_0(x)}, \forall x \in (-\infty, \infty), \tag{1.16}$$

where

$$\begin{aligned} M &= \frac{\alpha\beta M_1}{1+\alpha} e^{\beta M_0}, \\ M_0 &= \sup_{-\infty<x<\infty} |S_0(x)|, M_1 = \sup_{-\infty<x<\infty} |S_0'(x)|. \end{aligned} \tag{1.17}$$

The main result of this paper is the following theorem.

THEOREM 3.1. *Under the hypotheses (H1)-(H3),problem (1.1)-(1.2) admits a unique global smooth solution on $t \geq 0$, and the following hold for any $t \geq 0, -\infty < x < \infty$:*

$$\begin{aligned} &\rho(t,x) > 0, \overline{c}(t,x) \leq h(x), \\ &\frac{\partial u}{\partial x} - \alpha e^{\beta S} |\frac{\partial \overline{c}}{\partial x}| \geq M h(x) e^{2\beta S}. \end{aligned} \tag{1.18}$$

REMARK. *At the case of isentropic gas $S_0(x) \equiv S_0$, condition (1.16) is the sufficient and necessary condition for the global existence of smooth solution.*

In fact, at this case, $M = 0$ and (1.16) is reduced to

$$u_0'(x) - \alpha|c_0'(x)| \geq 0. \tag{1.19}$$

That is, the Riemann invariants $u - \alpha c$ and $u + \alpha c$ are monotone nondecreasing at the initial time, this is the necessary condition for the existence of global smooth solution(cf. [3]). Therefore, Theorem 3.1 also gives an another proof of the corresponding results in papers [5-6], and removes the condition "$\gamma \leq 3$" in paper [6].

§2. MAXIMUM PRINCIPLE FOR HYPERBOLIC SYSTEMS

In order to prove our main theorem, in this section, we first consider the following Cauchy problem for semilinear hyperbolic systems:

$$\frac{\partial u_i}{\partial t} + \lambda_i(t,x)\frac{\partial u_i}{\partial x} = f_i(t,x,u)(i = 1,\cdots,n), \tag{2.1}$$

$$t = 0 : u_i = u_i^0(x)(i = 1,\cdots,n), -\infty < x < \infty, \tag{2.2}$$

where $u = (u_1,\cdots,u_n)$ is unknown function. Suppose that $\lambda_i(t,x) \in C^1$ and bounded, $u_i^0(x) \in C^0(i = 1,\cdots,n), f_i(t,x,u) \in C^0(i = 1,\cdots,n)$ and local *Lip*-continuous with respect to u, i.e., $\forall K > 0, \exists C > 0$ such that if $|u| \leq K$, $|\overline{u}| \leq K$, then

$$|f_i(t,x,u) - f_i(t,x,\overline{u})| \leq C|u - \overline{u}|(i = 1,\cdots,n), \tag{2.3}$$

then we can define the C^0 solution of problem (2.1)-(2.2), that is, a C^0 function $u = u(t,x)$ is called a C^0 solution of problem (2.1)-(2.2), if it satisfies the following integral system:

$$u_i(t,x) = u_i^0(\xi_i(t,x)) + \int_0^t f_i(\tau, X_i, u(\tau, X_i))d\tau(i = 1,\cdots,n), \tag{2.4}$$

where $X_i = X_i(\tau; t, x)$ is the ith characteristic curve which passes through the point (t, x), i.e.

$$\frac{dX_i(\tau;t,x)}{d\tau} = \lambda_i(\tau, X_i(\tau;t,x)), X_i(t;t,x) = x, \tag{2.5}$$

and

$$\xi_i(t,x) = X_i(0;t,x). \tag{2.6}$$

By means of the usual iterative scheme, we can prove it easily that there exists a suitably small $\delta > 0$, such that on the local domain $R(\delta) = \{(t,x)|0 \le t \le \delta, -\infty < x < \infty\}$, problem (2.1)-(2.2) admits a unique C^0 solution and the solution is stable in the sense of the following:

if $u_\epsilon(t,x)$ is the C^0 solution of the following problem

$$\begin{aligned} &\frac{\partial u_i}{\partial t} + \lambda_i(t,x)\frac{\partial u_i}{\partial x} = f_i(t,x,u)(i=1,\cdots,n), \\ &t=0: u_i = u_{i\epsilon}^0(x)(i=1,\cdots,n), -\infty < x < \infty, \end{aligned} \tag{2.7}$$

and $u_\epsilon^0(x)$ converges uniformly to $u_0(x)$ as $\epsilon \to 0$, then the solution $u_\epsilon(t,x)$ also converges uniformly to $u(t,x)$ as $\epsilon \to 0$.

Now we turn to the following specific problem

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + \lambda_1(t,x)\frac{\partial u_1}{\partial x} = a_1(t,x,u_1,u_2)(u_2-u_1), \\ &\frac{\partial u_2}{\partial t} + \lambda_2(t,x)\frac{\partial u_2}{\partial t} = a_2(t,x,u_1,u_2)(u_1-u_2), \\ &t=0: u_1 = u_1^0(x), u_2 = u_2^0(x), -\infty < x < \infty, \end{aligned} \tag{2.8}$$

suppose that

$$\lambda_1(t,x) \le \lambda_2(t,x), \forall t \ge 0, -\infty < x < \infty, \tag{2.9}$$

then from (2.5)-(2.6), we see that

$$\xi_2(t,x) \le \xi_1(t,x), \forall t \ge 0. -\infty < x < \infty. \tag{2.10}$$

We have

THEOREM 2.1. *Suppose that $\lambda_i(t,x) \in C^1(i=1,2)$ and bounded, $a_i(t,x,u,v) \in C^0(i=1,2)$ and local Lip-continuous w.r.t. u_1 and u_2, $u_i^0(x) \in C^0(i=1,2)$. Suppose further that there exists a function $m(t,x)$ such that for any fixed (t,x)*

$$a_i(t,x,u_1,u_2) \ge 0(i=1,2).\forall u_1,u_2 \ge m(t,x). \tag{2.11}$$

If $(u_1(t,x),u_2(t,x))$ is the C^0 solution of problem (2.8) on the domain $R(T) = \{(t,x)|0 \le t \le T, -\infty < x < \infty\}(T>0)$ and

$$\min_{\xi_2(t,x)\le\xi\le\xi_1(t,x)}(u_1^0(\xi),u_2^0(\xi)) \ge m(t,x), \forall (t,x) \in R(T), \tag{2.12}$$

then the following holds:

$$\begin{aligned} \min(u_1(t,x),u_2(t,x)) \ge &\min_{\xi_2(t,x)\le\xi\le\xi_1(t,x)}(u_1^0(\xi),u_2^0(\xi)) \\ \ge &m(t,x), \forall (t,x) \in R(T). \end{aligned} \tag{2.13}$$

Proof. By the stability of the C^0 solution of problem (2.8), we need only to prove that for any $\epsilon > 0$, if

$$\min_{\xi_2(t,x)\le\xi\le\xi_1(t,x)}(u_1^0(\xi),u_2^0(\xi)) \ge m(t,x)+\epsilon, \tag{2.14}$$

then (2.13) holds.

Since (2.13) always holds for $t = 0$, if (2.13) does not hold for some $t_0 > 0$, we can assume that there exist some (t_0, x_0), such that

$$\min(u_1(t_0,x_0),u_2(t_0,x_0)) = \min_{\xi_2(t_0,x_0)\le\xi\le\xi_1(t_0,x_0)}(u_1^0(\xi),u_2^0(\xi)) - \epsilon_0, \tag{2.15}$$

where

$$0 < \epsilon_0 < \epsilon. \tag{2.16}$$

It is easy to prove that there exists a $(\bar{t},\bar{x}), 0 < \bar{t} \le t_0, X_2(\bar{t};t_0,x_0) \le \bar{x} \le X_1(\bar{t};t_0,x_0)$, such that

$$\min(u_1(\bar{t},\bar{x}),u_2(\bar{t},\bar{x})) = \min_{\xi_2(t_0,x_0)\le\xi\le\xi_1(t_0,x_0)}(u_1^0(\xi),u_2^0(\xi)) - \epsilon_0, \tag{2.17}$$

and

$$\begin{aligned} \min(u_1(t,x),u_2(t,x)) > & \min_{\xi_2(t_0,x_0)\le\xi\le\xi_1(t_0,x_0)}(u_1^0(\xi),u_2^0(\xi)) - \epsilon_0, \\ & \forall t < \bar{t}, X_2(t;t_0,x_0) \le x \le X_1(t;t_0,x_0). \end{aligned} \tag{2.18}$$

Without lost of generality, we assume

$$u_1(\bar{t},\bar{x}) = \min(u_1(\bar{t},\bar{x}),u_2(\bar{t},\bar{x})). \tag{2.19}$$

Let $x_1(t) = X_1(t;\bar{t},\bar{x})$ and

$$\begin{aligned} & u_1(t) = u_1(t,x_1(t)), u_2(t) = u_2(t,x_1(t)), \\ & a_1(t) = a_1(t,x_1(t),u_1(t),u_2(t)), \end{aligned} \tag{2.20}$$

then from the first equation of (2.8), we see that

$$\frac{du_1(t)}{dt} = a_1(t)(u_2(t)-u_1(t)), \tag{2.21}$$

hence

$$u_1(\bar{t}) - u_1(t) = \int_t^{\bar{t}} a_1(\tau)(u_2(\tau)-u_1(\bar{t}))e^{-\int_\tau^t a_1(s)ds}d\tau. \tag{2.22}$$

From (2.11) and (2.14)-(2.19), we see that

$$a_1(t) \ge 0, u_1(t) > u_1(\bar{t}), u_2(t) > u_1(\bar{t}), \forall t < \bar{t}. \tag{2.23}$$

Hence the left hand of (2.22) is negative, while the right hand is positive. This is a contradiction. Therefore, (2.13) holds.

Finally, by the stability of the C^0 solution, it is easy to prove that (2.13) holds under the assumption (2.12). □

Similarly, we can prove

THEOREM 2.2. *Suppose that* $\lambda_i(t,x) \in C^1(i=1,2)$ *and bounded,* $a_i(t,x,u,v) \in C^0(i=1,2)$ *and local Lip-continuous w.r.t.* u_1 *and* u_2, $u_i^0(x) \in C^0(i=1,2)$. *Suppose further that there exist two functions* $m(t,x)$ *and* $M(t,x)$ *such that for any fixed* (t,x)

$$a_i(t,x,u_1,u_2) \geq 0(i=1,2), \forall m(t,x) \leq u_1, u_2 \leq M(t,x). \tag{2.24}$$

If $(u_1(t,x), u_2(t,x))$ *is the* C^0 *solution of problem (2.8) on* $R(T)$ *and*

$$\begin{aligned} \min_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} (u_1^0(\xi), u_2^0(\xi)) &\geq m(t,x), \forall (t,x) \in R(T), \\ \max_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} (u_1^0(\xi), u_2^0(\xi)) &\leq M(t,x), \forall (t,x) \in R(T), \end{aligned} \tag{2.25}$$

then the followinf hold:

$$\begin{aligned} \min(u_1(t,x), u_2(t,x)) &\geq \min_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} (u_1^0(\xi), u_2^0(\xi)), \forall (t,x) \in R(T), \\ \max(u_1(t,x), u_2(t,x)) &\leq \max_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} (u_1^0(\xi), u_2^0(\xi)), \forall (t,x) \in R(T). \end{aligned} \tag{2.26}$$

§3. THE PROOF OF THE MAIN THEOREM

By means of the local classical solutions existence theorem, in order to prove Theorem 3.1, we need only to obtain the uniform a priori estimate of the C^1 norm of the solution on the domain where the classical solution exists.

We first prove following lemmas.

LEMMA 3.1. *For problem (1.1)-(1.2), the following hold on the domain where the classical solution exists:*

$$\begin{aligned} \frac{\partial z}{\partial t} - \lambda \frac{\partial z}{\partial x} &= a(t,x,z,w)(w-z), \\ \frac{\partial w}{\partial t} + \lambda \frac{\partial w}{\partial x} &= b(t,x,z,w)(z-w), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} z &= e^{-2\beta S}(\frac{\partial u}{\partial x} - \alpha e^{\beta S} \frac{\partial \overline{c}}{\partial x}), \\ w &= e^{-2\beta S}(\frac{\partial u}{\partial x} + \alpha e^{\beta S} \frac{\partial \overline{c}}{\partial x}), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} a(t,x,z,w) &= \frac{1}{2}(1+1/\alpha)e^{2\beta S}(z - \frac{\alpha\beta}{1+\alpha} e^{-\beta S} \cdot \overline{c} \frac{\partial S}{\partial x}), \\ b(t,x,z,w) &= \frac{1}{2}(1+1/\alpha)e^{2\beta S}(w + \frac{\alpha\beta}{1+\alpha} e^{-\beta S} \cdot \overline{c} \frac{\partial S}{\partial x}). \end{aligned} \tag{3.3}$$

Proof. Let

$$\phi = \frac{\partial u}{\partial x} - \alpha e^{\beta S}\frac{\partial \overline{c}}{\partial x}, \quad (3.4)$$
$$\psi = \frac{\partial u}{\partial x} + \alpha e^{\beta S}\frac{\partial \overline{c}}{\partial x},$$

then from (1.7), we obtain

$$\frac{\partial \phi}{\partial t} - c\rho\frac{\partial \phi}{\partial x} = \frac{\rho}{2}(1+1/\alpha)(\psi-\phi)\phi - 3\beta/2c\rho\frac{\partial S}{\partial x}\phi - \beta/2c\rho\frac{\partial S}{\partial x}\psi, \quad (3.5)$$
$$\frac{\partial \psi}{\partial t} + c\rho\frac{\partial \phi}{\partial x} = \frac{\rho}{2}(1+1/\alpha)(\phi-\psi)\psi + \beta/2c\rho\frac{\partial S}{\partial x}\phi + 3\beta/2c\rho\frac{\partial S}{\partial x}\psi.$$

Hence, (3.1) can be obtained immediately. □

LEMMA 3.2. *For problem (1.1)-(1.2), the following hold on the domain where the classical solution exists:*

$$\begin{aligned}
&\frac{\partial}{\partial t}(e^{2\beta S}\rho) - \lambda\frac{\partial}{\partial x}(e^{2\beta S}\rho) = -w\cdot(e^{2\beta S}\rho)^2,\\
&\frac{\partial \overline{c}}{\partial t} - \lambda\frac{\partial \overline{c}}{\partial x} = -\frac{1}{\alpha}w\rho\cdot e^{2\beta S}\overline{c},\\
&\frac{\partial \overline{c}}{\partial t} + \lambda\frac{\partial \overline{c}}{\partial x} = -\frac{1}{\alpha}z\rho\cdot e^{2\beta S}\overline{c},\\
&\frac{\partial u}{\partial t} - \lambda\frac{\partial u}{\partial x} = -c\rho w\cdot e^{2\beta S},\\
&\frac{\partial u}{\partial t} + \lambda\frac{\partial u}{\partial x} = c\rho z\cdot e^{2\beta S}.
\end{aligned} \quad (3.6)$$

Proof. Frome (1.1),(1.10) and (3.2), lemma 3.2 can be proved similarly. □

The proof of theorem 3.1. In order to obtain the uniform a priori estimate of C^1 norm of the solution, we need only to prove that if problem (1.1)-(1.2) admits a smooth solution on

$$R(T) = \{(t,x)|0 \le t \le T, -\infty < x < \infty\}, T > 0, \quad (3.7)$$

then the C^1 norm of the solution on $R(T)$ does not depend on T.

Firstly, from the third equation of (1.1) and the initial data, we see that

$$S(t,x) = S_0(x), \forall t \ge 0, -\infty < x < \infty, \quad (3.8)$$

hence, the C^1 norm of $S(t,x)$ is uniform bounded. Next, let

$$z_0(x) = z(0,x), w_0(x) = w(0,x), \quad (3.9)$$

then condition (1.16) is equivalent to

$$z_0(x) \geq Mh(x), w_0(x) \geq Mh(x), -\infty < x < \infty. \tag{3.10}$$

There are two cases:

Case 1. $||S_0'|| = 0$, or $S_0(x) \equiv S_0$. From (3.8), we have $S(t,x) \equiv S_0$, then we see from (3.3) that

$$a(t,x,z,w) \geq 0, b(t,x,z,w) \geq 0, \forall z \geq 0, w \geq 0. \tag{3.11}$$

Hence, by theorem 2.1 and (3.10), we have

$$z(t,x) \geq 0, w(t,x) \geq 0, \forall (t,x) \in R(T). \tag{3.12}$$

Let $\lambda_1 = -\lambda, \lambda_2 = \lambda$, we can get (2.9)-(2.10) similarly. Since from lemma 3.2 we see that along the characteristic curves $e^{2\beta S}\rho$, $\bar{c}$ and u satisfy the Riccati and linear ordinary differential equations respectively, noting (3.8), (3.12) and (1.17), it is easy to see that $\forall (t,x) \in R(T)$

$$\begin{aligned}
&0 < \rho(t,x) \leq e^{4\beta M_0} \sup_{-\infty<x<\infty} \rho_0(x), \\
&0 < \bar{c}(t,x) \leq \min(\bar{c}_0(\xi_1(t,x), \bar{c}_0(\xi_2(t,x)), \\
&\inf_{-\infty<x<\infty} u_0(x) \leq u(t,x) \leq \max_{-\infty<x<\infty} u_0(x).
\end{aligned} \tag{3.13}$$

Set

$$M_2 = \sup\{z_0(x), w_0(x) | -\infty < x < \infty\}, \tag{3.14}$$

then by theorem 2.2, we have

$$z(t,x) \leq M_2, w(t,x) \leq M_2, \forall (t,x) \in R(T). \tag{3.15}$$

Hence from (3.2), it is easy to see that $\frac{\partial u}{\partial x}$ and $\frac{\partial \bar{c}}{\partial x}$ are uniform bounded on $R(T)$. Thus we get the uniform estimate of C^1 norm of the solution on $R(T)$.

Case 2. $||S_0'|| > 0$. Since we will prove that on $R(T)$

$$\min(z(t,x), w(t,x)) \geq M \min_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} h(x) > 0, \tag{3.16}$$

we first assume that (3.12) holds on $R(T)$, then same as *Case* 1, (3.13) holds. By the second inequality of (3.13) and (1.14)-(1.15), (1.17), we see that

$$\begin{aligned}
&|\frac{\alpha\beta}{1+\alpha} e^{-\beta S(t,x)} \bar{c}(t,x) \frac{\partial S}{\partial x}(t,x)| \\
&\leq \frac{\alpha\beta M_1}{1+\alpha} e^{\beta M_0} \min(\bar{c}_0(\xi_1(t,x)), \bar{c}_0(\xi_2(t,x))) \\
&\leq M \min(h(\xi_1(t,x)), h(\xi_2(t,x))) = M \min_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} h(x),
\end{aligned} \tag{3.17}$$

hence from (3.3), we have

$$a(t,x,z,w) \geq 0, b(t,x,z,w) \geq 0, \forall z, w \geq M \min_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} h(t,x), \tag{3.18}$$

By theorem 2.1 and (3.10), we get

$$\min(z(t,x), w(t,x)) \geq \min_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} (z_0(x), w_0(x)) \geq M \min_{\xi_2(t,x)\leq\xi\leq\xi_1(t,x)} h(x). \tag{3.19}$$

This is also showed that our assumption of (3.12) is reasonable. Similar to *Case* 1, $\frac{\partial u}{\partial x}$ and $\frac{\partial \bar{c}}{\partial x}$ are uniform bounded on $R(T)$. Thus we also get the uniform estimate of C^1 norm of the solution on $R(T)$. Therefore problem (1.1)-(1.2) admits a unique global C^1 solution on $t \geq 0$. □

From the proof of the above theorem, we can obtain the following theorem.

THEOREM 3.2. *For problem (1.1)-(1.2), if the initial data are C^1, there is no vacuum state at the initial time, then the vacuum state will never occur on the domain where the classical solution exists. That is, the vacuum state dose not occur before the shock waves arise.*

Proof. From the first equation of (3.6), we see that along the first characteristic curves, $e^{2\beta S}\rho$ satisfies a Riccati equation. Hence, if $\rho > 0$ at the initial time, then $\rho > 0$ on the domain where the classical solution exists. □

REFERENCES

[1] F. JOHN, *Formation of singularities in one-dimensional nonlinear waves propagation,*, Comm. Pure Appl. Math.,, 27 (1974), pp. 377-405.

[2] T.P. LIU, *Development of singularities in the nonlinear wave for quasilinear partial differential equations*, Jour. Diff. Equations, 30 (1979), pp. 92-111.

[3] P.D. LAX, *Development of singularities of solutions of nonlinear hyperbolic partial differential equations*, Jour. of Math. Phys., 5 (1964), pp. 611-613.

[4] LI TA-TSIEN, *Global regularty and formation of singularities of solution to first order quasilinear hyperbolic systems*, Proceedings of the Royal Society of Edinburgh, 87A (1981), pp. 255-261.

[5] LIN LONGWEI, *On the vacuum state for the equations of isentropic gas dynamics(in Chinese)*, Journal of Hua Chiao University, 2 (1984), pp. 1-4.

[6] LI TA-TSIEN AND ZHAO YANCHUN, *Vacuum problems for the systems of one-dimensional isentropic flow(in Chinese)*, Chinese Quarterly Journal of Math., 1, pp. 41-46.

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