ILL-POSEDNESS RESULTING FROM SLIP AS A POSSIBLE EXPLANATION OF MELT FRACTURE

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IMA Preprint Series # 536

June 1989

IMA Preprint Series #536

Ill-posedness at the boundary

for elastic solids sliding under Coulomb friction

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Key Words: Sliding friction, Agmon's condition, Ill-posedness
AMS(MOS) classifications: 35R25, 73H10, 73T05

Abstract

We consider an incompressible neo-Hookean elastic solid sliding on a rigid surface under the influence of Coulomb friction. It is shown that illposedness at the boundary due to failure of Agmon's condition can occur. If the friction coefficient is greater than one, this is the case even in the limit of linear elasticity. The effect of a dependence of the friction force on the sliding velocity is also considered.

1. Introduction and formulation of the problem

It is well known that the well-posedness of initial-value problems for partial differential equations requires, in addition to conditions on the PDE's themselves, an algebraic restriction on the boundary conditions, which is known as Agmon's condition. If this condition is violated, then short-wavelength perturbations localized near the boundary can grow in a catastrophic manner. The significance of Agmon's condition for free-surface problems in elasticity has long been recognized; we refer to [2] for a recent paper and further references to the literature. To the author's knowledge, however, frictional contact has not been investigated in this context. In this paper, we shall demonstrate that Agmon's condition can indeed be violated in this case. If this situation actually occurs in experiments, it should manifest itself in short scale irregularities on the sliding surface. The instability discussed in this paper is probably related to instabilities which have been found in finite element studies of sliding friction [1].

We consider an incompressible neo-Hookean elastic medium in two space dimensions. We choose to work in Eulerian coordinates and use Cauchy stress and velocity as our basic variables. Let $\mathbf{v} = (u, v)$ denote the velocity, p the pressure, and

$$\mathbf{T} = \begin{pmatrix} \sigma & \tau \\ \tau & \gamma \end{pmatrix} \tag{1}$$

the extra stress tensor. The equations of motion are

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = \text{div } \mathbf{T} - \nabla p,$$

$$\text{div } \mathbf{v} = 0. \tag{2}$$

With $\mathbf{F}$ denoting the deformation gradient, the constitutive law for a neo-Hookean material is

$$\mathbf{T} = \mu(\mathbf{F}\mathbf{F}^T - \mathbf{1}). \tag{3}$$

After taking the material time derivative, (3) takes the form (note that the material time derivative of $\mathbf{F}$ equals $\nabla\mathbf{v}\mathbf{F}$)

$$\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{T} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T = \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T). \tag{4}$$

The medium is filling the half-space $y > 0$, and we are interested in small perturbations of uniform shear. If κ is the shear, the stress in the unperturbed state can be determined from (3) as

$$\mathbf{T} = \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon & 0 \end{pmatrix} = \begin{pmatrix} \mu\kappa^2 & \mu\kappa \\ \mu\kappa & 0 \end{pmatrix}. \tag{5}$$

At the boundary $y = 0$, the material is sliding under the influence of Coulomb friction; let U denote the (unperturbed) sliding velocity. We now consider small perturbations of steady sliding with stresses given by (5). After linearization, equations (4) and (2) yield the system

$$\begin{gathered}
\frac{\partial \sigma}{\partial t} + U\frac{\partial \sigma}{\partial x} - 2(\Sigma + \mu)\frac{\partial u}{\partial x} - 2\Upsilon\frac{\partial u}{\partial y} = 0, \\
\frac{\partial \tau}{\partial t} + U\frac{\partial \tau}{\partial x} - \Upsilon(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) - (\Sigma + \mu)\frac{\partial v}{\partial x} - \mu\frac{\partial u}{\partial y} = 0, \\
\frac{\partial \gamma}{\partial t} + U\frac{\partial \gamma}{\partial x} - 2\Upsilon\frac{\partial v}{\partial x} - 2\mu\frac{\partial v}{\partial y} = 0, \\
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} - \frac{\partial \sigma}{\partial x} - \frac{\partial \tau}{\partial y} = 0, \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \frac{\partial \tau}{\partial x} - \frac{\partial \gamma}{\partial y} = 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.
\end{gathered} \tag{6a}$$

We separate variables and seek solutions proportional to $\exp(i\alpha x + \lambda t)$ (w.l.o.g. we shall consider $\alpha > 0$). The system (6a) now takes the form

$$\begin{gathered}
(\lambda + i\alpha U)\sigma - 2(\Sigma + \mu)i\alpha u - 2\Upsilon u' = 0, \\
(\lambda + i\alpha U)\tau - (\Sigma + \mu)i\alpha v - \mu u' = 0, \\
(\lambda + i\alpha U)\gamma - 2\Upsilon i\alpha v - 2\mu v' = 0, \\
\rho(\lambda + i\alpha U)u + i\alpha p - i\alpha\sigma - \tau' = 0, \\
\rho(\lambda + i\alpha U)v + p' - i\alpha\tau - \gamma' = 0, \\
i\alpha u + v' = 0.
\end{gathered} \tag{6b}$$

Here a prime denotes differentiation with respect to y. We seek solutions of (6b) which decay exponentially for $y > 0$ and satisfy the boundary conditions

$$\begin{gathered}
v = 0, \\
\tau = \nu(p - \gamma).
\end{gathered} \tag{7}$$

The second equation in (7) is the linearization of the Coulomb friction law, which relates the shear stress at the wall to the applied load. We are interested in solutions of (6b) and (7) for which Re $\lambda > 0$. Since λ is proportional to α, such solutions imply blow-up of shortwave disturbances at the boundary and ill-posedness; the real parts of the eigenvalues are not bounded from above. We note that the boundary value problem given by (6b) and (7) is not self-adjoint, and in general $\lambda + i\alpha U$ is neither purely real nor purely imaginary, in contrast to the case of free-surface boundary conditions.

2. Ill-posedness for Coulomb friction

The most economic way to solve (6b) and (7) is by introducing the streamfunction and vorticity. Let $u = \psi'$, $v = -i\alpha\psi$ and $\zeta = u' - i\alpha v = \psi'' - \alpha^2\psi$. In (6b), we take the derivative of the fourth equation minus $i\alpha$ times the fifth equation to eliminate p. We then multiply by $\lambda + i\alpha U$ and use the first three equations of (6b) to substitute for σ, τ and γ. In this way we obtain

$$\rho(\lambda + i\alpha U)^2\zeta + \alpha^2(\Sigma + \mu)\zeta - 2i\alpha\Upsilon\zeta' - \mu\zeta'' = 0. \tag{8}$$

Equation (8) admits solutions $\exp(-\beta y)$, where

$$\rho(\lambda + i\alpha U)^2 + \alpha^2(\Sigma + \mu) + 2i\alpha\beta\Upsilon - \mu\beta^2 = 0, \text{ and hence}$$

$$\beta = \frac{i\alpha\Upsilon \pm \sqrt{-\alpha^2\Upsilon^2 + \alpha^2\mu(\mu + \Sigma) + \mu\rho(\lambda + i\alpha U)^2}}{\mu}. \tag{9}$$

As long as λ is not purely imaginary, one of the two values given by (9) has positive and one has negative real part. Since we want solutions which decay for $y > 0$, we must choose the value with positive real part. Henceforth β denotes this value.

The vorticity has the form $\zeta = \zeta_0 \exp(-\beta y)$ and the streamfunction is readily obtained as

$$\psi = \frac{\zeta_0}{\beta^2 - \alpha^2} e^{-\beta y} + \psi_0 e^{-\alpha y}. \tag{10}$$

The first equation in (7) yields

$$\psi(0) = 0, \text{ and hence } \psi_0 = -\frac{\zeta_0}{\beta^2 - \alpha^2}. \tag{11}$$

At the boundary $y = 0$, we can now compute the derivatives of the velocities,

$$u = \psi' = (\beta - \alpha)\psi_0, \; v = 0, \; u' = \psi'' = (\alpha^2 - \beta^2)\psi_0, \; v' = -i\alpha\psi' = i\alpha(\alpha - \beta)\psi_0,$$
$$u'' = \psi''' = (\beta^3 - \alpha^3)\psi_0, \tag{12}$$

the stresses,

$$\sigma = \frac{2}{\lambda + i\alpha U}((\Sigma + \mu)i\alpha u + \Upsilon u') = \frac{2}{\lambda + i\alpha U}((\Sigma + \mu)i\alpha(\beta - \alpha) + \Upsilon(\alpha^2 - \beta^2))\psi_0,$$

$$\tau = \frac{1}{\lambda + i\alpha U}((\Sigma + \mu)i\alpha v + \mu u') = \frac{\mu}{\lambda + i\alpha U}(\alpha^2 - \beta^2)\psi_0,$$

$$\gamma = \frac{2}{\lambda + i\alpha U}(\Upsilon i\alpha v + \mu v') = \frac{2\mu}{\lambda + i\alpha U} i\alpha(\alpha - \beta)\psi_0, \tag{13}$$

and the pressure,

$$p = \sigma + \frac{\tau'}{i\alpha} - \frac{\rho}{i\alpha}(\lambda + i\alpha U)u$$

$$= \Big[\frac{1}{\lambda + i\alpha U}((\Sigma+\mu)i\alpha(\beta-\alpha)+2\Upsilon(\alpha^2-\beta^2)+\frac{\mu}{i\alpha}(\beta^3-\alpha^3)) - \frac{\rho}{i\alpha}(\lambda+i\alpha U)(\beta-\alpha)\Big]\psi_0. \tag{14}$$

We insert these values into the second boundary condition (7) and multiply by the factor $i\alpha(\lambda + i\alpha U)/((\alpha - \beta)\psi_0)$. This yields the condition

$$i\alpha\mu(\alpha + \beta) = \nu((\Sigma + 3\mu)\alpha^2 + 2\Upsilon i\alpha(\alpha + \beta) - \mu(\alpha^2 + \alpha\beta + \beta^2)$$
$$+\rho(\lambda + i\alpha U)^2). \tag{15}$$

By combining (15) with (9), we obtain

$$i\alpha\mu(\alpha + \beta) = \nu(\mu\alpha^2 + 2\Upsilon i\alpha^2 - \mu\alpha\beta), \tag{16}$$

and hence

$$\beta = \frac{(\nu - i)\mu + 2i\nu\Upsilon}{(\nu + i)\mu}\alpha. \tag{17}$$

The real part of β as given by (17) turns out positive if

$$\mu\nu^2 + 2\nu\Upsilon - \mu > 0, \tag{18}$$

or, if

$$\frac{\Upsilon}{\mu} = \kappa > \frac{1 - \nu^2}{2\nu}. \tag{19}$$

Once β has been determined from (19), we can find λ from (9). We note that λ occurs only in the combination $(\lambda + i\alpha U)^2$ and hence one solution for λ always has positive real part unless $(\lambda + i\alpha U)^2$ is real and negative. It is easy to check that this can only happen in special cases. In general, therefore, we have ill-posedness if (19) holds.

Equation (19) gives a critical value of the shear, beyond which ill-posedness occurs. This critical value becomes negative if $\nu > 1$. In this case, therefore, we have ill-posedness even for linear elasticity.

3. Friction dependent on slip velocity

In this section, we shall incorporate a dependence of the friction force on the sliding velocity. Heuristically, one expects instability if the friction force is a decreasing function of slip velocity. As we shall see, things are actually a little more complicated.

We confine our attention to the case where the inertial terms in the equation of motion can be neglected. The linearized condition (7) is modified to

$$\tau = \nu(p - \gamma) + \epsilon u. \tag{20}$$

We repeat the analysis above. Since inertia are neglected, equation (9) becomes

$$\beta = \frac{i\alpha\Upsilon + \alpha\sqrt{\mu(\Sigma + \mu) - \Upsilon^2}}{\mu}. \tag{21}$$

In place of equation (16), we obtain

$$\begin{aligned} i\alpha\mu(\alpha + \beta) &= \nu(\mu\alpha^2 + 2\Upsilon i\alpha^2 - \mu\alpha\beta) \\ &\quad -\epsilon i\alpha(\lambda + i\alpha U). \end{aligned} \tag{22}$$

This yields

$$\text{Re } \lambda = -\frac{\mu}{\epsilon}(\alpha + \text{Re } \beta + \nu \text{ Im } \beta) + 2\frac{\nu\Upsilon}{\epsilon}\alpha$$

$$= \frac{\mu\alpha}{\epsilon}\left(-1 - \sqrt{\frac{\mu(\Sigma + \mu) - \Upsilon^2}{\mu^2}} + \nu\frac{\Upsilon}{\mu}\right) = \frac{\mu\alpha}{\epsilon}(-2 + \nu\kappa). \tag{23}$$

If the shear is less than $2/\nu$, then we have indeed instability if ϵ is negative and stability (for the particular class of modes considered here) if ϵ is positive. However, the situation is reversed if the shear is greater than $2/\nu$.

Acknowledgement

This research was completed while the author was visiting the Institute for Mathematics and its Applications at the University of Minnesota. Financial support from the IMA and from the National Science Foundation under Grant No. DMS-8796241 is gratefully acknowledged.

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