A VARIATIONAL INEQUALITY ASSOCIATED WITH A LUBRICATION PROBLEM

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§0. **Introduction.** In this paper, we study a variational inequality which describes a lubrication problem in a journal bearing. For the physical background, we refer the reader to Bayada[3] and Bayada and Chambat[4] for details. Given a constant $a \in (0,1)$ and a smooth function $\theta(y)$ defined in $[0,1]$ such that

$$(1-a)/(1+a) < \theta(y) < 1, \forall y \in [0,1],$$

let $h(x) = 1 + a\cos x$, $g(x,y) = h(x) - (1+a)\theta(y)$, and introduce the sets(see Figure 1)

$$\begin{aligned}
&\Omega = (0,2\pi) \times (0,1), \quad R_1 = (0,\pi) \times (0,1), \quad R_2 = (\pi,2\pi) \times (0,1),\\
&\Omega_1 = \{(x,y) \in R_1; g(x,y) > 0\}, \quad \Omega_2 = \Omega \setminus \overline{\Omega_1}, \quad T = \partial\Omega_1 \cap \Omega,\\
&\partial_0\Omega = \{0\} \times (0,1), \quad \partial_1\Omega = \partial\Omega \setminus \partial_0\Omega.
\end{aligned}$$

Let

$$K_0 = \{\phi \in H^1(\Omega); \phi \geq 0 \text{ a.e. in } \Omega_2, \phi = 0 \text{ on } \partial_1\Omega\}.$$

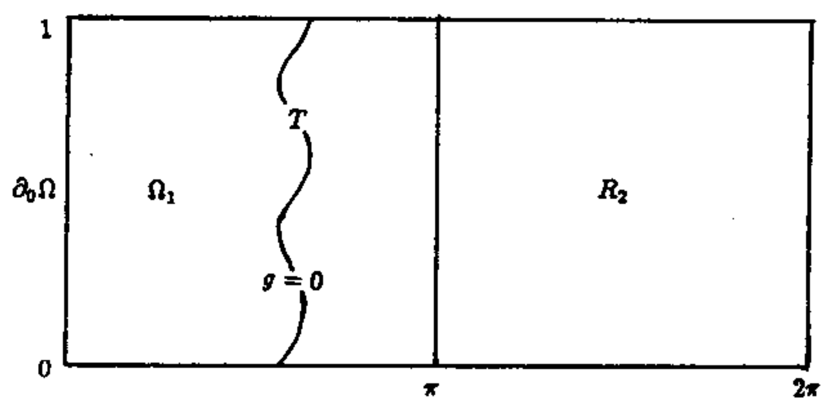

Figure 1

Note that $T \subset R_1$ and $g = 0$ on T by the assumption on θ. The following classical formulation of the lubrication problem was posed in [4].

Problem(P_c). *Find a non-negative continuous function $u(x,y)$ in $\bar{\Omega}$ and two disjoint free boundaries S_1 and S_2 such that the following conditions hold:*

$$S_1 \subset \Omega_1, S_2 \subset R_2, \tag{0.1}$$

$$-\nabla \cdot (h^3 \nabla u) = a \sin x \quad in \quad \Omega^+ \equiv \{u > 0\}, \tag{0.2}$$

$$u = 0 \quad on \quad \partial\Omega, \tag{0.3}$$

$$u = 0 \quad on \quad S \equiv \partial\Omega^+ \cap \Omega = S_1 \cup S_2, \tag{0.4}$$

$$h^3 \frac{\partial u}{\partial \nu} = g\, \nu_x \quad on \quad S_1, \tag{0.5}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad on \quad S_2, \tag{0.6}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$, ν is the unit outer normal to $\partial\Omega^+$, $\nu = (\nu_x, \nu_y)$, and $\partial u/\partial \nu$ is defined from the right-hand side of S_1 and from the left-hand side of S_2.

Hereafter $A^+(u)$ or A^+ will denote the set $A \cap \{u > 0\}$. It is anticipated that the set $\{u = 0\}$ lies to the left of S_1 and to the right of S_2. To solve this problem, Bayada and Chambat [4] considered the following variational inequality which is a weak formulation of Problem(P_c).

Problem(P). *Find $u \in K_0$ such that*

$$\int_\Omega h^3\, \nabla u \cdot \nabla(\phi - u) \geq \int_{\Omega_1} I_{\Omega_1^+(u)}\, g\, (\phi - u)_x + \int_{\Omega_2} g\, (\phi - u)_x, \tag{0.7}$$

for all $\phi \in K_0$, where I_A denotes the characteristic function of A.

In this paper the subscript x or y will always represent the partial derivative of functions with respect to that variable. Note that we do not require functions in K_0 vanish on $\partial_0\Omega$. In [4], the authors proved existence of a solution to the ϵ-problem(see Lemma 1.1 in §1). However, they have not been able to use it to establish existence of a solution to Problem(P) (unless they a priori assume that the free boundary is Lipschitz). We instead start with a weaker formulation:

Problem(P_0). *Find (u, γ), $u \in K_0$, $\gamma \in L^\infty(\Omega_1)$, $0 \leq \gamma \leq 1$ a.e. in Ω_1 and $\gamma \subset H(u)$, such that*

$$\int_\Omega h^3\, \nabla u \cdot \nabla(\phi - u) \geq \int_{\Omega_1} \gamma\, g\, (\phi - u)_x + \int_{\Omega_2} (a \sin x)(\phi - u), \tag{0.8}$$

for all $\phi \in K_0$, where $H(u)$ is the Heaviside graph of u (i.e., $H(t) = 1$ if $t > 0$, $= 0$ if $t < 0$, and $= [0,1]$ if $t = 0$).

We shall establish existence of a solution to Problem(P_0) and prove a regularity theorem for solutions to Problem(P_0) in §1. In §2 we shall be using, among other methods, the comparison argument of [4] to prove, under some condition on θ, that any solution of Problem(P_0) vanishes on all of $\partial\Omega$; we shall also derive some properties of the free boundary and the non-coincidence set Ω^+. In §3 we prove that $\gamma = I_{\Omega_1^+(u)}$ for any solution (u, γ) of Problem(P_0) and establish some regularity results of the free boundary. Since $g = 0$ on T by assumption, the second integrals on the right-hand sides of (0.7) and (0.8) are equal. Therefore, the two problems (P_0) and (P) are equivalent. Finally, in §4 we shall prove that there exists only one solution to Problem(P_0).

§1. **Existence and regularity.** For any $\epsilon > 0$, let

$$H_\epsilon(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ t/\epsilon & \text{if} \quad 0 \le t \le \epsilon, \\ 1 & \text{if} \quad t > \epsilon. \end{cases}$$

LEMMA 1.1. *There exists a $u_\epsilon \in K_0$, $u_\epsilon \ge 0$ in Ω, such that the inequality*

$$\int_\Omega h^3 \nabla u_\epsilon \cdot \nabla(\phi - u_\epsilon) \ge \int_{\Omega_1} H_\epsilon(u_\epsilon)\, g\, (\phi - u_\epsilon)_x + \int_{\Omega_2} (a \sin x)(\phi - u_\epsilon) \tag{1.1}$$

holds for all $\phi \in K_0$. Moreover,

$$\int_\Omega |\nabla u_\epsilon|^2 \le C, \tag{1.2}$$

where C is a positive constant independent of ϵ.

The proof of Lemma 1.1 is given in [4].

Taking $\phi = u_\epsilon \pm \zeta$ with $\zeta \in C_0^\infty(\Omega_1)$ in (1.1), we deduce that u_ϵ is a weak solution of the equation

$$-\nabla \cdot (h^3 \nabla u_\epsilon) = -(H_\epsilon(u_\epsilon)\, g)_x \quad \text{in} \quad \Omega_1. \tag{1.3}$$

Next, taking $\zeta \in C^\infty(\Omega_1)$, $\zeta = 0$ on $\partial\Omega_1 \setminus \partial_0\Omega$, we obtain

$$\int_{\partial_0\Omega} h^3 \frac{\partial u_\epsilon}{\partial x} \zeta = \int_{\partial_0\Omega} H_\epsilon(u_\epsilon) g \zeta,$$

or equivalently,

$$\frac{\partial u_\epsilon}{\partial x} = \frac{1 - \theta(y)}{(1+a)^2} H_\epsilon(u_\epsilon) \quad \text{on} \quad \partial_0\Omega. \tag{1.4}$$

From (1.3), (1.4) and $u_\epsilon = 0$ on $\partial_1\Omega$ it follows by elliptic regularity theory(cf. [7] and [10]) that

$$\|u_\epsilon\|_{C^\alpha(\overline{\Omega'})} \le C, \forall \Omega' \subset\subset \overline{\Omega_1} \setminus T, \forall \alpha \in (0,1), \tag{1.5}$$

where C is a positive constant independent of ϵ.

Theorem 1.2. *There exists a solution (u,γ) of Problem(P_0).*

Proof. By (1.2) and (1.5), there is a sequence $\{\epsilon_n\}$, $\epsilon_n \downarrow 0$ as $n \to \infty$, such that

$$\begin{aligned} &u_{\epsilon_n} \to u \quad \text{weakly in} \quad H^1(\Omega), \\ &u_{\epsilon_n} \to u \quad \text{uniformly on any compact subset of} \quad \overline{\Omega_1} \setminus T, \\ &u_{\epsilon_n} \to u \quad \text{strongly in} \quad L^2(\Omega) \quad \text{and a.e. in} \quad \Omega, \\ &H_{\epsilon_n}(u_{\epsilon_n}) \to \gamma \quad \text{weakly in} \quad L^2(\Omega_1) \quad \text{and weakly* in} \quad L^\infty(\Omega_1). \end{aligned} \tag{1.6}$$

Also, $\gamma = 1$ a.e. in Ω_1^+ and $0 \le \gamma \le 1$ a.e. in Ω_1.

For any $\phi \in K_0$ we have

$$\int_\Omega h^3 \nabla u_{\epsilon_n} \cdot \nabla \phi \to \int_\Omega h^3 \nabla u \cdot \nabla \phi, \tag{1.7}$$

$$\int_{\Omega_2} (a \sin x)(\phi - u_{\epsilon_n}) \to \int_{\Omega_2} (a \sin x)(\phi - u), \tag{1.8}$$

$$\int_{\Omega_1} H_{\epsilon_n}(u_{\epsilon_n})\, g\, \phi_x \to \int_{\Omega_1} \gamma\, g\, \phi_x, \tag{1.9}$$

as $n \to \infty$, and

$$\int_\Omega h^3 |\nabla u|^2 \le \liminf_{n\to\infty} \int_\Omega h^3 |\nabla u_{\epsilon_n}|^2. \tag{1.10}$$

Introduce the function

$$G_\epsilon(t) = \begin{cases} 0 & \text{if} \quad t \le 0 \\ t^2/(2\epsilon) & \text{if} \quad 0 \le t \le \epsilon \\ t - \epsilon/2 & \text{if} \quad t \ge \epsilon; \end{cases}$$

it satisfies $G_\epsilon'(t) = H_\epsilon(t)$. Clearly

$$\begin{aligned} |G_{\epsilon_n}(u_{\epsilon_n}) - u| &\le |G_{\epsilon_n}(u_{\epsilon_n}) - G_{\epsilon_n}(u)| + |G_{\epsilon_n}(u) - u| \\ &\le |u_{\epsilon_n} - u| + |G_{\epsilon_n}(u) - u|. \end{aligned} \tag{1.11}$$

Hence, using the fact that $g = 0$ on T, we get

$$\begin{aligned} -\int_{\Omega_1} H_{\epsilon_n}(u_{\epsilon_n})\, g\, u_{\epsilon_n,x} &= -\int_{\Omega_1} G'_{\epsilon_n}(u_{\epsilon_n})\, u_{\epsilon_n,x}\, g \\ &= -\int_{\Omega_1} \frac{\partial}{\partial x}[G_{\epsilon_n}(u_{\epsilon_n})]\, g \\ &= \int_{\Omega_1} G_{\epsilon_n}(u_{\epsilon_n})\, g_x + \int_{\partial_0\Omega} G_{\epsilon_n}(u_{\epsilon_n}) g \\ &\to \int_{\Omega_1} u g_x + \int_{\partial_0\Omega} u g \quad \text{as} \quad n \to \infty, \end{aligned} \tag{1.12}$$

by (1.6) and (1.11). Also,

$$\int_{\Omega_1} u g_x + \int_{\partial_0 \Omega} u g = -\int_{\Omega_1} g\, u_x = -\int_{\Omega_1} \gamma\, g\, u_x, \tag{1.13}$$

since $\gamma = 1$ in $\{u > 0\}$. Letting $\epsilon \to 0$ in (1.1) and using (1.7-10) and (1.12-13), we conclude that (u, γ) is a solution of Problem(P_0). □

We shall denote by $B_r(P)$ the ball with center at P and radius r.

THEOREM 1.3. *For any solution (u, γ) of Problem(P_0), $u \in C^\alpha(\bar{\Omega})$ for all $\alpha \in (0,1)$ and u is real analytic in Ω^+.*

Proof. Let (u, γ) be a solution of Problem(P_0). Substituting $\phi = u^+ \equiv \max(u, 0)$ in (0.8), we obtain that $u \geq 0$ in Ω. Let

$$K = \{\phi \in K_0; \phi \geq 0 \text{ a.e. in } \Omega\}.$$

Then (u, γ) satisfies the variational inequality (0.8) for all $\phi \in K$. Since $\tilde{g} \equiv \gamma g I_{\Omega_1}$ and $\tilde{h} \equiv a \sin x I_{\Omega_2}$ are functions in $L^\infty(\Omega)$, it follows from Theorem 2.9 in [9] that

$$u \in C^\alpha(\bar{\Omega}), \forall \alpha \in (0,1).$$

Note that u satisfies the equation

$$-\nabla \cdot (h^3 \nabla u) = a \sin x \quad \text{in} \quad \Omega_1^+ \cup \Omega_2^+. \tag{1.14}$$

Let $P \in T$ be such that $u(P) > 0$. Then there is a positive number r_0 such that $u > 0$ in $B = B_{r_0}(P)$. For any $\zeta \in C_0^\infty(B)$, the function $\phi = u \pm \epsilon\zeta$ is in K_0 for any small positive ϵ. By (0.8)

$$\int_B h^3 \nabla u \cdot \nabla \zeta = \int_{\Omega_1 \cap B} g \zeta_x + \int_{\Omega_2 \cap B} (a \sin x)\zeta = \int_B (a \sin x)\zeta. \tag{1.15}$$

It follows that u satisfies (1.14) also in B; and consequently,

$$-\nabla \cdot (h^3 \nabla u) = a \sin x \quad \text{in} \quad \Omega^+ \tag{1.16}$$

and by elliptic regularity theory u is analytic in Ω^+. □

§2. **Preliminaries.** Let (u, γ) be a solution of Problem(P_0). We shall denote the free boundary by $S = S(u, \gamma)$, i.e., $S = \partial\Omega^+ \cap \Omega$. Let

$$\tilde{K} = \{\phi \in H_0^1(\Omega); \phi \geq 0 \text{ a.e. in } \Omega\}.$$

We recall the classical Reynolds variational inequality:

PROBLEM($\tilde{P}$). *Find $\tilde{u} \in \tilde{K}$ such that*

$$\int_\Omega h^3 \nabla\tilde{u} \cdot \nabla(\phi - \tilde{u}) \geq \int_\Omega (a \sin x)(\phi - \tilde{u}), \forall \phi \in \tilde{K}. \tag{2.1}$$

It is well-known that there exists a unique solution $\tilde{u}$ of Problem($\tilde{P}$) such that $\tilde{u} \in W^{2,p}(\Omega) \cap W^{2,\infty}_{\text{loc}}(\Omega)$ for any $1 < p < \infty$. Let $\tilde{\Omega} = \{\tilde{u} > 0\}$ and $\tilde{S} = \partial\tilde{\Omega} \cap \Omega$. The following lemma shows that the free boundary $\tilde{S}$ is not empty.

LEMMA 2.1. *There is no positive solution for the Dirichlet problem*

$$\begin{aligned} -\nabla \cdot (h^3 \nabla u) &= a \sin x \quad in \quad \Omega, \\ u &= 0 \quad on \quad \partial\Omega. \end{aligned} \tag{D}$$

Proof. Suppose that there exists a solution u of (D) such that $u > 0$ in Ω. Integrating the partial differential equation over Ω and applying the divergence theorem, we obtain

$$-\int_{\partial\Omega} h^3 \frac{\partial u}{\partial \nu} = \int_\Omega a \sin x = 0. \tag{2.2}$$

Since $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, we have

$$\frac{\partial u}{\partial \nu} \leq 0 \quad \text{on} \quad \partial\Omega, \tag{2.3}$$

and, by the strong maximum principle,

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega \cap \partial R_1, \tag{2.4}$$

since $-\nabla \cdot (h^3 \nabla u) \geq 0$ in R_1. It follows that the left-hand side of (2.2) is positive, a contradiction. □

THEOREM 2.2. *The solution $\tilde{u}$ of Problem($\tilde{P}$) is symmetric with respect to $y = 1/2$ and satisfies*

$$\tilde{u}_y \leq 0 \quad in \quad Q \equiv (0, 2\pi) \times (1/2, 1).$$

The free boundary $\tilde{S}$ is non-empty and is contained in R_2. Consequently, $\tilde{S} \cap Q$ is an x-graph which is given by

$$y = \tilde{\rho}(x) = \sup\{\eta \in (1/2, 1); \tilde{u}(x, \eta) > 0\}$$

for $x \in (\pi, 2\pi)$ such that $u(x, 1/2) > 0$, where $\tilde{\rho}$ is real analytic in $\{x; \tilde{\rho}(x) < 1\}$. Moreover, $\tilde{u}(x_0, 1/2) = 0$ for some $x_0 \in (\pi, 2\pi)$ implies that $\tilde{u} \equiv 0$ in $[x_0, 2\pi] \times [0, 1]$, i.e., $\tilde{S} \cap \{y = 1/2\}$ is a single point.

Proof. The first part of the theorem can be proved by using a reflection argument with the aid of the maximum principle. Utilizing a method used in [8](see also the proof of

Theorem 3.3 below), one can obtain the Lipschitz continuity of $\tilde{\rho}$. Then from the regularity results for the free boundary of variational inequality (cf. [6, Chap.2]) the analyticity of $\tilde{\rho}$ follows. Finally, the last statement of the theorem follows from the maximum principle. □

Now, we introduce the function

$$\sigma(x) = \int_0^x \frac{a(1+\cos\xi)}{h^3(\xi)} d\xi. \tag{2.5}$$

Note that $\sigma(0) = 0$, $\sigma(x) > 0$ if $x > 0$, and σ satisfies the equation

$$-\nabla \cdot (h^3 \nabla \sigma) = a \sin x.$$

By the strong maximum principle

$$\tilde{u} < \sigma \quad \text{in} \quad \Omega, \quad \frac{\partial \tilde{u}}{\partial x} < \frac{\partial \sigma}{\partial x} = \frac{2a}{(1+a)^3} \quad \text{on} \quad \partial_0 \Omega, \tag{2.6}$$

since $\tilde{u} = \sigma = 0$ on $\partial_0\Omega$ and $-\nabla \cdot (h^3 \nabla(\tilde{u} - \sigma)) = 0$ in $\tilde{\Omega}$. We define a function $\tilde{\theta}(y)$ on $(0, 1)$ by

$$\frac{\partial \tilde{u}}{\partial x}(0, y) = \frac{1 - \tilde{\theta}(y)}{(1+a)^2}. \tag{2.7}$$

Then from (2.6) and (2.7) it follows that $\tilde{\theta}$ satisfies the condition

$$\tilde{\theta} > \frac{1-a}{1+a}. \tag{2.8}$$

Let

$$\tilde{g}(x, y) = h(x) - (1+a)\tilde{\theta}(y). \tag{2.9}$$

We claim that $\tilde{u}$ satisfies the variational inequality:

$$\int_\Omega h^3 \nabla \tilde{u} \cdot \nabla(\phi - \tilde{u}) \geq \int_\Omega \tilde{g}(\phi - \tilde{u})_x, \forall \phi \in K_0. \tag{2.10}$$

Indeed, since $\tilde{S}$ is smooth, for any $\phi \in K_0$, we have

$$\begin{aligned} \int_\Omega h^3 \nabla \tilde{u} \cdot \nabla(\phi - \tilde{u}) &= \int_{\tilde{\Omega}} h^3 \nabla \tilde{u} \cdot \nabla(\phi - \tilde{u}) \\ &= \int_{\tilde{\Omega}} (a \sin x)(\phi - \tilde{u}) - \int_{\partial_0 \Omega} (1+a)^3 \frac{\partial \tilde{u}}{\partial x} \phi. \end{aligned} \tag{2.11}$$

On the other hand, the right-hand side of (2.10) can be computed as

$$\begin{aligned} \int_\Omega \tilde{g}(\phi - \tilde{u})_x &= \int_\Omega (a \sin x)(\phi - \tilde{u}) - \int_{\partial_0 \Omega} (1+a)(1 - \tilde{\theta})\phi \\ &= \int_{\tilde{\Omega}} (a \sin x)(\phi - \tilde{u}) + \int_{\Omega \setminus \tilde{\Omega}} (a \sin x)\phi - \int_{\partial_0 \Omega} (1+a)(1 - \tilde{\theta})\phi. \end{aligned} \tag{2.12}$$

Since $\sin x < 0$ for $x \in (\pi, 2\pi)$ and $\phi \geq 0$ in R_2, (2.10) follows from (2.7), (2.11) and (2.12).

The following theorem is as same as Theorem 5.1 in [4]. Here we give a different proof.

Theorem 2.3. *If $\theta \le \tilde{\theta}$, then $u \le \tilde{u}$ in Ω and hence $u = 0$ on $\partial\Omega$.*

Proof. Recall that u satisfies

$$\int_\Omega h^3 \nabla u \cdot \nabla(\phi - u) \ge \int_{\Omega_1} \gamma g(\phi - u)_x + \int_{\Omega_2} g(\phi - u)_x, \forall \phi \in K_0. \qquad (2.13)$$

Since $\phi = u \pm (u - \tilde{u})^+ \in K_0$, $\gamma = 1$ if $u > 0$ and $(u - \tilde{u})^+ = 0$ if $u = 0$, we get

$$\int_\Omega h^3 \nabla u \cdot \nabla (u - \tilde{u})^+ = \int_\Omega g[(u - \tilde{u})^+]_x. \qquad (2.14)$$

Also, taking $\phi = \tilde{u} + (u - \tilde{u})^+ \in K_0$ in (2.10),

$$\int_\Omega h^3 \nabla \tilde{u} \cdot \nabla (u - \tilde{u})^+ \ge \int_\Omega \tilde{g}\,[(u - \tilde{u})^+]_x. \qquad (2.15)$$

Subtracting (2.15) from (2.14) we obtain

$$\int_\Omega h^3 |\nabla (u - \tilde{u})^+|^2 \le \int_\Omega (g - \tilde{g})[(u - \tilde{u})^+]_x. \qquad (2.16)$$

Since $g \ge \tilde{g}$ by assumption and $(g - \tilde{g})_x = 0$, an integration by parts of the integral on the right-hand side of (2.16) gives

$$\int_\Omega (g - \tilde{g})[(u - \tilde{u})^+]_x = - \int_{\partial_0 \Omega} (g - \tilde{g}) u \le 0.$$

Hence $(u - \tilde{u})^+ = 0$ and the theorem follows. □

Remark 2.4. Note that $0 < \tilde{\theta} < 1$ was proved in [4]. However, (2.8) must be verified in order to ensure the existence of θ in the hypotheses of Theorem 2.3.

Next, we shall prove some properties of the free boundary S. Let

$$K = \{\psi \in H^1(\Omega_2); \psi = u \quad \text{on} \quad T, \psi = 0 \quad \text{on} \quad \partial\Omega_2 \setminus T, \psi \ge 0 \quad \text{a.e. in} \quad \Omega_2\}.$$

Notice that the set K is a closed convex subset of $H^1(\Omega_2)$. By the matching lemma (cf. [6, p.31]), the function

$$\phi = \begin{cases} u & \text{in} \quad \Omega_1 \\ \psi & \text{in} \quad \Omega_2 \end{cases}$$

is in K_0 for any $\psi \in K$. Hence, by (0.8),

$$\int_{\Omega_2} h^3 \, \nabla u \cdot \nabla (\psi - u) \ge \int_{\Omega_2} (a \sin x)(\psi - u)$$

for all $\psi \in K$. From general regularity results for variational inequalities(cf. [6, p.44]) it follows that

$$u \in C^{1,1}_{\text{loc}}(\Omega_2). \tag{2.17}$$

LEMMA 2.5. *Let $\Omega_{21} = \{(x,y) \in R_1; g(x,y) < 0\} = R_1 \backslash \overline{\Omega_1}$. Then $u > 0$ in $\Omega_{21} \cup \{x = \pi\}$ and consequently S is a union of two disjoint sets S_1 and S_2 such that $S_1 \subset \Omega_1 \cup T$ and $S_2 \subset R_2$.*

Proof. In Ω_{21}, u satisfies the inequality

$$-\nabla \cdot (h^3 \nabla u) \geq a \sin x > 0.$$

From the non-negativity of u and the strong maximum principle it follows that

$$u > 0 \quad \text{in} \quad \Omega_{21}.$$

Hence S is a union of two disjoint sets S_1 and S_2 such that $S_1 \subset \Omega_1 \cup T$ and $S_2 \subset R_2 \cup \{x = \pi\}$. We claim that $S_2 \subset R_2$. If there is a point $(x_0, y_0) \in S_2$ with $x_0 = \pi$, then $u(x_0, y_0) = 0$ and $\nabla u(x_0, y_0) = 0$ by (2.17). On the other hand, by the Hopf boundary point lemma, we have

$$\frac{\partial u}{\partial x} < 0 \quad \text{at} \quad (x_0, y_0),$$

which is a contradiction. □

PROPOSITION 2.6. *If $\theta \leq \tilde{\theta}$ then $S_2 \neq \emptyset$.*

Proof. This is a simple consequence of Theorems 2.2 and 2.3. □

PROPOSITION 2.7. *If $\theta \leq \tilde{\theta}$ and θ is not identically equal to $\tilde{\theta}$, then $S_1 \neq \emptyset$.*

Proof. Suppose that $S_1 = \emptyset$. Then $u > 0$ in Ω_1 and $\gamma \equiv 1$ in Ω_1. For any $\phi \in \tilde{K} \subset K_0$ we have

$$\int_\Omega h^3 \nabla u \cdot \nabla(\phi - u) \geq \int_{\Omega_1} g(\phi - u)_x + \int_{\Omega_2} (a \sin x)(\phi - u). \tag{2.18}$$

Since

$$\int_{\Omega_1} g(\phi - u)_x = \int_{\Omega_1} (a \sin x)(\phi - u),$$

using $g = 0$ on T and $\phi = u = 0$ on $\partial\Omega$, we conclude from (2.18) that u is a solution of Problem($\tilde{P}$). Therefore, $u \equiv \tilde{u}$ by the uniqueness of solution to Problem($\tilde{P}$). Since

$$\frac{\partial u}{\partial x} = \frac{1 - \theta}{(1+a)^2} \quad \text{and} \quad \frac{\partial \tilde{u}}{\partial x} = \frac{1 - \tilde{\theta}}{(1+a)^2} \quad \text{on} \quad \partial_0 \Omega,$$

we obtain that $\theta \equiv \tilde{\theta}$, a contradiction. □

LEMMA 2.8. *The set Ω_2^+ is connected.*

Proof. Suppose that there is a component G of Ω_2^+ such that $G \subset R_2$. Since

$$-\nabla\cdot(h^3\nabla u) = a\sin x < 0 \quad \text{in} \quad G,$$
$$u = 0 \quad \text{on} \quad \partial G,$$

it follows from the maximum principle that $u < 0$ in G, a contradiction. □

LEMMA 2.9. *$\gamma_x \geq 0$ in Ω_1 in the distribution sense.*

Proof. The idea of proof is essentially due to Alt[1]. For any $\zeta \in C_0^\infty(\Omega_1)$, the function $\phi = u \pm \zeta$ is in K_0. Hence we obtain

$$\int_{\Omega_1} h^3\nabla u\cdot\nabla\zeta = \int_{\Omega_1} \gamma\, g\zeta_x. \tag{2.19}$$

It follows that

$$-\nabla\cdot(h^3\nabla u) + \gamma\, g_x + \gamma_x\, g = 0 \quad \text{in} \quad \Omega_1 \tag{2.20}$$

in the distribution sense.

For any $\zeta \in C_0^\infty(\Omega_1), \zeta \geq 0$ and $\epsilon > 0$, we have

$$\int_{\Omega_1} h^3\nabla u\cdot\nabla[\min(u,\epsilon\zeta)] = \int_{\Omega_1} \gamma\, g[\min(u,\epsilon\zeta)]_x,$$

or

$$\epsilon\int_{\Omega_1\cap\{u>\epsilon\zeta\}} h^3\nabla u\cdot\nabla\zeta + \int_{\Omega_1\cap\{u\leq\epsilon\zeta\}} h^3|\nabla u|^2 = \int_{\Omega_1} g[\min(u,\epsilon\zeta)]_x, \tag{2.21}$$

since $u > 0$ implies that $\gamma = 1$. The right-hand side of (2.21) can be computed as

$$\begin{aligned}\int_{\Omega_1} g[\min(u,\epsilon\zeta)]_x &= -\int_{\Omega_1} g_x[\min(u,\epsilon\zeta)]\\ &= -\int_{\Omega_1^+} g_x[\min(u,\epsilon\zeta)]\\ &= -\epsilon\int_{\Omega_1^+} g_x\zeta + \int_{\Omega_1^+} g_x[\epsilon\zeta - \min(u,\epsilon\zeta)]\\ &= -\epsilon\int_{\Omega_1^+} \gamma\, g_x\zeta + \epsilon\int_{\Omega_1^+} g_x(\zeta - u/\epsilon)^+.\end{aligned}$$

Therefore, we obtain

$$\int_{\Omega_1\cap\{u>\epsilon\zeta\}} h^3\nabla u\cdot\nabla\zeta + \int_{\Omega_1^+} \gamma\, g_x\zeta \leq \int_{\Omega_1^+} g_x(\zeta - u/\epsilon)^+. \tag{2.22}$$

Since the integral

$$\int_{\Omega_1^+} g_x(\zeta - u/\epsilon)^+$$

tends to zero as $\epsilon \to 0$, letting $\epsilon \to 0$ in (2.22), we get

$$\int_{\Omega_1} h^3 \nabla u \cdot \nabla \zeta + \int_{\Omega_1} \gamma g_x \zeta \le 0, \tag{2.23}$$

here we used the facts that $\gamma \ge 0$, $\zeta \ge 0$, and $g_x < 0$ in Ω_1. Recalling that $g > 0$ in Ω_1, the lemma follows from (2.20) and (2.23). □

In the sequel, we let $\rho(y) = \arccos\{[(1+a)\theta(y) - 1]/a\}$. Note that $(x, y) \in T$ if and only if $x = \rho(y)$.

LEMMA 2.10. *There is an upper semi-continuous function ρ_1 such that*

$$R_1^+ = \{(x, y) \in R_1; x > \rho_1(y)\}. \tag{2.24}$$

Proof. If $\Omega_1^+ = \emptyset$, then we can choose $\rho_1 = \rho$ by Lemma 2.5. Suppose that $\Omega_1^+ \ne \emptyset$. Given $(x_0, y_0) \in \Omega_1^+$, there is a positive constant δ such that the ball $B_0 \equiv B_\delta(x_0, y_0)$ is contained in Ω_1^+, since Ω_1^+ is open by the continuity of u. Hence $\gamma = 1$ in B_0. By Lemma 2.9, we deduce that $\gamma = 1$ in the set

$$Q \equiv \{(x, y) \in \Omega_1; x > x_0, |y - y_0| < \delta\} \cup B_0.$$

Hence u satisfies

$$-\nabla \cdot (h^3 \nabla u) = a \sin x \quad \text{in} \quad Q,$$

and by the strong maximum principle, $u > 0$ in Q. We claim that $u > 0$ at the point $P = (\rho(y_0), y_0)$. For contradiction, we suppose that $u(P) = 0$. Let n be the unit outer normal to $\partial\Omega_1$ and denote by u^l and u^r the function u in Ω_1 and Ω_2 respectively. From Lemmas 2.5 it follows that there is a ball $B = B_r(P)$ for some $r \in (0, \delta)$ such that $u > 0$ in $B \setminus T$. For any $\zeta \in C_0^\infty(B)$ such that $\zeta \ge 0$ in B, by (0.8), we have

$$\int_B h^3 \nabla u \cdot \nabla \zeta \ge \int_{\Omega_1 \cap B} g \zeta_x + \int_{\Omega_2 \cap B} (a \sin x) \zeta. \tag{2.25}$$

The left-hand side of (2.25) can be computed as

$$\begin{aligned}
\int_B h^3 \nabla u \cdot \nabla \zeta &= \int_{B\cap\Omega_1} h^3 \nabla u^l \cdot \nabla \zeta + \int_{B\cap\Omega_2} h^3 \nabla u^r \cdot \nabla \zeta \\
&= -\int_B \nabla \cdot (h^3 \nabla u) \zeta + \int_{T\cap B} h^3 \frac{\partial u^l}{\partial n} \zeta - \int_{T\cap B} h^3 \frac{\partial u^r}{\partial n} \zeta \\
&= \int_B (a \sin x) \zeta + \int_{T\cap B} h^3 \left(\frac{\partial u^l}{\partial n} - \frac{\partial u^r}{\partial n}\right) \zeta.
\end{aligned}$$

Using $g = 0$ on T, the right-hand side of (2.25) is equal to

$$\int_B (a \sin x)\zeta.$$

Hence we obtain from (2.25)

$$\int_{T\cap B} h^3\left(\frac{\partial u^l}{\partial n} - \frac{\partial u^r}{\partial n}\right)\zeta \geq 0,$$

i.e.,

$$\frac{\partial u^l}{\partial n} - \frac{\partial u^r}{\partial n} \geq 0 \text{ on } T \cap B.$$

This is a contradiction, since $\partial u^l/\partial n < 0$ and $\partial u^r/\partial n > 0$ at P by the Hopf boundary point lemma. Hence $u(P) > 0$.

Defining

$$\begin{aligned} \rho_1(y_0) &= \inf\{x; u(x, y_0) > 0, (x, y_0) \in \Omega_1\} \quad \text{if} \quad (x_0, y_0) \in \Omega_1^+ \quad \text{for some} \quad x_0, \\ &= \rho(y_0) \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad u(x, y_0) = 0, \forall (x, y_0) \in \Omega_1, \end{aligned}$$

we conclude that (2.24) holds. Finally, the upper semi-continuity of ρ_1 follows from the continuity of u. □

As a consequence of Lemmas 2.8 and 2.10, we have:

THEOREM 2.11. *The set Ω^+ is connected and*

$$S_1 = \{(x, y); x = \rho_1(y), y \in (0, 1) \text{ such that } \rho_1(y) > 0\}.$$

§3. Regularity of free boundary. In the rest of this paper we shall assume that

$$\theta < \tilde{\theta}; \tag{3.1}$$

then by Theorem 2.3, Propositions 2.6 and 2.7 $u = 0$ on $\partial\Omega$, $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

LEMMA 3.1. *Let $0 < y_1 < y_2 < 1$ and $x_0 > 0$ such that*

$$(a) \quad x_0 \leq \min_{y \in [y_1, y_2]} \rho(y),$$

$$(b) \quad u = 0 \quad \text{on} \quad (0, x_0) \times \{y_1, y_2\}.$$

Then

$$\int_Q (-h^3 u_x + \gamma g) \leq 0.$$

Proof. We adopt a method from [5]. Introduce $Q \equiv (0, x_0) \times (y_1, y_2)$. Then $Q \subset \Omega_1$ by assumption (a). Let $\zeta \in H^1(Q) \cap C^0(\bar{Q})$ be a non-negative function such that $\zeta = 0$ on $x = x_0$. Then for any $\epsilon > 0$ the function

$$\phi = \begin{cases} u \pm \min(u, \epsilon\zeta) & \text{in} \quad Q \\ u \quad \text{elsewhere} \end{cases}$$

is in K_0 by assumption (b). Hence

$$\int_Q h^3 \nabla u \cdot \nabla[\min(u, \epsilon\zeta)] = \int_Q \gamma g[\min(u, \epsilon\zeta)]_x,$$

or

$$\epsilon \int_{Q \cap \{u > \epsilon\zeta\}} h^3 \nabla u \cdot \nabla\zeta + \int_{Q \cap \{u \le \epsilon\zeta\}} h^3 |\nabla u|^2 = \int_Q g[\min(u, \epsilon\zeta)]_x,$$

since $u > 0$ implies that $\gamma = 1$. Therefore, we obtain

$$\int_{Q \cap \{u > \epsilon\zeta\}} h^3 \nabla u \cdot \nabla\zeta - \int_Q g\zeta_x \le -\int_Q g[(\zeta - u/\epsilon)^+]_x. \tag{3.2}$$

The right-hand side of (3.2) can be computed as

$$\begin{aligned} -\int_Q g[(\zeta - u/\epsilon)^+]_x &= -\int_{y_1}^{y_2} \int_0^{x_0} g[(\zeta - u/\epsilon)^+]_x \, dx \, dy \\ &= \int_{y_1}^{y_2} \int_0^{x_0} g_x(\zeta - u/\epsilon)^+ \, dx \, dy \\ &\quad - \int_{y_1}^{y_2} [g(\zeta - u/\epsilon)^+](x_0, y) \, dy + \int_{y_1}^{y_2} [g(\zeta - u/\epsilon)^+](0, y) \, dy \\ &\le \int_{y_1}^{y_2} g(0, y)\zeta(0, y) \, dy, \end{aligned}$$

using the facts that $\zeta = 0$ on $x = x_0$ and $g_x < 0$ in Q. Letting $\epsilon \to 0$ in (3.2), we obtain

$$\int_Q h^3 \nabla u \cdot \nabla\zeta - \int_Q g\zeta_x \le \int_{y_1}^{y_2} g(0, y)\zeta(0, y) \, dy. \tag{3.3}$$

Now, let $\sigma(x) = x_0 - x$ in $[0, x_0]$ and $= 0$ for $x > x_0$. Define

$$\eta_\delta(y) = \begin{cases} 1 & \text{if} \quad y \in [y_1 + \delta, y_2 - \delta], \\ 0 & \text{if} \quad y \notin (y_1, y_2), \\ \text{linear} & \text{if} \quad y \in [y_1, y_1 + \delta] \cup [y_2 - \delta, y_2], \end{cases}$$

for any small positive constant δ. Then the function

$$\phi^\delta_\pm(x,y) = u(x,y) \pm \sigma(x)\eta_\delta(y)$$

is in K_0. Let

$$\zeta^\delta(x,y) = \sigma(x)[1-\eta_\delta(y)].$$

Then $\zeta^\delta \in H^1(Q)\cap C^0(\bar{Q})$ and satisfies

$$\zeta^\delta \geq 0 \quad \text{in} \quad Q, \quad \zeta^\delta = 0 \quad \text{on} \quad x = x_0.$$

Therefore, we obtain

$$\begin{aligned}\int_Q(-h^3u_x+\gamma g) &= \int_Q(h^3\nabla u\cdot\nabla\sigma - \gamma g\sigma_x)\\ &= \int_Q\left[h^3\nabla u\cdot\nabla(\sigma\eta_\delta) - \gamma g(\sigma\eta_\delta)_x\right]\\ &+ \int_Q\left[h^3\nabla u\cdot\nabla[\sigma(1-\eta_\delta)] - \gamma\, g[\sigma(1-\eta_\delta)]_x\right].\end{aligned} \tag{3.4}$$

The first integral on the right-hand side of (3.4) vanishes since $\phi^\delta_\pm \in K_0$. From (3.3) it follows that the second integral on the right-hand side of (3.4) is less than or equal to

$$\int_{y_1}^{y_2} g(0,y)x_0[1-\eta_\delta(y)]\,dy + \int_Q(1-\gamma)g[\sigma(1-\eta_\delta)]_x. \tag{3.5}$$

Since the second integral in (3.5) is less than or equal to zero and the first integral in (3.5) tends to zero as δ tends to zero, the lemma follows. □

LEMMA 3.2. *If $u=0$ in a ball $B=B_r(x_0,y_0)$ which is contained in Ω_1 for some $r>0$, then $\gamma=0$ a.e. in $\tilde{B}$, where*

$$\tilde{B} = \{(x,y); x<x_0, |y-y_0|<r\}\cup B.$$

Proof. For any $y_1<y_2$ with $y_1\geq y_0-r$ and $y_2\leq y_0+r$, by Lemmas 2.10 and 3.1, we have

$$\int_Q(-h^3u_x+\gamma\, g)\leq 0, \tag{3.6}$$

where $Q=(0,x_0+\bar{x})\times(y_1,y_2)\subset\tilde{B}$ and $\bar{x}=\min(\sqrt{r^2-(y_2-y_0)^2},\sqrt{r^2-(y_1-y_0)^2})$. Since $u=0$ in $\tilde{B}$ by Lemma 2.10, (3.6) implies that

$$\int_Q\gamma\, g\leq 0.$$

From the facts $g > 0$ and $\gamma \geq 0$ in Ω_1 it follows that $\gamma = 0$ a.e. in Q. Therefore, $\gamma = 0$ a.e. in $\tilde{B}$. □

Let δ be any positive constant. For any set A, let

$$A^* = A \cup E, \quad \text{where} \quad E = (-\delta, 0] \times (0, 1).$$

Extend u and γ to be zero in E. Notice that $u \in H_0^1(\Omega^*)$ by Theorem 2.3. For any $\zeta \in C_0^\infty(\Omega_1^*)$ the function $\phi = u \pm \zeta$ restricted to Ω is in K_0. Hence

$$\int_{\Omega_1} h^3 \nabla u \cdot \nabla \zeta = \int_{\Omega_1} \gamma g \zeta_x.$$

Therefore, we obtain

$$\int_{\Omega_1^*} h^3 \nabla u \cdot \nabla \zeta = \int_{\Omega_1^*} \gamma g \zeta_x,$$

i.e., u satisfies the equation

$$-\nabla \cdot (h^3 \nabla u) + (\gamma g)_x = 0 \text{ in } \Omega_1^* \tag{3.7}$$

in the weak sense. Now, we are ready to prove the Lipschitz continuity of $S_1 \setminus T$.

THEOREM 3.3. *The free boundary $S_1 \setminus T$ is locally Lipschitz continuous.*

Proof. Let $P \in S_1 \setminus T$ and let r_0 be a positive number such that the ball $B_0 = B_{r_0}(P)$ is contained in Ω_1. Adapting a method of Alt[2], we introduce the function

$$w(x, y) = \int_{-\delta}^{x} h^3(\xi) u(\xi, y)\, d\xi. \tag{3.8}$$

Then $u(x, y) = 0$ if and only if $w(x, y) = 0$. We have

$$\begin{aligned} w_x &= h^3\, u, \\ w_{xx} &= h^3\, u_x + 3\frac{h'}{h} w_x, \\ w_y(x, y) &= \int_{-\delta}^{x} h^3(\xi) u_y(\xi, y)\, d\xi. \end{aligned}$$

We approximate γ by a sequence $\{\gamma_n\}$ of functions in $C^\infty(\Omega_1^*) \cap L^\infty(\Omega_1^*)$ such that $\gamma_n = 0$ in a neighborhood of $\{x = -\delta, 0 < y < 1\}$. For each n let u_n be the solution of

$$\begin{cases} -\nabla \cdot (h^3 \nabla u_n) + (\gamma_n g)_x = 0 \quad \text{in} \quad \Omega_1^* \\ u_n = u \quad \text{on} \quad \partial \Omega_1^*. \end{cases} \tag{3.9}$$

Note that $u_n \to u$ uniformly in $\overline{\Omega_1^*}$ and strongly in $H^1(\Omega_1^*)$. Let w_n be defined as in (3.8) for u being replaced by u_n. We compute using (3.9) and $u_{n,x}(-\delta, y) = \gamma_n(-\delta, y) = 0$

$$\begin{aligned} w_{n,yy}(x,y) &= \int_{-\delta}^{x} [h^3(\xi) u_{n,y}(\xi, y)]_y \, d\xi \\ &= -\int_{-\delta}^{x} [h^3(\xi) u_{n,\xi}(\xi, y)]_\xi \, d\xi + \int_{-\delta}^{x} (\gamma_n g)_\xi(\xi, y) \, d\xi \\ &= -w_{n,xx}(x,y) + 3(\frac{h'}{h} w_{n,x})(x,y) + (\gamma_n g)(x,y). \end{aligned}$$

Hence we obtain

$$-\Delta w_n + 3\frac{h'}{h} w_{n,x} = -\gamma_n g \quad \text{in} \quad \Omega_1^*. \tag{3.10}$$

Therefore, for any $\zeta \in C_0^\infty(\Omega_1^*)$ we have

$$\int_{\Omega_1^*} \nabla w_n \cdot \nabla \zeta + \int_{\Omega_1^*} 3\frac{h'}{h} w_{n,x} \zeta + \int_{\Omega_1^*} \gamma_n g \zeta = 0. \tag{3.11}$$

Letting $n \to \infty$ in (3.11), we obtain that w satisfies

$$-\Delta w + 3\frac{h'}{h} w_x = -\gamma g \quad \text{in} \quad \Omega_1^* \tag{3.12}$$

in the weak sense. Since $\gamma \in L^\infty(\Omega_1^*)$, (3.12) holds in the strong sense. Recall that $0 \le \gamma \le 1$ and $\gamma = 1$ if $w > 0$. Hence w satisfies the variational inequality

$$\begin{aligned} -\Delta w + 3\frac{h'}{h} w_x &\ge -g, \\ w &\ge 0, \\ w[-\Delta w + 3\frac{h'}{h} w_x + g] &= 0 \end{aligned} \tag{3.13}$$

a.e. in Ω_1 and $w_x > 0$ in Ω_1^+. Note that $w \in C^{1,1}_{\text{loc}}(\Omega_1)$ by general regularity results for variational inequalities(cf. [6, p.44]).

For any $r \in (0, r_0)$, let $B_r = B_r(P)$ and $B_{r/2} = B_{r/2}(P)$. Let ζ be a smooth function such that $0 \le \zeta \le 1$, $\zeta = 1$ on ∂B_r and $\zeta = 0$ in $B_{r/2}$. Following the method used in [8], we consider the function

$$v(x,y) = -b w_x(x,y) + c w_y(x,y) + w(x,y) - d\zeta(x,y),$$

where b, c and d are constants with $b > 0$, $|c|$ small and $d > 0$. Since

$$\begin{aligned} -\Delta w &= -3\frac{h'}{h} w_x - g, \\ \Delta w_x &= 3\frac{h'}{h} w_{xx} + 3\frac{h''}{h} w_x - 3\frac{h'^2}{h^2} w_x - a \sin x, \\ -\Delta w_y &= -3\frac{h'}{h} w_{xy} + (1+a)\theta' \end{aligned}$$

in $B_r \cap \Omega_1^+$, we obtain

$$-\Delta v + 3\frac{h'}{h}v_x = 3bw_x(\frac{h''}{h} - \frac{h'^2}{h^2}) - ba\sin x \\ + c(1+a)\theta' - g + d(\Delta\zeta - 3\frac{h'}{h}\zeta_x) \tag{3.14}$$

in $B_r \cap \Omega_1^+$. Since $a \sin x \geq \delta$ in B_0 for some positive constant δ and there is a positive constant M such that

$$||w||_{C^{1,1}(B_0)} \leq M,$$

one can choose r so small that the first two terms on the right-hand side of (3.14) is less than or equal to $-b\delta/2$. Hence we obtain

$$-\Delta v + 3\frac{h'}{h}v_x \leq 0 \quad \text{in} \quad B_r \cap \Omega_1^+, \tag{3.15}$$

if b is sufficiently large and $|c|$ and d are sufficiently small. For the boundary conditions, we have

$$v = -d\zeta \leq 0 \quad \text{on} \quad S_1,$$
$$v \leq cw_y + d/2 - d \leq 0 \quad \text{on} \quad \partial B_r \cap \Omega_1^+ \cap \{w \leq d/2\},$$

if $|c|$ is sufficiently small, and

$$v \leq -b\inf_\Gamma w_x + d/2 + \sup w - d \leq 0 \quad \text{on} \quad \Gamma \equiv \partial B_r \cap \Omega_1^+ \cap \{w > d/2\},$$

if b is sufficiently large. Therefore, it follows from (3.15) and the maximum principle that $v \leq 0$ in $B_r \cap \Omega_1^+$. In particular, we have

$$-bw_x + cw_y \leq 0 \quad \text{in} \quad B_{r/2}. \tag{3.16}$$

and the theorem follows. □

Remark 3.4. From (3.16) and the regularity theorems for the free boundary of variational inequality(cf. [6, Chap. 2]) it follows that $S_1 \setminus T$ is in $C^{m+1,\alpha}_{\text{loc}}$ if $g \in C^{m,\alpha}$, where $m \geq 1$ and $0 < \alpha < 1$. Notice that the inhomogeneous term $-g$ in (3.13) being strictly uniformly negative is needed to get C^1 regularity.

Remark 3.5. The proof of Theorem 3.3 cannot be applied at the points $P \in S_1 \cap T$, since we do not know whether u satisfies an elliptic equation in a neighborhood of P.

In the sequel, we let $\pi_y(A)$ be the projection of A onto the y-axis.

THEOREM 3.6. *$\rho_1(y)$ is continuous.*

Proof. In view of Theorem 3.3, it remains to prove the continuity of ρ_1 only at points $y \in \pi_y(S_1 \cap T)$. Suppose for contradiction that ρ_1 is not continuous at y_0 for some $y_0 \in \pi_y(S_1 \cap T)$. Since ρ_1 is upper semi-continuous, there is a sequence $y_n \to y_0$ such that

$$x_0 \equiv \lim_{n\to\infty} \rho_1(y_n) < \rho_1(y_0) = \rho(y_0).$$

Set $x_n = \rho_1(y_n)$, $P_0 = (x_0, y_0)$ and $P_n = (x_n, y_n)$. Let r_0 be a positive number such that $B = B_{r_0}(P_0) \subset \Omega_1$. Then there are positive constants δ and M such that

$$a \sin x \geq \delta \quad \text{in} \quad B, \quad ||w||_{C^{1,1}(B)} \leq M.$$

Without loss of generality we may assume that $P_n \in B_{r_0/2}(P_0), \forall n \geq 1$. From the proof of Theorem 3.3 there exist positive constants ω and r depending only on δ and M such that $u > 0$ in $A_\omega(P_n) \cap B_r(P_n)$ for all n, where $A_\omega(P_n)$ is the cone with axis parallel to x-axis, vertex at P_n and opening angle ω. This is a contradiction, since $u = 0$ on $\{0 \leq x \leq \rho(y_0), y = y_0\}$ by Lemma 2.10. Hence the theorem follows. □

COROLLARY 3.7. *For any solution (u, γ) of Problem(P_0),*

$$\gamma = I_{\Omega_1^+} \quad \text{a.e. in} \quad \Omega_1.$$

Proof. Let (x_0, y_0) be a point in Ω_1 with $x_0 < \rho_1(y_0)$. From Theorem 3.6 it follows that there exists a positive number r such that the ball $B = B_r(x_0, y_0)$ is contained in the set $\{u = 0\}$. By Lemma 3.2, we obtain that $\gamma = 0$ a.e. in $\tilde{B}$, where $\tilde{B}$ is defined as in Lemma 3.2. Since the point (x_0, y_0) is arbitrary and the measure of the set $\{(x, y); x = \rho_1(y)\}$ is zero, the corollary follows. □

PROPOSITION 3.8. *The set $\{y; \rho_1(y) = \rho(y)\}$ has no interior points.*

Proof. Suppose that there exist y_0 and $\delta > 0$ such that

$$\rho_1(y) = \rho(y), \forall y \in (y_0 - \delta, y_0 + \delta).$$

Let $P = (\rho(y_0), y_0)$ and $B = B_\delta(P)$. Recall the notations n, u^r and u^l in the proof of Lemma 2.10. Then by the Hopf boundary lemma we have

$$\frac{\partial u^r}{\partial n} > 0 \quad \text{on} \quad B \cap T.$$

On the other hand, for any $\zeta \in C_0^\infty(B)$ with $\zeta \geq 0$, since $u = \gamma = 0$ in $B \cap \Omega_1$, we have

$$\int_{B\cap\Omega_2} h^3 \nabla u \cdot \nabla\zeta \geq \int_{B\cap\Omega_2} (a \sin x)\zeta. \tag{3.17}$$

The left-hand side of (3.17) can be computed as

$$\int_{B\cap\Omega_2} h^3 \nabla u \cdot \nabla\zeta = \int_{B\cap\Omega_2} (a \sin x)\zeta - \int_{B\cap T} h^3 \frac{\partial u^r}{\partial n}\zeta.$$

Hence we obtain from (3.17) that $\partial u^r/\partial n \leq 0$ on $B \cap T$, a contradiction. □

PROPOSITION 3.9. *The closure of $\pi_y(S_1)$ is $[0,1]$.*

Proof. Suppose that there is y_0 such that $y_0 \in (0,1) \setminus A$, where $A = \overline{\pi_y(S_1)}$. Then there exists a positive constant δ such that $J \equiv (y_0 - \delta, y_0 + \delta) \subset (0,1) \setminus A$. Hence we have $\rho_1(y) = 0, \forall y \in J$. Let $B = B_\delta(0, y_0)$. For any $\zeta \in C_0^\infty(B)$, since the function $u \pm \zeta$ restricted to Ω is in K_0 and $u > 0$ in $B \cap \Omega$, we have

$$\int_{B\cap\Omega_1} h^3 \nabla u \cdot \nabla \zeta = \int_{B\cap\Omega_1} g\,\zeta_x,$$

or

$$\int_{B\cap\Omega_1} (a \sin x)\zeta - \int_{B\cap\partial_0\Omega} h^3\, u_x \zeta = \int_{B\cap\Omega_1} (a \sin x)\zeta - \int_{B\cap\partial_0\Omega} g\zeta.$$

Hence

$$u_x = \frac{1-\theta}{(1+a)^2} \quad \text{on} \quad B \cap \partial_0\Omega. \tag{3.18}$$

From (2.7), (3.18) and Theorem 2.3 it follows that

$$\frac{1-\theta(y)}{(1+a)^2} \leq \frac{1-\tilde{\theta}(y)}{(1+a)^2}, \forall y \in J.$$

That is we have $\theta(y) \geq \tilde{\theta}(y)$ for all $y \in J$. This contradicts to the assumption that $\theta < \tilde{\theta}$ in $(0,1)$. Therefore, the set $(0,1)\setminus A$ is empty. Since A is closed, the proposition follows. ▯

Remark 3.10. If one can show that $S_1 \cap T = \emptyset$ and S_2 is smooth, then by Theorem 3.3 u will be a classical solution of Problem(P_c) for any solution (u, γ) of Problem(P_0). Note that if $S_1 \cap T \neq \emptyset$ then S_1 cannot be smooth near any point $P \in S_1 \cap T$. Indeed, otherwise we have $\partial u/\partial \nu < 0$ at P by the Hopf boundary point lemma. But from (0.5) and the fact $g = 0$ on T it follows that $\partial u/\partial \nu = 0$ at P, a contradiction. Note also that if one can prove that S_2 is a y-graph, then S_2 will be locally real analytic by the same reasoning as in Theorem 2.2.

§4. Uniqueness. Let (u_1, γ_1) and (u_2, γ_2) be two solutions of Problem(P_0). We shall prove that $u_1 = u_2$ and $\gamma_1 = \gamma_2$ using the method of Carrillo and Chipot(cf. [5]). Let ρ_i be the function defined in Lemma 2.10 corresponding to the solution $(u_i, \gamma_i), i = 1,2$, and let

$$u_0 = \min(u_1, u_2), \quad \gamma_0 = \min(\gamma_1, \gamma_2), \quad \rho_0 = \min(\rho_1, \rho_2),$$
$$D_i = \{y \in (0,1); \rho_i(y) > \rho_0(y)\}, i = 1,2.$$

LEMMA 4.1. *For any $\zeta \in H^1(\Omega) \cap C^0(\bar{\Omega})$ such that $\zeta \geq 0$ in Ω, we have*

$$\int_\Omega h^3 \nabla(u_1 - u_0) \cdot \nabla\zeta - \int_{\Omega_1} (\gamma_1 - \gamma_0) g \zeta_x \leq \int_{D_2} (g\zeta)(\rho_1(y), y)\, dy, \tag{4.1}$$

$$\int_\Omega h^3 \nabla(u_2 - u_0) \cdot \nabla\zeta - \int_{\Omega_1} (\gamma_2 - \gamma_0) g \zeta_x \leq \int_{D_1} (g\zeta)(\rho_2(y), y)\, dy. \tag{4.2}$$

Proof. Let ϵ be an arbitrary small positive number. Since $\phi = u_1 \pm \min(u_1 - u_0, \epsilon\zeta) \in K_0$, we have

$$\int_\Omega h^3 \nabla u_1 \cdot \nabla[\min(u_1 - u_0, \epsilon\zeta)] = \int_{\Omega_1} \gamma_1 g[\min(u_1 - u_0, \epsilon\zeta)]_x + \int_{\Omega_2} (a \sin x)[\min(u_1 - u_0, \epsilon\zeta)]. \tag{4.3}$$

On the other hand, $\phi = u_2 + \min(u_1 - u_0, \epsilon\zeta) \in K_0$ implies that

$$\int_\Omega h^3 \nabla u_2 \cdot \nabla[\min(u_1 - u_0, \epsilon\zeta)] \geq \int_{\Omega_1} \gamma_2 g[\min(u_1 - u_0, \epsilon\zeta)]_x + \int_{\Omega_2} (a \sin x)[\min(u_1 - u_0, \epsilon\zeta)]. \tag{4.4}$$

Substracting (4.4) from (4.3), we get

$$\int_\Omega h^3 \nabla(u_1 - u_2) \cdot \nabla[\min(u_1 - u_0, \epsilon\zeta)] \leq \int_{\Omega_1} (\gamma_1 - \gamma_2) g[\min(u_1 - u_0, \epsilon\zeta)]_x. \tag{4.5}$$

We write $u_1 - u_2 = (u_1 - u_0) + (u_0 - u_2)$ and $\gamma_1 - \gamma_2 = (\gamma_1 - \gamma_0) + (\gamma_0 - \gamma_2)$. Then from (4.5) and the identities $(u_0 - u_2)(u_1 - u_0) = 0$ and $(\gamma_0 - \gamma_2)(u_1 - u_0) = 0$ it follows that

$$\int_\Omega h^3 \nabla(u_1 - u_0) \cdot \nabla[\min(u_1 - u_0, \epsilon\zeta)] \leq \int_{\Omega_1} (\gamma_1 - \gamma_0) g[\min(u_1 - u_0, \epsilon\zeta)]_x. \tag{4.6}$$

Since

$$\int_{\Omega_1} (\gamma_1 - \gamma_0) g[\min(u_1 - u_0, \epsilon\zeta)]_x = \epsilon \int_{\Omega_1} (\gamma_1 - \gamma_0) g\zeta_x - \epsilon \int_{\Omega_1} (\gamma_1 - \gamma_0) g[(\zeta - \frac{u_1 - u_0}{\epsilon})^+]_x$$

and

$$\int_\Omega h^3 \nabla(u_1 - u_0) \cdot \nabla[\min(u_1 - u_0, \epsilon\zeta)] = \epsilon \int_{\Omega \cap \{u_1 - u_0 > \epsilon\zeta\}} h^3 \nabla(u_1 - u_0) \cdot \nabla\zeta + \int_{\Omega \cap \{u_1 - u_0 \leq \epsilon\zeta\}} h^3 |\nabla(u_1 - u_0)|^2,$$

we obtain from (4.6)

$$\int_{\Omega \cap \{u_1 - u_0 > \epsilon\zeta\}} h^3 \nabla(u_1 - u_0) \cdot \nabla\zeta \leq \int_{\Omega_1} (\gamma_1 - \gamma_0) g\zeta_x - \int_{\Omega_1} (\gamma_1 - \gamma_0) g\, [(\zeta - \frac{u_1 - u_0}{\epsilon})^+]_x. \tag{4.7}$$

By Lemma 2.10 and Corollary 3.7, the second integral on the right-hand side of (4.7) can be computed as

$$\begin{aligned}
-\int_{\Omega_1}(\gamma_1-\gamma_0)g[(\zeta-\frac{u_1-u_0}{\epsilon})^+]_x &= -\int_{\Omega_1\cap\{u_1>0\}\cap\{u_2=0\}} g[(\zeta-\frac{u_1}{\epsilon})^+]_x \\
&= -\int_{D_2}\int_{\rho_1(y)}^{\rho_2(y)} g[(\zeta-\frac{u_1}{\epsilon})^+]_x\,dx\,dy \\
&= \int_{D_2}\int_{\rho_1(y)}^{\rho_2(y)} g_x(\zeta-\frac{u_1}{\epsilon})^+\,dx\,dy \\
&\quad -\int_{D_2}[g(\zeta-\frac{u_1}{\epsilon})^+](\rho_2(y),y)\,dy \\
&\quad +\int_{D_2}[g(\zeta-\frac{u_1}{\epsilon})^+](\rho_1(y),y)\,dy \\
&\to \int_{D_2}(g\zeta)(\rho_1(y),y)\,dy \quad \text{as} \quad \epsilon\to 0.
\end{aligned}$$

Letting $\epsilon\to 0$ in (4.7), then (4.1) follows. The proof for (4.2) is similar. □

Now, we are ready to prove the following uniqueness theorem.

THEOREM 4.2. *There exists a unique solution to Problem(P_0).*

Proof. Choose a point P on $\{y=0\}\cap\partial\Omega_{21}$ such that the ball $B=B_r(P)$ only intersects Ω_{21} for some $r>0$. Note that $B^+(u_i)=B\cap\{y>0\}$ for $i=1,2$ by Lemma 2.5. Let $\Gamma=\partial B\cap\{y>0\}$ and let σ be the solution of problem:

$$\begin{cases}
\nabla\cdot(h^3\nabla\sigma)=0 \quad \text{in} \quad B,\\
\sigma=1 \quad \text{on} \quad \partial B\cap\{y\geq -r/4\},\\
0\leq\sigma<1 \quad \text{on} \quad \partial B\cap\{y<-r/4\}.
\end{cases}$$

Notice that $0\leq\sigma\leq 1$ in B by the maximum principle and $\partial\sigma/\partial\nu>0$ on Γ by the Hopf boundary point lemma. Then, by the divergence theorem, we have

$$\int_\Gamma h^3(u_1-u_0)\frac{\partial\sigma}{\partial\nu}=\int_{B+}h^3\nabla(u_1-u_0)\cdot\nabla\sigma, \tag{4.8}$$

since

$$\nabla\cdot[h^3\nabla\sigma(u_1-u_0)]=\nabla\cdot(h^3\nabla\sigma)(u_1-u_0)+h^3\nabla(u_1-u_0)\cdot\nabla\sigma$$

in B^+ and $u_1=u_0=0$ on $y=0$. We extend σ to be 1 outside B and still denote it by σ. Then we obtain from (4.8) that

$$\int_\Gamma h^3(u_1-u_0)\frac{\partial\sigma}{\partial\nu}=\int_\Omega h^3\nabla(u_1-u_0)\cdot\nabla\sigma-\int_{\Omega_1}(\gamma_1-\gamma_0)g\sigma_x. \tag{4.9}$$

For any $\delta > 0$ small, let η_δ be a smooth function such that $0 \le \eta_\delta \le 1$ and

$$\begin{aligned}\eta_\delta(x,y) &= 1 \quad \text{if} \quad (x,y) \in A \equiv \Omega_1^+(u_0) \cup \Omega_2 \cup (\partial\Omega_1 \setminus \partial_0\Omega) \\ &= 0 \quad \text{if} \quad d((x,y),A) > \delta,\end{aligned}$$

where $d((x,y),A)$ is the distance between the point (x,y) and the set A. Since $\phi = u_1 \pm (1-\eta_\delta)\sigma \in K_0$, we have

$$\int_\Omega h^3 \nabla u_1 \cdot \nabla[(1-\eta_\delta)\sigma] = \int_{\Omega_1} \gamma_1 g[(1-\eta_\delta)\sigma]_x. \tag{4.10}$$

Since $u_0(1-\eta_\delta) = 0$ and $\gamma_0(1-\eta_\delta) = 0$, we get

$$\int_\Omega h^3 \nabla u_0 \cdot \nabla[(1-\eta_\delta)\sigma] = \int_{\Omega_1} \gamma_0 g\,[(1-\eta_\delta)\sigma]_x. \tag{4.11}$$

From (4.1) and (4.9-11) it follows that

$$\begin{aligned}\int_\Gamma h^3(u_1-u_0)\frac{\partial\sigma}{\partial\nu} &= \int_\Omega h^3\nabla(u_1-u_0)\cdot\nabla(\eta_\delta\sigma) - \int_{\Omega_1}(\gamma_1-\gamma_0)g(\eta_\delta\sigma)_x \\ &\le \int_{D_2}(g\eta_\delta\sigma)(\rho_1(y),y)\,dy \\ &\to 0 \quad \text{as} \quad \delta \to 0.\end{aligned} \tag{4.12}$$

Since $u_1 \ge u_0$, (4.12) implies that $u_1 = u_0$ on Γ. The same reasoning gives that $u_2 = u_0$ on Γ. By decreasing r, we conclude that $u_1 = u_2$ in B^+. Hence $u_1 = u_2$ in $\Omega^+(u_1)$ and $\Omega^+(u_1) = \Omega^+(u_2)$ by Theorem 1.3, Theorem 2.11 and the unique continuation theorem for real analytic function. Therefore, we conclude that $(u_1,\gamma_1) = (u_2,\gamma_2)$. This completes the proof of the theorem. □

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