MATHEMATICAL PROBLEMS ASSOCIATED WITH THE ELASTICITY OF LIQUIDS

By

D. D. Joseph

IMA Preprint Series # 520

April 1989

MATHEMATICAL PROBLEMS ASSOCIATED WITH THE ELASTICITY OF LIQUIDS

D. D. Joseph

Department of Aerospace Engineering and Mechanics
University of Minnesota, Minneapolis, MN 55455

This lecture is in three parts:

1. Physical phenomena associated with hyperbolicity and change of type;
2. Conceptual ideas associated with effective viscosities and rigidities and the origins of viscosity in elasticity;
3. Mathematical problems associated with hyperbolicity and change of type.

The ideas which I will express in this lecture are a very condensed form of ideas which have been put forward in various papers and most completely in my forthcoming book "Fluid Dynamics of Viscoelastic Liquids" which is to be published in 1989 by Springer-Verlag. The mathematical theory of hyperbolicity and change of type is associated with models with an instantaneous elastic response. Basically, this means that there is no Newtonian like part of the constitutive equation. The theory for these models as it is presently known is in my book. I am persuaded that further development of this subject lies in the realm of physics rather than mathematics. The main issues are centered around the idea of the effective viscosity and rigidity and the measurements of slow speeds, topics which are discussed in this paper in a rather more discursive than mathematical manner.

1. Physical phenomena associated with hyperbolicity and change of type

It is well known that small amounts of polymer in a Newtonian liquid can have big effects on the dynamics of flow. Drag reductions of the order of 80% can be achieved by adding polymers in concentrations of fifty parts per million to water. This minute addition does not change the viscosity of the liquid but evidently has a strong effect on other properties of the liquid which have as yet been inadequately identified.

We are going to consider some effects of adding minute quantities of polyethylene oxide to water on the flow over wires. The first experiments were on uniform flow with velocity U across small wires, flow over a cylinder. James and Acosta [1970] measured the heat transferred from three wires of diameter D=0.001, 0.002 and 0.006 inches. They used three different molecular weights of polymers in water (WSR 301, 205 and coagulant) in concentrations ϕ ranging from 7 parts to 400 parts per million by weight, the range of extreme dilution, in the drag reduction range. They found a critical velocity U_c in all cases except the case of most extreme dilution ϕ=6.62 ppm, as is shown in figure 1. A brief summary of the results apparent in this figure follows.

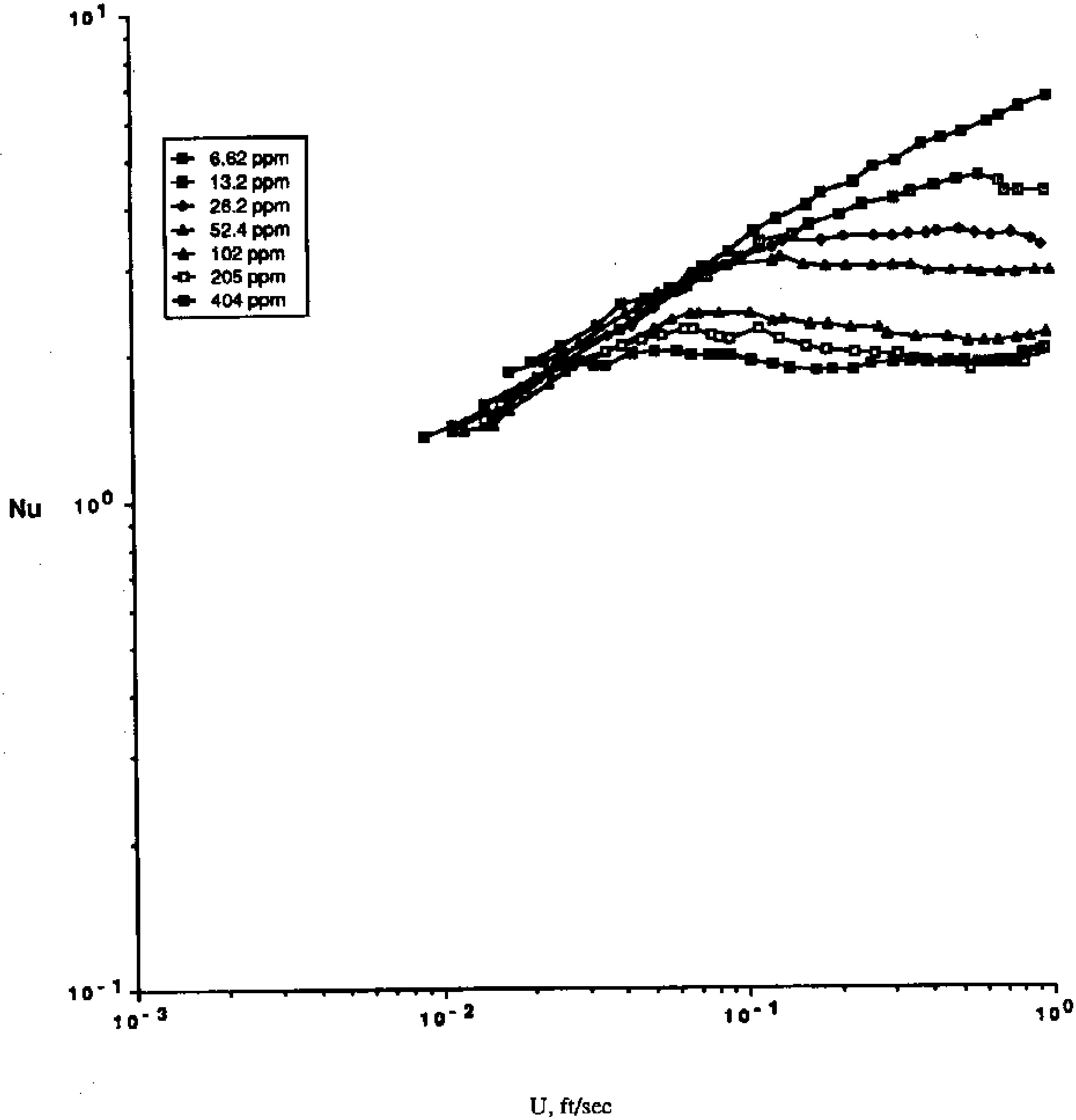

Figure 1. Heat transfer from heated wires in the cross-flow of WSR-301 (after James and Acosta, 1987). The critical U is independent of wire diameter.

1. There is a critical value U_c for all but the most dilute solutions: When $U<U_c$, the Nusselt number Nu(U) increases with U as in a Newtonian fluid. For $U>U_c$, the Nusselt number becomes independent of U as in figure 1.

2. U_c is independent of the diameter of the wire. This is remarkable. It suggests that U_c is a material parameter depending on the fluid alone.

3. U_c is a decreasing function of ϕ, the concentration. It is useful to note once again that in the range of ϕ between 6 ppm to 400 ppm, the viscosity is essentially constant and equal to the viscosity of water.

Ambari, Deslouis, and Tribollet [1984] obtained results for the mass transfer from 50 micron wires in a uniform flow of aqueous polyox (coagulant) solution in concentrations of 50, 100, and 200 parts per million. Their results are essentially identical to those obtained by James and Acosta [1970]; there is a critical U_c, a decreasing function of ϕ, signalling a qualitative change for the dependence of the mass transport of U, from a Newtonian dependence when $U<U_c$, to a U independent value for $U>U_c$. Their values of U_c for the break in the mass transport curve are just about the same as the value of U_c found by James and Acosta for heat transfer.

Ultman and Denn [1970] suggested that $U_c=c=\sqrt{\eta/\lambda\rho}$ where η is the viscosity, λ the relaxation time, and ρ is the density of a fluid whose extra stress $\boldsymbol{\tau}=\mathbf{T}+p\mathbf{1}$ satisfies Maxwell's equation

$$\lambda U\partial\tau/\partial x + \tau = \mu[\nabla \mathbf{u} = \nabla \mathbf{u}^T] \tag{1}$$

where $\mathbf{u}$ is the velocity. They used the molecular theory of Bueche to find the value of the relaxation time λ_B for the 52.4 ppm solution and they found that a $0.7\lambda_B$ would give $\sqrt{\eta/0.7\lambda_B} = U_c \sim 2.9$ cm/sec., that is, their estimate of λ_B from Bueche's theory is almost good enough to give $c=U_c$. Their calculation of the time of relaxation cannot be relevant, however, because in the Bueche theory λ_B does not go to zero with the concentration ϕ. The zero ϕ value of λ_B can be interpreted as a relaxation time for a single polymer in a sea of solvent. The relaxation time of one polymer cannot be the relaxation time of the solution in the limit in which the polymer concentration tends to zero, because in this limit the solution is all solvent.

Joseph, Riccius and Arney [1986] measured c=2.48 cm/sec in a 50 ppm, WSR 301 aqueous solution. This measurement supports the idea that $U_c=c$. We are trying now to measure wave speeds in extremely dilute solutions in the drag reduction range. We find considerable scatter in our data in these low viscosity solutions and are at present uncertain about the true value of the effective wave speed, including the values which we reported earlier.

The hypothesis that $U_c=c$ is consistent with the following argument about the dependence of the wave speed on concentration. In the regime of extreme dilution, the viscosity does not change with concentration. However, there appears to be a marked effect on the average time of relaxation which increases with concentration. It follows then that the wave speed $c=\sqrt{\eta/\rho\lambda}$ must decrease with concentration ϕ.

Konuita, Adler and Piau [1980] studied the flow around a 0.206 mm wire in an aqueous polyox solution (500 ppm, WSR-301) using laser-Doppler techniques. They found a kind of shock wave in front of the cylinder, like a bow shock. They say that the velocity of the fluid is zero in a region fluid in front of the stagnation point. Basically they say that there is no flow, or very slow flow near the cylinder. The formation of the shock occurs at a certain finite speed, perhaps U_c. This type of shock is consistent with the other observations in the sense that with a stagnant region around the cylinder, the transport of heat and mass could take place only by diffusion, without convection. This explains why there is no dependence of the heat and mass transfer on the velocity when it exceeds a critical value.

I estimated the critical speed, using the data of Konuita, Adler and Piau, and I estimated the wave speed c by extrapolating from our measurements in the polyox solutions at different concentrations. These estimates are reported in my new book "Fluid Dynamics of Viscoelastic Liquids." They are consistent with the notion of a supercritical shock transition at $U_c=c$.

Another striking phenomenon which appears to be associated with a supercritical transition is delayed die swell. It is well known that polymeric liquid will swell when extruded from small diameter pipes. The swelling can be very large, four, even five times the diameter of the jet. This swelling is still not well understood even when there is no delay. Joseph, Matta and Chen [1987] have carried out experiments on 19 different polymer solutions. They found that there is a critical value of the extrusion velocity U_c such that when $U<U_c$, the swell occurs at the exit, but when $U>U_c$ the swell is delayed, as in figure 2. If U is taken as the centerline velocity in the pipe, then the transition is always supercritical with $U_c>c$. The length of the delay increases with U. The velocity in the jet after the swell of jet has fully swelled is subcritical $U_f<U$ where U_f is the final U. This is something like a hydraulic jump with supercritical flow ahead of the delay and subcritical flow behind it.

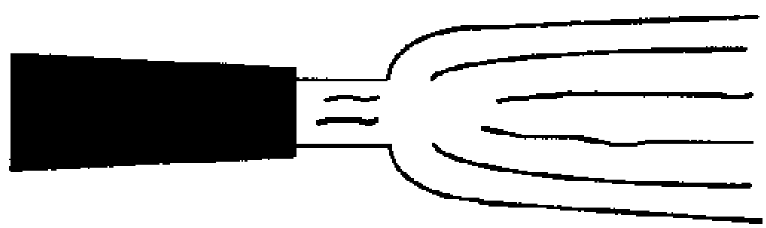

Figure 2. Delayed die swell.

Yoo and Joseph [1985] studied Poiseuille flow of an upper convected Maxwell model through a plane channel. Ahrens, Yoo and Joseph [1987] studied the same problem in a round pipe. In both cases, we get a hyperbolic region of flow in the center of the pipe when the centerline velocity U_m, equal to 2U in the Maxwell model, is greater than the wave speed c. This gives theoretical support to the idea that delayed die swell is a supercritical phenomenon.

There is a marked difference between the shape of the swell when it is delayed between different polymer solutions. The shape seems to correlate with a relaxation time

$$\lambda = \tilde{\mu}/G_c \qquad (2)$$

where $\tilde{\mu}$ is the zero shear rate viscosity and G_c is the rigidity. We get G_c from measuring c

$$c^2 = G_c/\rho \ . \qquad (3)$$

When λ is large, say $\lambda \geq 0(10^{-3}$ sec), the delay is sharp, as in figure 2. When the relaxation times are small, $\lambda \leq 0(10^{-4}$ sec), the delay is smoothed; in the extreme cases it is difficult to see that the swell is actually delayed.

We can say the Newtonian fluids are fluids with very large values of λ. In the case of delayed die swell, the smoothing of the swell is probably associated by the effect of smoothing due to an effective viscosity which arises from rapidly relaxing modes which have already relaxed when the delayed swell commences. Very viscous liquids always exhibit relaxation or non-Newtonian effects because even though the relaxation is fast, there is so much to relax.

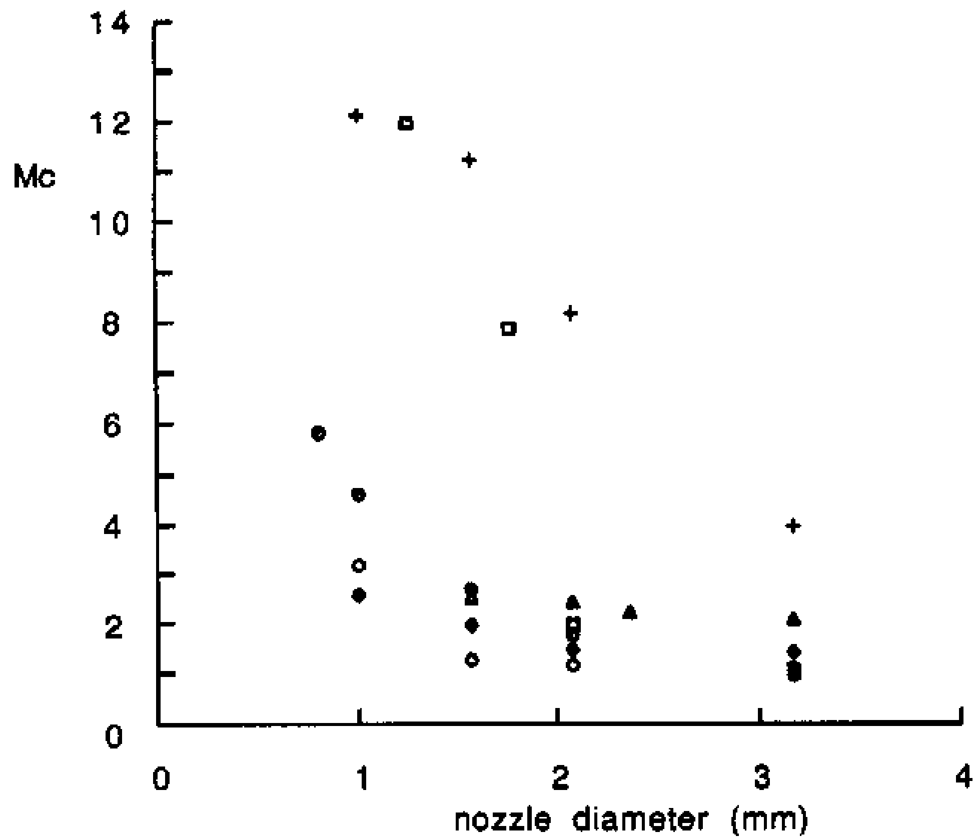

Figure 3. Mach number vs. pipe diameter.

○ 1.3% CMC □ 9.8% ELVACITE ◇ 6% PIB/D

● 2.5% POLYOX + 12.1% K-125 △ M-1

In figure 3, we plotted the critical Mach number

$$M_c = 2U_c/c$$

against the diameter of the pipe. In all cases $M_c \geq 1$, nearly. The value $M_c=1$ seems to be some form of asymptote for large values of the pipe diameter d. We do not understand why different fluids have such different M_c vs. d curves. We have thought about the consequences of shear thinning, which are important for some of the test liquids, in trying to collapse the experimental curves for different liquids into one curve, but we have not been successful.

2. Conceptual ideas

Nonlinear constitutive modeling is a jungle. The possible responses of the material to stresses are too complicated to describe by one explicit expression. General expressions are too abstract to be of direct use and are always insufficiently general to describe everything. Linearizing around rest is good because many different models collapse to one. The nonlinear parameters go away. Moreover, the elasticity of liquids is preeminently associated with propagation of small amplitude waves into rest.

We start with Boltzmann's expression for the extra stress τ which has been generalized to contain a Newtonian term

$$\tau = 2\mu D\,[u(x,t)] + 2\int_0^\infty G(s) D[u(x,\ t-s)]ds \tag{4}$$

where $\mathbf{u}$ is the velocity, $\mathbf{D}$ is the symmetric part of grad $\mathbf{u}$ and G(s) is positive, bounded and monotonically decreasing to zero. The actual stress $\mathbf{T}=-p\mathbf{1}+\tau$ differs from τ by a "pressure" p. Equation (4) is the most general linear functional of grad $\mathbf{u}$ in a fluid. To name a fluid, we need a Newtonian viscosity μ and a shear relaxation modulus G(s). We get Jeffreys' model from (4) when we write $G(s)=\frac{\eta}{\lambda}\exp(-s/\lambda)$ and Jeffreys' model reduces to Maxwell's if also $\mu=0$.

Now we consider viscosity. In steady flow, $\mathbf{u}$ is independent of t and comes out of the integral in (4). We get

$$\tau = 2\tilde{\mu}\mathbf{D}[\mathbf{u}(x)] \tag{5}$$

where $\tilde{\mu}=\mu+\eta$ is the static or zero shear viscosity and $\eta = \int_0^\infty G(s)ds$, the area under G(s), is the elastic viscosity. We have a viscosity inequality $\tilde{\mu}\geq\eta$ with equality when there is no Newtonian viscosity $\mu=0$.

Now we consider elasticity $\mu=0$, writing

$$\mathbf{D}[\mathbf{u}(x,\ t-s)] = -\frac{\partial}{\partial s}\mathbf{E}\,[\xi(x,\ t-s)] \tag{6}$$

where ξ is a displacement and $\mathbf{E}$ is the infinitesimal strain. If it were possible to make a step in strain without flow, and it isn't possible, we would have $\mathbf{D}[\mathbf{u}(x,t)] = \mathbf{E}_0(x)\delta(t)$ for Dirac δ. Then, from (4), with $\mu=0$,

$$\tau = 2G(t)\,\mathbf{E}_0(x)$$

and you can see why G(t) is called the stress relaxation function and G(0) the rigidity or shear modulus. Another way to see elasticity with $\mu=0$ is to write

$$\tau = 2\int_0^\infty -\frac{\partial}{\partial s}\{G(s)\mathbf{E}\,[\xi\,(x,\ t-s)]\}ds + 2\int_0^\infty G'(s)\mathbf{E}\,[\xi\,(x,\ t-s)]\,ds\ . \tag{7}$$

Now we can suppose that G(s) decays ever so slowly so that the second integral will tend to zero while the first gives rise to linear elasticity for an incompressible solid

$$\tau = 2G(0)\,\mathbf{E}\,[(\xi\,(x,\ t)]. \tag{8}$$

Now we restore the Newtonian viscosity and we note that this viscosity smooths discontinuities. For example, in the problem of the suddenly accelerated plate, the boundary at y=0 below a semi-infinite plate is suddenly put into motion, sliding parallel to itself with a uniform speed. If $\mu=0$, this problem is governed by a telegraph equation. The news of the change in the boundary value from zero to constant velocity propagates into the interior by a damped wave with a velocity $c=\sqrt{G(0)/\rho}$. The amplitude of the velocity shock decays exponentially. A short while after the wave passes, the solution at the given y looks diffusive. If $\mu\neq0$, and is small, a sharp front cannot propagate. Instead we get a shock layer whose thickness is proportional to $\sqrt{\mu y/\tilde{\mu}}$ and the solution, as in the Newtonian fluid, is felt instantly everywhere. We get a diffusive signal plus a wave. The wave could be dominant in the dynamics if μ is small.

Actually diffusion is impossible because it requires that a pulse initiated at any point be felt instantly everywhere. This same defect hold for all models with $\mu\neq0$, like Jeffreys'. Propagation should proceed as waves.

Poisson, Maxwell, Poynting and others thought that $\mu=0$ ultimately. It's all a matter of time scales. Short range forces between molecules of a liquid give rise to weak clusters of molecules which resist fast deformations elastically, then relax. Liquids are closer to solids than to gases. Liquid molecules do not bounce around with a mean free path, they move cooperatively.

So what is the difference between two liquids with the same η, one appearing viscous (Newtonian) and the other elastic? Maxwell thought that viscous liquids were actually elastic, with high rigidity and a single fast time of relaxation. To fix his idea in your mind, we compare two liquids with the same viscosity η, satisfying Maxwell's model with $G(s)=G(0)\exp(-s/\lambda)$, $G(0)=\eta/\lambda$. To have the same η the Newtonian liquid would have a relatively large $G(0)$ and a small time λ of relaxation. The trouble with Maxwell's model, if not his idea, is that a single time of relaxation is against experiments which can never be made to fit a single time of relaxation.

There are many different times of relaxation. Experiments indicate that many liquids respond to high frequency ultrasound like a solid organic glass with

$$G(0)\sim10^9\text{Pa},\ c=\sqrt{G(0)/\rho}\sim10^5\ \text{cm/sec}. \tag{9}$$

This type of estimation is valid for a huge range of liquids, from olive oil to high molecular weight silicon oils. With this time of relaxation and such a high rigidity, all the liquids would look Newtonian, with t much greater than $\tilde{\mu}/G(0)$, which is of the order of 10^{-10} sec. in olive oil, and is perhaps 10^{-6} in some high viscosity silicon oils. In fact, we see much longer lasting responses which come about because there are different times of relaxation. Small molecules relax rapidly, giving rise to large rigidity $G(0)$ and fast speed. Large molecules and polymers relax slowly, giving rise to a smaller effective rigidity $G_\mu(0)$, effective viscosity μ and slow speed

$$c=c_\mu=\sqrt{G_\mu(0)/\rho}\,. \tag{10}$$

To get this firmly in mind, we can think of a kernel with values like those given by (9), sketched in figure 4.

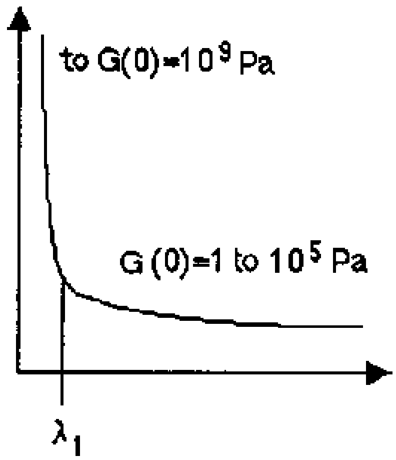

Figure 4. G(s), fast relaxation (say 10^{-10} sec) followed by a slow relaxation (say 10^{-4} sec).

We may inquire if at $t \gg 10^{-10}$ sec the relaxed fast modes have a dynamical effect. Yes, they give rise to an effective viscosity. We may as well collapse the glassy mode into a one-sided delta function $\mu\delta(s)$ where $\mu = G(0)\lambda_1$, or some fraction of this. This is our effective viscosity and our construction shows that is not unique. This is a very interesting concept, but it is not amenable to experiments that we know.

It is useful to define a time unit in terms of the slowest relaxation, say $\tilde{\mu}/G_c$. This gives rise to an internal clock, with a material time defined by the slowest relaxation. This time may be slow or fast on the external clock. To get this idea, think of the analog for the transport of heat. Heat is transported in solids by fast waves. The fastest wave is associated with electrons with relaxation times of 10^{-13} sec, then by lattice waves (phonons) with relaxation times of 10^{-11} sec. Both times are surpassingly short on our clock. However, at 10^{-13} sec, the electrons have all relaxed (and they give rise to diffusion) whilst the phonons have not begun to relax. Of course, it's more interesting when the slow relaxation is not too fast on our clock, as is true for viscoelastic fluids.

The notion of an external and internal clock is an appealing idea for expressing the difference between different theories of fading memory. Some theories, like Maxwell's and the more mathematical one by Coleman and Noll [1960] use an external clock; in rapid deformations the fluid responds elastically; in slow deformations the response is viscous. Fast and slow are measured in our time, on the external clock. Such theories rule out transient Newtonian responses. Models with $\mu \neq 0$, like Jeffreys', or the more mathematical one by Saut and Joseph [1983], are disallowed. To get $\mu \neq 0$ back in, even though ultimately $\mu = 0$, we need an effective μ, associated with an internal clock.

3. Mathematical theory

When the fluid is elastic the governing equations are partly hyperbolic. The hyperbolic theory makes sense when the Newtonian viscosity is zero or small relative to the static viscosity $\tilde{\mu}$. For very fast deformations in which the fluid responds momentarily like a glass, the equations always exhibit properties of hyperbolic response, waves and change of type. However, the glassy response takes place in times too short to notice. Hence, the hyperbolic theory is not useful where it is exact. The hyperbolic theory is useful when we get an elastic response at times we read on our clock, in the domain of the effective theory. Hence, the hyperbolic theory is useful where it is not exact.

Most of the mathematical work has been done with fluids like Maxwell's and for plane flows. These problems are governed by six quasilinear equations in six unknowns. The unknowns are two velocity components, three components of the stress, and a pressure. The continuity equation, two momentum equations and three equations for the stress govern the evolution of the six variables. The stress equations are like Maxwell's

$$\lambda\left[\frac{\partial\tau}{\partial t}+\mathbf{u}\cdot\nabla\tau+\tau\Omega-\Omega\tau-a(\mathbf{D}\tau+\tau\mathbf{D})\right]=2\eta\mathbf{D}+\ell$$

where $\mathbf{D}$ is the symmetric part and Ω the antisymmetric part of ∇u, $-1\le a\le 1$ and ℓ are lower order terms, algebraic in the system variables. This system may be analyzed for type in the usual way. We get a 6th order system and it factors into three quadratic roots. Two of the roots are imaginary so that the system is not hyperbolic. The streamlines are characteristic, with double roots so that the system is not strictly hyperbolic. The third quadratic factor depends on the unknown solution, algebraically, and it can be real or complex, depending on the solution. We say that such a solution with mixed roots is of composite type. Some variables are elliptic, some are hyperbolic.

It turns out that the pair of roots which depend on the unknown solution and can change type are associated with the vorticity equation, a second order nonlinear PDE. This equation is either elliptic or it is hyperbolic, depending on the solution. It is not of composite type, but is classical, like the equation for the potential in gas dynamics.

We can think of the unsteady vorticity equation and the steady vorticity equation. The analysis of the two has greatly different consequences. The unsteady equation is ill-posed when it is elliptic and well-posed when it is hyperbolic. Ill-posed problems are catastrophically unstable to short waves, with growth rates which go to infinity with the wave number. The conditions on the stress which lead to ill-posed problems can be determined by the method of frozen coefficients, as was first done by Rutkevich [1969]. It turns out that the Maxwell models with $a=\pm1$ cannot be ill-posed on smooth solutions, but the other models do become ill-posed for certain flows.

The problem of change of type in steady flow is different. The vorticity in steady flow can be of mixed type with elliptic and hyperbolic regions, as in transonic flow. The physical implications of these mixed "transonic" fields are not yet perfectly understood, though many examples have been calculated.

There are many models, other than those like Maxwell's, in which vorticity is the key variable. It is the only variable which is either strictly elliptic or strictly hyperbolic. The stream function satisfies Laplace's equation, the velocity and the stresses are of composite type. The stresses do not satisfy a hyperbolic equation and it is wrong to speak of the propagation of stress waves.

There are other models in which the vorticity is not the key variable. However, when these models are linearized around rest, one finds again that the steady vorticity equation is either elliptic or hyperbolic, and the unsteady vorticity equation is always hyperbolic. Hence it is precisely waves of vorticity which propagate into rest.

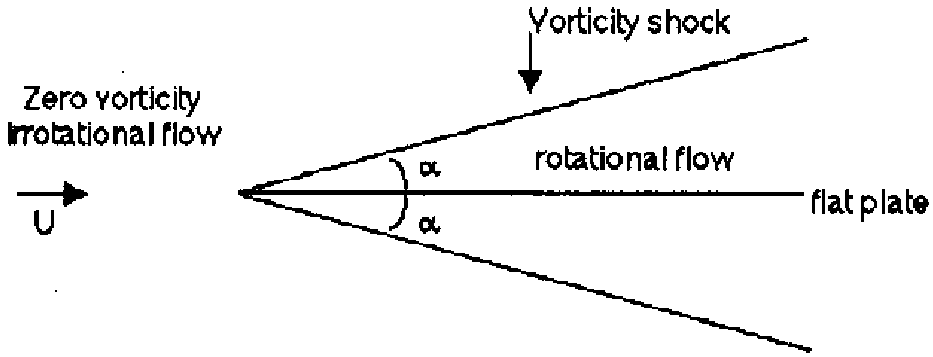

Figure 5. Mach wedge for the vorticity, $\tan \alpha = (1-M^2)^{-1/2}$.

The mathematical consequences of composite roots are clearly evident in the recent solution of L. E. Fraenkel [1987] of the problem of linearized supercritical flow over a flat plate. The linearization here is around the uniform flow which exists at infinity, as in Oseen's problem for the Navier-Stokes equation. Fraenkel's solution shows that there is a Mach wedge of vorticity ζ centered on the leading edge of the plate. The vorticity in front of this wedge is zero and it is not zero behind the wedge [see figure 5]. Surprisingly, the vorticity jumps from zero to infinity at the wedge, but the singularity is integrable. We have rotational flow behind the shock and irrotational flow in front of the shock. The stream function satisfies $\nabla^2\Psi=-\zeta$ where $\zeta=0$ in front of the shock. Therefore, we may write $\Psi=\Psi_1+\Psi_2$, $\nabla^2\Psi_2=-\zeta$, $\nabla^2\Psi_1=0$. To satisfy the boundary conditions on the plate, we must have a nonzero potential field Ψ_1. In fact Ψ_1 satisfies a Dirichlet boundary for the region outside a strip on the positive x axis.

The potential flow decays to uniform flow as one moves upstream, but the delay is slow. There is no upstream influence in the fully hyperbolic flow of a gas over a flat plate. The upstream influence of the flat plate in the flow of a Newtonian fluid is almost negligible. The persistence of Ψ_1 is a consequence of its ellipticity, ultimately to the fact that the first order system is of composite type. This type of solution may be new in mathematical physics.

The velocity and the stresses decompose into harmonic and vortical parts. Hence these fields are all of composite type. Only the vorticity is pure, strictly hyperbolic in the linearized problem of flow past bodies. The velocity and stresses are continuous across the shock. The normal derivative of the velocity, the normal and shear stress are also continuous, but the tangential derivative of the tangential components of velocity and stress are discontinuous. The elliptic component of our composite system is associated with a huge upstream influence.

As a final matter I should like to discuss the question of the blow up of certain variables in unsteady flow of nonlinear models of a viscoelastic fluid. For one-dimensional shearing flows this question reduces to whether the vorticity or the velocity blows up as a consequence of nonlinear evolution starting from smooth data. The general belief, without proof, is that the blow up of vorticity will lead to a shock of velocity and the blow up of velocity to a shock in an integral of the velocity, say the stream function. I am going to develop my ideas in a rather more general context, calling attention to the difference between first order quasilinear systems in which derivative $\delta \mathbf{u}/\partial x_i$ of a system variable $\mathbf{u}$ appears linearly and a nonlinear first order system in which the first derivatives appear in a nonlinear way, say $\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y}$.

We can reduce a system of N first order nonlinear partial differential equations in γ independent variable to a system of $(\gamma+1)N$ quasilinear equations. A general system of N first order PDE's in two independent variables can be expressed as

$$F_i\,(x, y, u_1, \ldots, u_n, p_1, \ldots, p_n, q_1, \ldots, q_n) = 0 \qquad i = 1, \ldots, n \tag{12}$$

where

$$p_i = \frac{\partial u_i}{\partial x} \qquad q = \frac{\partial u_i}{\partial y}$$

are introduced as additional unknowns. We have 3N variables and 3N equations but one of the equations is nonlinear rather than quasilinear. The system can be reduced to a quasilinear one by differentiation but the reduction is not unique. One symmetric reduction is: $F_i = 0$ is an identity in x and y jointly, hence

$$\frac{dF_i}{dy} = \frac{\partial F_i}{\partial y} + \frac{\partial F_i}{\partial u_\ell} q_\ell + \frac{\partial F_i}{\partial q_\ell}\frac{\partial q_\ell}{\partial y} + \frac{\partial F_i}{\partial p_\ell}\frac{\partial p_\ell}{\partial y} = 0\,, \tag{13}$$

$$\frac{dF_i}{dx} = \frac{\partial F_i}{\partial x} + \frac{\partial F_i}{\partial u_\ell} p_\ell + \frac{\partial F_i}{\partial q_\ell}\frac{\partial q_\ell}{\partial x} + \frac{\partial F_i}{\partial p_\ell}\frac{\partial p_\ell}{\partial x} = 0\,, \tag{14}$$

Equation (12) implies that

$$\frac{\partial F_i}{\partial p_j} p_j + \frac{\partial F_i}{\partial q_j} q_j = \frac{\partial F_i}{\partial p_j}\frac{\partial u_j}{\partial x} + \frac{\partial F_i}{\partial q_j}\frac{\partial u_j}{\partial y} \tag{15}$$

We put this system into a symmetric form by writing $\partial p_\ell/dy = \partial q_\ell/dx$ in (14) and (15). Then we put the principal part on the right and the lower order terms on the left

$$-\left(\frac{\partial F_i}{\partial y} + \frac{\partial F_i}{\partial u_j} q_j\right) = \frac{\partial F_i}{\partial p_j}\frac{\partial q_j}{\partial x} + \frac{\partial F_i}{\partial q_j}\frac{\partial q_j}{\partial y} \tag{16}$$

$$-\left(\frac{\partial F_i}{\partial x} + \frac{\partial F_i}{\partial u_j} p_j\right) = \frac{\partial F_i}{\partial p_j}\frac{\partial p_j}{\partial x} + \frac{\partial F_i}{\partial q_j}\frac{\partial p_j}{\partial y} \tag{17}$$

Equations (15), (16), and (17) are 3N equations for the 3N unknowns.

The principal parts of each of the equations (15), (16), and (17) are identical. Each one determines the same characteristic directions. We have

$$\frac{\partial F_i}{\partial p_j}\frac{\partial a_j}{\partial x} + \frac{\partial F_i}{\partial q_j}\frac{\partial a_j}{\partial y} = \text{lot}\,. \tag{18}$$

Hence the characteristics $\lambda = dy/dx$ are determined from

$$\det\left[\lambda\,\frac{\partial F_i}{\partial p_j} - \frac{\partial F_i}{\partial q_j}\right] = 0\,. \tag{19}$$

Equation (19) has N roots. The N nonlinear first order PDE's give rise to N characteristic roots for the quasilinear system arising from differentiating the nonlinear system once with respect to each independent variable.

If we generate (19) by the method of simple jumps, we can state that real characteristic directions are the loci for discontinuities in the derivatives of p_i and q_i. This means that the second derivatives of u_i suffer jumps in the nonlinear case and first derivatives jump in the quasilinear case. The first derivatives are smooth when second derivatives jump so that we get one more derivative of smoothness in the nonlinear case.

It appears that a more far-reaching conclusion following along lines of the last paragraph may be true. Compare quasilinear and nonlinear first order systems which allow blow up in finite time. The solutions are smooth before the blow up time. To find blow up we look for intersecting characteristics. First derivatives blow up in quasilinear systems, second derivatives in nonlinear systems. This conjecture is true for some special one-dimensional models of flow of a viscoelastic fluid which have been studied by M. Slemrod [1985] and Renardy, Hrusa, and Nohel [1987].

Acknowledgement

This work was supported by the Army Research Office, National Science Foundation and Department of Energy.

References

Ahrens, M., Yoo, J. Y. and Joseph, D. D. 1987 Hyperbolicity and change of type in the flow of viscoelastic fluids through pipes, *J. Non-Newtonian Fluid Mech.* **24**, 67–83.

Ambari, A., Deslouis, C. and Tribollet, B. 1984 Coil-stretch transition of macromolecules in laminar flow around a small cylinder. *Chem. Eng. Commun.* 29, 63–78.

Coleman, B. and Noll, W. 1960 An approximation theorem for functionals with applications in continuum mechanics, *Arch. Rat. Mech. Anal.* **6**, 355–370.

Fraenkel, L. E. 1987 Some results for a linear, partly hyperbolic model of viscoelastic flow past a plate, in *Material Instabilities in Continuum Mechanics and Related Mathematical Problems* (ed. J. M. Ball) Clarendon Press: Oxford.

James, D. and Acosta, A. J. 1970 The laminar flow of dilute polymer solutions around circular cylinders, *J. Fluid Mech.* **42**, 269–288.

Joseph, D. D., Matta, J. and Chen, K. 1987 Delayed die swell, *J. Non-Newtonian Fluid Mech.* **24**, 31–65.

Konuita, A., Adler, P. M. and Piau, J. M. 1980 Flow of dilute solutions around circular cylinders. *J. Non-Newtonian Fluid Mech.* **7**, 101–106.

Renardy, M., Hrusa, W. J. and Nohel, J. A. 1987 *Mathematical Problems in Viscoelasticity*. Longman: Harlow, UK.

Rutkevich, M. I. 1969 Some general properties of the equations of viscoelastic fluid dynamics. *PMM* 33, 42–51.

Saut, J. C. and Joseph, D. D. 1983 Fading memory, *Arch. Rat. Mech. Anal.* **81**, 53–95.

Slemrod, M. 1985 Breakdown of smooth shearing flow in viscoelastic fluids for two constitutive equations; the vortex sheet vs. the vortex shock. Appendix to the paper by Joseph, D. D. Hyperbolic phenomena in the flow of viscoelastic fluids, in *Viscoelasticity and Rheology* (eds. A. S. Lodge, J. Nohel and M. Renardy), Academic Press.

Ultman, J. S. and Denn, M. M. 1970 Anomalous heat transfer and a wave phenomenon in dilute polymer solutions. *Trans. Soc. Rheol.* **14**, 307–317.

Yoo, J. and Joseph, D. D. 1985 Hyperbolicity and change of type in the flow of viscoelastic fluids through channels. *J. Non-Newtonian Fluid Mech.* **19**, 15–41.