COMPLEX ANALYTIC DYNAMICS

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PRELIMINARY VERSION for the Thurston CBMS conference. Any suggestions for the final version would be greatly appreciated. The figures not contained in the supplement will hopefully be distributed at the conference. The reader's guide has not been written yet.
The global study of the iteration of complex analytic maps is a beautiful combination of complex analysis and dynamical systems. The powerful analysis yields results which, at first, seem highly improbable and which are unimaginable in other kinds of dynamical systems. Often the indecomposable, completely invariant sets are fractals (à la Mandelbrot [M1]) because, in fact, they are quasi-self-similar (Sullivan [S3]). Sometimes they are nowhere differentiable Jordan curves whose Hausdorff dimension is greater than one (Sullivan [S4] and Ruelle [R]). Yet they are generated by a single analytic function

\[ z_{n+1} = R(z_n) \]

of a single complex variable.

The study of this subject began during the First World War. Both P. Fatou and G. Julia independently published a number of Compte rendu notes, and then both wrote long mémoires - Julia [J] in 1918 and Fatou [F1 - F3] in 1919 and 1920. At that time, they had at their disposal a new theorem of Montel which gave a sufficient condition for the normality of a family of meromorphic functions in terms of their images. They applied the theory of normal families to the dynamical system to prove some remarkable results.

Recently there has been an explosion of interest in the subject, and many mathematicians have made substantial contributions. In fact, significant progress is still being made, and it is impossible to predict where and when it will end. The aim here is to give a rapid introduction to the subject in light of these achievements. We do not intend to survey all of the new work since most of it is still in progress, but we do include bibliographic notes which can serve as a reader's guide to those results with which we are familiar. However, no claims of completeness are implied or intended.
Much of the paper is an exposition of the classical work of Fatou and Julia. We give a complete proof of the fundamental result (Theorem (5.15)) that the Riemann sphere disjointly splits into two sets - the closure of the repelling periodic points and the set of normal (i.e. stable) points. Along the way, we discuss the dynamics in a neighborhood of a periodic point (Section 3) and the global consequences of Montel's Theorem (Section 4). The decomposition theorem is proven in Section 5. Sections 6 and 8 contain other noteworthy classical results. The theorems in Section 9 were also known classically although they were not formulated in quite the same manner.

Sections 7 and 10 contain expositions of recent work. Sullivan ([S1] and [S2]) has completed a classification of the dynamics in the domains of normality. We summarize this classification in Section 7. Section 9 is an exposition of the recent work of Douady and Hubbard [DH] and Mandelbrot [M1-M3] on the dynamics of quadratic polynomials. A brief survey of this work has been included because it is a wonderful example of how "simple" dynamical systems can have complicated dynamics. Moreover, the family of quadratic polynomials is varied enough to illustrate most of the ideas presented here as well as many other surprising phenomena. Although the material in Sections 7, 9, and 10 depends on the previous sections, the reader may find it helpful to read these sections before tackling some of the more involved proofs in the preceding ones.

The first section simply establishes notation and recalls theorems from complex analysis and the theory of Riemann surfaces which will be used freely throughout the article.
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1. Background and Notation

We are interested in studying the dynamics of a discrete dynamical system of the Riemann sphere \( \mathcal{C} \) generated by one holomorphic transformation

\[
(1.1) \quad R : \mathcal{C} \to \mathcal{C}.
\]

In other words, our phase space will be the unique, simply connected, closed Riemann surface \( \mathcal{C} = \mathbb{C} \cup \{ \infty \} \) which is homeomorphic to the two-dimensional sphere

\[
S^2 = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \}.
\]

We shall usually use the variables \( z \) and \( w = \frac{1}{z} \) to represent the two standard coordinate charts on \( \mathcal{C} \) determined by stereographic projection. Then any holomorphic (analytic) map \( R \) of \( \mathcal{C} \) can be written in the form

\[
(1.2) \quad R(z) = \frac{p(z)}{q(z)}.
\]

where \( p(z) \) and \( q(z) \) are polynomials with complex coefficients and no common factors. Hence, there is a one-to-one correspondence between rational functions (1.2) and holomorphic maps (1.1). The poles of the rational function are simply the points of \( \mathcal{C} \) which are mapped to infinity. The reader should consult Hille [Hi], Ahlfors [Al], Narasimhan [N], or Farkas and Kra [FK] for more details.

The degree \( \deg(R) \) of any continuous map \( R : S^2 \to S^2 \) is a homotopy invariant which measures how many times \( R \) wraps \( S^2 \) around itself. In our context, the degree of \( R \) can be calculated in two ways. If \( R(z) \) is written in the form (1.2), then

\[
\deg(R) = \max \{ \deg(p), \deg(q) \}.
\]
Also the degree of \( R \) is the number (counted with multiplicity) of inverse images of any point of \( \mathbb{C} \). The Fatou - Julia theory applies to rational maps \( R \) whose degree is at least two.

A dynamical system is formed by the repeated application (iteration) of \( R \) to \( \mathbb{C} \).

**DEFINITION.** Given a point \( z \in \mathbb{C} \), the sequence \( \{z_n\} \) is inductively defined by

\[
z_{n+1} = R(z_n).
\]

This sequence is called the **forward orbit** of \( z_0 \) and is denoted \( O^+(z_0) \).

There are many questions that one could ask about any given orbit \( O^+(z_0) \).

What are the limit points of \( O^+(z_0) \)? What are the topological and analytic properties of the set of all limit points of \( O^+(z_0) \)? What is the limiting distribution of \( O^+(z_0) \)? How do the set of iterated preimages \( O^-(z_0) \) distribute themselves? In this paper, we shall mostly confine our attention to topological questions.

An elementary way to distinguish different orbits is to count the number of points in the orbit.

**DEFINITION.** If \( z_n = z_0 \) for some \( n \), then \( z_0 \) is a **periodic point** and \( O^+(z_0) \) is a **periodic orbit** (often called a periodic cycle or simply a cycle). If \( n \) is the first natural number such that \( z_n = z_0 \), then \( n \) is called the **period** of the orbit. Usually, if the period of an orbit is one, we call \( z_0 \) a **fixed point** rather than a periodic point.

It is surprising how much of the theory in question is related to the distribution of periodic points. To illustrate the dynamical concepts we have
just introduced, we make a few observations about the example $z + z^2$, which (in many ways) is the simplest example in the subject.

(1.3) EXAMPLE. Let $R(z) = z^2$. The behavior of the orbits depends upon where they lie relative to the unit circle. The orbit of any point inside the circle approaches the origin (which is a fixed point), and any orbit outside the circle approaches $\infty$. Moreover, $R(\infty) = \infty$, so $\infty$ is another fixed point of the map $R$. Finally, to completely describe the dynamics of $R$, we need to describe the action of $R$ on the unit circle. But, on the unit circle,

$$R(e^{i\theta}) = e^{i2\theta}, \quad \theta \in \mathbb{R},$$

So even though the map has a rich orbit structure, it is still a system which is amenable to computation. The reader who is unfamiliar with $R|S^1$ should try to describe this rich structure. It is not trivial.

Before we continue with our review, we should introduce the following notation which can be the source of some confusion in this subject.

NOTATION. The symbol $R^n$ denotes the $n$-fold composition

$$R^n = R \circ R \circ \ldots \circ R$$

of the function $R$ with itself. Since we can multiply functions as well as compose them, we must be careful. The notation $R^2$ represents $R \circ R$ rather than the function $S(z) = [R(z)]^2$.

In the theory there is a notion of the equivalence of two systems: conjugation. If $R(z)$ and $S(z)$ are two rational functions and $M(z)$ is a Mobius transformation such that the diagram
commutes, then $R$ and $S$ are (analytically) **conjugate**. Note that if $R$ and $S$ are conjugate by $M$, then $R^n$ and $S^n$ are also conjugate by $M$. So $R$ and $S$ are holomorphically the "same" dynamical system. Hence, the answer to any topological or conformal question about the system generated by $S$ can be obtained from the answer to the same question applied to $R$ by applying the map $M$.

(1.4) EXAMPLE. To illustrate the usefulness of this notion, we consider an arbitrary quadratic polynomial

$$R(z) = az^2 + 2bz + d.$$ 

We can conjugate $R(z)$ to some quadratic polynomial $p(z)$ of the form $p(z) = z^2 + c$. Let $M(z) = az + b$ and $c = ad + b - b^2$. Then we compute

$$M^{-1} \circ p \circ M(z) = M^{-1}((az + b)^2 + c)$$

$$= M^{-1}(a^2z^2 + 2abz + b^2 + c)$$

$$= \frac{(a^2z^2 + 2abz + b^2 + c) - b}{a}$$

$$= R(z).$$

As a result, we find that we need only study the class of quadratic polynomials of the form

$$z + z^2 + c$$

to understand the dynamics of all quadratic polynomials. Hence, the space of
"all" dynamical systems of quadratic polynomials is really \( \mathbb{C} \) rather than \( \mathbb{C}^3 \) as it might at first appear. Moreover, as we shall see in Section 10, there are technical reasons why this formulation is useful.

This is all we need to say in terms of background to the dynamical theory we will present. For convenience, we review a few concepts from complex analysis which are essential.

The notion of a normal family is closely connected with compactness of a set of analytic maps in the topology of uniform convergence on compact subsets. The following definition uses the spherical metric on \( \overline{\mathbb{C}} \).

**DEFINITION.** Let \( U \) be an open subset of \( \overline{\mathbb{C}} \) and \( F = \{ f_i \mid i \in I \} \) be a family of meromorphic functions defined on \( U \) with values in \( \overline{\mathbb{C}} \) (\( I \) is any index set). The family \( F \) is a normal family if every sequence \( f_n \) contains a subsequence \( f_{n_j} \) which converges uniformly on compact subsets of \( U \).

It is awkward to relate this definition to any dynamical property, but Arzela's Theorem relating normal families to equicontinuous ones shows that there are quite a few connections.

**DEFINITION.** Let \( X \) be a metric space with the metric \( d \). A family of functions \( \{ f_i : X \to X \} \) is an equicontinuous family if, given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( d(x,y) < \delta \) implies \( d(f_i(x_1), f_i(x_2)) < \varepsilon \) for all \( i \).

(1.5) **THEOREM.** The family \( \{ f_i : U \to \overline{\mathbb{C}} \} \) of meromorphic functions is a normal family if and only if it is an equicontinuous family on every compact subset of \( U \).

Hence, normal families have values which do not diverge under iteration. In Section 2, we will be more precise about this connection.
In Section 4, we shall also need the following sufficient condition for normality.

(1.6) THEOREM. Let \( \{f_i: U \to \mathbb{C}\} \) be a meromorphic family of functions. If the family is locally uniformly bounded on \( U \), then it is a normal family.  

The reader is referred to Ahlfors' book [A1] for a proof of Arzela's Theorem and a more complete discussion of normal families.

We recall the elementary version of Schwarz's lemma.

NOTATION. Given \( z \in \mathbb{C} \) and \( r > 0 \), we denote the open disk centered at \( z \) of radius \( r \)

\[ \{z' \mid |z' - z| < r\} \]

by \( D_r(z) \). The symbol \( D_r \) will represent the same set as \( D_r(0) \).

(1.7) THEOREM. Let \( f:D_1 \to D_1 \) be an analytic function such that \( f(0) = 0 \).

Then

(a) \( |f(z)| < |z| \) for all \( z \in D_1 \)

and

(b) \( |f'(0)| < 1 \).

If equality holds in either (a) for any \( z \in D_1 - \{0\} \) or in (b), then

\[ f(z) = e^{i\theta}z \] where \( \theta \in \mathbb{R} \).  

On the whole, we shall not need much of the theory of Riemann surfaces. However, in Section 4, we use the uniformization theorem - the cornerstone of the classification of Riemann surfaces. Recall that a Riemann surface is a one (complex) dimensional manifold. The uniformization theorem classifies all simply connected Riemann surfaces.
(1.8) THEOREM. Every simply connected Riemann surface is conformally equivalent to either $\mathbb{C}$, $D_1$, or $\mathbb{C}$.

In our context, we will only encounter Riemann surfaces which are open subsets of $\mathbb{C}$ or which are $\mathbb{C}/L$ where $L$ is a lattice. The reader should use the uniformization theorem to show that, if $U$ is a domain in $\mathbb{C}$ which has at least three boundary points, then its universal cover $U$ is conformally equivalent to $D_1$.

There are many good references on the subject of Riemann surfaces. Abikoff [Ab1] and [Ab2], Ahlfors [A2], and Farkas and Kra [FK] are particularly appropriate for the uniformization theorem.
2. The Dynamical Dichotomy of Fatou and Julia

Fatou and Julia studied the iteration of rational maps $R: \mathbb{C} \rightarrow \mathbb{C}$ under the assumption that $\deg(R) > 2$. Basically they focused on a disjoint invariant decomposition of $\mathbb{C}$ into two sets. One of these sets is often called the Julia set. The other set does not have a standard name, and in this paper, we shall refer to it as the Fatou set.

DEFINITION. A point $z \in \mathbb{C}$ is an element of the Fatou set $F(R)$ of $R$ if there exists a neighborhood $U$ of $z$ in $\mathbb{C}$ such that the family of iterates $\{R^n|U\}$ is a normal family. The Julia set $J(R)$ is the complement of the Fatou set.

The classical papers denote the Julia set by $\mathbb{F}$, but since the term "Julia set" is now commonly used, we shall denote it by $J(R)$. The Fatou set did not have an explicit name classically, and it is now sometimes referred to as the domain of equicontinuity.

In this section, we shall prove that the Julia set is always nonempty, but the reader should beware that the Fatou set can be void. In Section 3, we will show that the Julia set of the map

$$z + \frac{(z^2 + 1)^2}{4z(z^2 - 1)}$$

is the entire Riemann Sphere (this is Latté's example - see (3.2)).

The Fatou set is an open set by definition, and proving that it is completely invariant is straightforward. In other words, if $z \in F(R)$, then $R(z) \in F(R)$ and $R^{-1}(z) \in F(R)$. Consequently, the Julia set is also completely invariant and compact.
If we consider the example $R(z) = z^2$ again and take a point $z_0$ inside the unit circle, then there exists an open disk $U$ around $z_0$ on which the sequence $R^n|_U$ converges to the constant function $C(z) = 0$ for all $z \in U$. Hence, the interior of the unit circle is a subset of the Fatou set. Likewise, the exterior is also a subset of the Fatou set although the limit function is different. The unit circle is, however, equal to the Julia set because the family $\{R^n\}$ is not equicontinuous on any open set which intersects the circle.

The Julia set of the map $S(z) = z^2 - 2$ is also simple to describe. It is the interval $[-2,2]$ on the real line. This is not so easy to see from the definitions, but the reader may enjoy investigating this example before we prove this later (see (7.12.1)).

The above two examples are deceiving. In general, Julia sets are not smooth. The following pictures illustrate the types of complexity that Julia sets usually possess.

See Supplement

(2.1) FIGURE. Douady's Rabbit - The Julia set of the map $z + z^2 + c$ where $c$ satisfies $c^3 + 2c^2 + c + 1$ and $\text{Im}(c) > 0$. We shall explain this unusual choice of $c$ in (3.6).

Most Julia sets have this fractal structure.

See Supplement

(2.2) FIGURE. This is the Julia set of the quadratic $p(z) = z^2 - 0.3125$. It is a Jordan curve. In addition, Sullivan [S4] has shown that it is a quasi-circle whose Hausdorff dimension is greater than one. Ruelle's results [R] on the real analytic nature of Hausdorff dimension are also relevant.
(2.3) FIGURE. The Julia set of $p(z) = z^2 + .3$. Theorem (9.7) applies to this map; therefore, $J(p)$ is a Cantor set. The next figure shows some of the finer detail.

(2.4) FIGURES. These figures illustrate the fractal nature of the Julia set in the previous figure. Figure $b$ is an enlargement of the small box in Figure $a$.

(2.5) FIGURE. Newton's method applied to the polynomial equation $z^3 - 1 = 0$ yields the rational function

$$N(z) = \frac{2z^3 + 1}{3z^2}$$

The orbit of almost any initial value will converge to a root of the equation. The Julia set is the set of initial values for which the method fails.

The above pictures are neither the most accurate nor the most elaborate ones available. There are many methods with which one can draw Julia sets, and the tradeoffs are (as usual) quality versus computer power and time. In Section 3, we shall encounter situations where the more expensive methods seem (at our present state of knowledge) absolutely necessary. But for our purposes, the accuracy of the above pictures shall suffice. The reader should, however, seek out some of the more intricate pictures which have been published recently and which will be in forthcoming publications. This author is aware of the articles
of Curry, Garnett, and Sullivan [CGS], Douady [D], Douady and Hubbard [DH], and Mandelbrot [M1], [M2], [M3], and this is probably a very incomplete list of all such publications.

We conclude our introduction to the dichotomy with the next theorem.

\textbf{(2.6) THEOREM.} The Julia set is nonempty.

\textbf{Proof.} Suppose $J(R) = \emptyset$. Then $F(R) = \overline{\mathbb{C}}$ and, therefore, the family $\{R^n\}$ is normal on $\overline{\mathbb{C}}$. A convergent subsequence $R^{n_i}$ converges uniformly to a holomorphic limit function $S$. Since $S$ is holomorphic on all of $\overline{\mathbb{C}}$, it must be rational. A contradiction can now be derived by considering the topological degrees of these maps. We have

\[
\begin{align*}
\text{deg}(S) & \leftarrow \text{deg}(R^{n_i}) \rightarrow \infty \\
\text{as } n_i & \rightarrow \infty.
\end{align*}
\]

But $\text{deg}(S)$ is finite. \hfill $\square$
3. Periodic Points

In this section we discuss the dynamics in a neighborhood of a periodic point. In particular, we give conditions which often determine in which set - $F$ or $J$ - the periodic point lies.

**DEFINITION.** Let $z_0$ be a periodic point of period $n$. Then the number $\lambda_{z_0} = (R^n)'(z_0)$ is the eigenvector of the periodic orbit.

Note that the chain rule implies that $\lambda_{z_0}$ is an invariant of $O^+(z_0)$.

**DEFINITION.** A periodic orbit $O^+(z_0)$ is:

1. **attracting** if $0 < |\lambda_{z_0}| < 1$,
2. **superattracting** if $\lambda_{z_0} = 0$,
3. **repelling** if $|\lambda_{z_0}| > 1$, or
4. **neutral** if $|\lambda_{z_0}| = 1$.

A. Attracting and Repelling Periodic Points.

Using the mean value theorem and Arzela's Theorem (1.5), it is easy to prove the following proposition.

(3.1) **PROPOSITION.** If $O^+(z_0)$ is a (super)attracting periodic orbit, then $O^+(z_0) \in J$.

At this point it may be worthwhile to the reader to go back to the example $z + z^2$ and check the proposition with our original analysis of the Julia set.
(3.2) LATTE'S EXAMPLE [L]. We can now show why $J(R) = \mathbb{T}$ for the map

$$R(z) = \frac{(z^2 + 1)^2}{4z(z^2 - 1)}.$$ 

Let $L$ be any discrete lattice in $\mathbb{C}$ of the form $n_1 w_1 + n_2 w_2$ with $n_i \in \mathbb{Z}$, $w_i \in \mathbb{C}$, and $w_2/w_1 \not\in \mathbb{R}$. The Weierstrass $p$ function is defined by

$$p(z) = \frac{1}{z^2} + \sum_{w \in L \atop w \neq 0} \left( \frac{1}{(z - w)^2} - \frac{1}{z^2} \right).$$

The $p$ function can be thought of as a function from a torus $T^2$ to $\mathbb{T}$ (see [Al] and [Co]). Multiplication by 2 preserves $L$, and consequently it induces a map $M$ from $T^2$ to $T^2$. It also preserves the equivalence relation generated by $p$ on $T^2$, so we get a rational map $R_L : \mathbb{T} \longrightarrow \mathbb{T}$.

$$
\begin{array}{ccc}
T^2 & \xrightarrow{M} & T^2 \\
p & \downarrow & \downarrow p \\
\mathbb{T} & \xrightarrow{R_L} & \mathbb{T}
\end{array}
$$

Suppose $z = q_1 w_1 + q_2 w_2$ with $q_i \in \mathbb{Q}$. Then $p(z)$ is either periodic or eventually periodic. Moreover, the norm of the eigenvalue of any periodic orbit is at least 2. Hence, we have a dense set of points in $T^2$ which map by $p$ to a dense set of $\mathbb{T}$ - all of which are in the Julia set of $R_L$. Since $J(R_L)$ is closed, it must equal the whole sphere $\mathbb{T}$. See Latte [L] or Herman [H3] for an explicit derivation of the above rational equation.

We know everything about the dynamics in a neighborhood of either an attracting or a repelling periodic point. The map $R$ is locally conjugate to its
(3.3) THEOREM. Let \( z_0 \) be an attracting periodic orbit of period \( n \). There exists a neighborhood \( U \) of \( z_0 \) and an analytic homeomorphism \( \phi : U \to D_r \) (for some \( r \)) such that \( \phi(z_0) = 0 \), \( \phi'(z_0) = 1 \) and the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{R^n} & U \\
\phi \downarrow & & \phi \\
D_r & \xrightarrow{} & D_r \\
\end{array}
\]

\[ z \to (\lambda z_0) z \]

commutes. \( \square \)

From this theorem we see that the neighborhood \( U \) is forward invariant i.e. \( R^n(U) \subset U \), and the orbit of every point in \( U \) is asymptotic to the periodic orbit \( O^+(z_0) \).

There are both analytic and geometric proofs of the existence of \( \phi \). The oldest proof is due to Koenigs in the Nineteenth Century. Basically he proved that the sequence of functions

\[
\frac{R^k(z) - z_0}{(\lambda z_0)^k}
\]

tends uniformly to a holomorphic function \( \phi \). Then it is easy to verify that

\[
\phi \circ R^n(z) = (\lambda z_0)(\phi(z))
\]

The reader is referred to Siegel and Moser [SM, p. ] for the technical details.

Theorem (3.3) also yields a local conjugacy in the repelling case. If \( O^+(z_0) \) is a repelling periodic orbit, then (3.3) can be applied to the inverse of \( R^n \) which has \( O^+(z_0) \) as an attracting periodic orbit. However, we get
less information because the orbit of every point \( z \in U \) except \( z_0 \) eventually leaves the neighborhood \( U \).

B. The Superattracting Case.

The superattracting periodic orbits are also locally conjugate to a simple map. Yet the dynamics of this simple map is much more interesting.

(3.4) THEOREM Let \( O^+(z_0) \) be a superattracting periodic orbit. Suppose

\[
(R^n)^{(k)}(z_0) \neq 0
\]

and

\[
(R^n)'(z_0) = (R^n)^{(2)}(z_0) = \ldots = (R^n)^{(k-1)}(z_0) = 0.
\]

Then there exists a neighborhood \( U \) of \( z_0 \) and an analytic homeomorphism \( \phi : U + D_r \) (for some \( r \)) such that \( \phi(z_0) = 0, \phi'(z_0) = 1 \), and the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{R^n} & U \\
\downarrow \phi & & \downarrow \phi \\
D_r & \xrightarrow{z + z^k} & D_r \\
\end{array}
\]

commutes.

Again we have a forward invariant neighborhood, and every orbit in the neighborhood is asymptotic to \( O^+(z_0) \). But in this case, the map \( z + z^k \) is not locally invertible. This gives us a great deal of information which will be particularly helpful in the analysis of the dynamics of polynomials.

This theorem is not as commonly known; consequently, we include the proof from Fatou [F1, pp. 187-189]. He credits Böttcher as the first person to
demonstrate the existence of the conjugacy, but no reference is given.

Before giving the proof, we recall a few facts about fractional exponents. The map \( z + z^k \) is a branched cover of \( \mathbb{C} \) by itself. If we have a map

\[ S: D_r \rightarrow \mathbb{C} \]

we can often find a map which we denote \( (S(z))^\frac{1}{k} \) such that

\[
\begin{array}{ccc}
  & & \mathbb{C} \\
  (S(z))^\frac{1}{k} & \downarrow & z \\
  z & \downarrow & + \\
  & \downarrow & z^k \\
 D_r & \rightarrow & \mathbb{C}
\end{array}
\]

commutes.

For example, if \( 0 \not\in S(D_r) \), then covering space theory gives us \( k \) maps which work. Also, if \( S(0) = 0 \) and \( S(z) \neq 0 \) for all \( z \in D_r - \{0\} \) and if the local degree of \( S \) around \( 0 \) is a multiple of \( k \), then there are also \( k \) maps that will work. The exponential notation is used in order to summarize a few properties of these maps such as

\[
\frac{1}{(s(z))^\frac{1}{2}} = (s(z))^\frac{1}{4}.
\]

Unfortunately we have chosen to use exponents in two different ways. Hence,

\[
(s^n(z))^\frac{1}{n} \neq S(z).
\]

The idea of the following proof of (3.4) is exactly the same as Koenig's proof. A sequence \( \{\phi_n\} \) of maps is produced by iterating the rational function \( n \) times and then taking the appropriate inverse of \( z + z^k \). This
sequence converges to the conjugacy.

Proof of (3.4). We prove the theorem in the case where \( n = 1 \) and \( z_0 = 0 \).

The power series expansion of \( R(z) \) is

\[
  a_0 z^k + a_{k+1} z^{k+1} + \ldots.
\]

If we conjugate \( R(z) \) by the map \( z \to bz \) where \( b^{k-1} = (a_k)^{-1} \), we get the map

\[
  S(z) = z^k + b_{k+1} z^{k+1} + \ldots.
\]

Proving the result for \( S(z) \) will suffice.

We always work inside a disk \( D_r \) such that \( S(D_r) \subset \text{int} (D_r) \) and

\[
  \lim_{n \to \infty} S^n(z) = 0 \quad \text{for all} \quad z \in D_r.
\]

We define

\[
  \phi_n(z) = \left( S^n(z) \right)^{1/k}
\]

choosing \( \phi_n'(0) = 1 \). If we prove that the sequence \( \{ \phi_n \} \) converges uniformly to a map \( \phi \), we have the desired conjugacy because

\[
  \phi \circ S = \lim_{n \to \infty} \left[ S^{n+1}(z) \right]^{1/k}
\]

and

\[
  [\phi(z)]^k = \lim_{n \to \infty} \left[ (S^{n+1}(z))^{k^{n+1}} \right]^k = \phi \circ S.
\]

To prove that \( \{ \phi_n \} \) converges uniformly on some neighborhood of 0, we introduce the function

\[
  H(z) = \frac{\phi_1(z)}{z}
\]
Note that \( H(0) = 1 \). Using this function we see that

\[
\frac{\phi_{n+1}(z)}{\phi_n(z)} = \prod_{i=0}^{n} \frac{1}{\phi_i(z)} = [H(S^n(z))]^{\frac{1}{kn}}
\]

Using the notation \( \phi_0(z) = z \), we have

\[
\phi_{n+1}(z) = z \prod_{i=0}^{n} \frac{\phi_{i+1}(z)}{\phi_i(z)} = z \prod_{i=0}^{n} [H(S^n(z))]^{\frac{1}{kn}}
\]

The uniform convergence of the sequence \( \{\phi_n\} \) can now be established by proving that the infinite product converges uniformly. This, in turn, is established by taking the logarithm of the infinite product (using the principal branch).

We get the infinite series

\[
(3.5) \quad \sum_{n=0}^{\infty} \frac{1}{k^n} \log (H(S^n(z)))
\]

Using the dynamical properties of \( S \) in \( D_r \), mentioned above, we know that

\[
|H(S^n(z)) - 1| < \frac{1}{2} \quad \text{and} \quad |H(S^n(z)) - 1| < C|S^n(z)| \quad \text{for some constant} \ C. \quad \text{If} \ |\alpha| < \frac{1}{2},
\]

\[
|\log (1+\alpha)| < |\alpha| + |\alpha|^2 + ... < 2|\alpha|
\]

so

\[
|\log H(S^n(z))| < 2C |S^n(z)|
\]

Rather than (3.5), we consider the series

\[
2C \sum_{n=0}^{\infty} \frac{|S^n(z)|}{k^n} < (2C) \sum_{n=0}^{\infty} \frac{r}{k^n}
\]

which clearly converges. \( \square \)
(3.6) REMARK. Douady's rabbit (Figure (2.1)) has a superattractive period 3 point at the origin. In fact, the choice of c was made so that the origin (which is the critical point) is a periodic point of period 3.

The following figure has a period 2 superattracting orbit contained in the shaded regions of the Fatou set.

| See Supplement |

(3.7) FIGURE. The Julia set of $z + z^2 - 1$. The two shaded components of the Fatou set are mapped one to the other.

C. Neutral Orbits.

The local dynamics in a neighborhood of a neutral periodic orbit is not so easy to describe. In fact, as we shall see, there are open questions relating to this case.

Again it is enough to describe the situation where $z_0$ is a fixed point. So we assume that $R(z_0) = z_0$ and $|\lambda z_0| = |R(z_0)| = 1$. If we try to conjugate $R|U$ to its derivative, then the diagram

$$
\begin{array}{ccc}
U & \xrightarrow{R} & U \\
\downarrow{\phi} & & \downarrow{\phi} \\
D_r & \xrightarrow{D_r} & D_r \\
\downarrow{z + (\lambda z_0)z}
\end{array}
$$

yields a functional equation

(SFE) \hspace{1cm} \phi \circ R(z) = (\lambda z_0)(\phi(z)) \hspace{1cm} .
This equation is called the Schröder Functional Equation. Fatou related solutions \( \phi \) of (SFE) to membership in the Fatou set.

(3.8) THEOREM. Let \( z_0 \) be a neutral fixed point. Then \( z_0 \in F \) if and only if the (SFE) has an analytic solution.

Obviously the same statement holds for a neutral periodic orbit of period \( n \) once \( R \) is replaced by \( R^n \) in the (SFE). We prove this theorem here although it is out of order logically. In the proof we shall use the fact that the Julia set contains at least three points. In fact, it is always uncountable because it is always a perfect set (see (4.8)).

Proof. If the (SFE) has a solution, then \( z_0 \in F \) can be verified directly from the definitions.

Suppose \( z_0 \in F \) and let \( U \) be the maximal domain such that \( z_0 \in U \) and \( U \subset F \). Since \( U \cap J = \emptyset \), the uniformization theorem (1.8) implies that the universal cover \( \tilde{U} \) of \( U \) is conformally equivalent to the unit disk \( D_1 \). We choose a cover \( p:D_1 \to U \) such that \( p(0) = z_0 \) and we choose a lift \( \tilde{R} \) of \( R \) on \( U \) so that \( \tilde{R}(0) = 0 \). Then \( \tilde{R}:D_1 \to D_1 \) and \( |(\tilde{R})'(0)| = 1 \). Applying Schwarz's lemma (1.7) yields the fact that \( \tilde{R}(z) = (\lambda z_0)z \). The universal cover \( p \) gives us our solution \( \phi \) to the (SFE).

\[ \square \]

REMARKS. (1) Note that the same proof can be used to prove that the existence of a topological conjugacy implies the existence of an analytic conjugacy.

(2) Another proof which does not use the fact that \( J \) is uncountable or the uniformization theorem was shown to me by P. Collet, R. de la Llave, and O. Lanford. Let
\[ \psi_n(z) = \frac{1}{n} \left( \sum_{p=0}^{n-1} \lambda^{-p} \, R^p(z) \right). \]

If the sequence \( \{R^p\} \) is normal around \( z_0 \), the \( \psi_n \) converge to a map \( \psi \) such that

\[ \psi_n \circ R = \lambda \psi. \]

An easy consequence of (3.8) concerns the case where \( \lambda z_0 \) is a root of unity.

(3.9) COROLLARY. If \( \lambda z_0 \) is a root of unity, then the (SFE) does not have a solution.

Proof. Again we only do the fixed point case. Suppose \( \lambda^1 = 1 \) and the (SFE) has a solution in a neighborhood \( U \) of \( z_0 \). Then

\[ \phi \circ R^1 \circ \phi^{-1} = \text{Id}, \]

and therefore \( R^1 = \text{Id} \) on \( U \). Since \( R \) is analytic, it must be the identity on \( \overline{U} \), contradicting our assumption that the degree of \( R \) is greater than one.

Both Fatou [F1, pp. 191-221] and Julia [J, pp. 297-311] extensively discussed the root of unity case, and they both credit Leau (These, 1897) with the initial analysis of the \( \lambda z_0 = 1 \) case. More recently, Camacho's [Ca] has classified the dynamics up to topological conjugacy. Without proof, we give a description of some of their results.

First we state the topological result.

(3.10) THEOREM. Let \( f(z) = \lambda z + a_2z^2 + a_3z^3 + \ldots \) be an analytic map in the neighborhood of the origin. Suppose \( \lambda^n = 1 \) and \( \lambda^m \neq 1 \) for \( 1 < m < n \). Then either \( f^n = \text{Id} \) or there is a local homeomorphism \( h \) and an integer \( k > 1 \)
such that \( h(0) = 0 \) and
\[
h \circ f \circ h^{-1}(z) = \lambda z(1 + z^k n).
\]

In order to understand what this means, it is necessary to analyze the dynamics of the map
\[
z + \lambda z(1 + z^k n).
\]
The condition \( z^k n \in \mathbb{R} \) describes a set of lines through the origin. Note that the map \( g(z) = z(1 + z^k n) \) leaves these lines invariant. Then \( z + \lambda z(1 + z^k n) \) is the composition of \( g \) with the periodic rotation \( z + \lambda z \). The dynamics of the standard form \( g \) in the cases \( k = 1, 2, \) and \( 3 \) are illustrated in the following figures.

See Supplement

(3.11) FIGURES. These figures illustrate the dynamics of \( z + z(1 + z^q) \) for \( q = 1, 2, \) and \( 3 \) respectively (top to bottom).

Using techniques similar to Camacho, Fatou and Julia also obtained some analytic information. These results are usually referred to as the Flower Theorem.

(3.12) THEOREM. Let \( f(z) = \lambda z + a_2 z^2 + \ldots \) be an analytic map in a neighborhood of the origin and suppose that \( \lambda^n = 1 \) and \( \lambda^m \neq 1 \) for \( 1 < m < n \). Then there is an integer \( k \) and \( nk \) analytic curves (petals) which are pairwise tangent at the origin. The petals are forward invariant and any orbit in a petal is asymptotic to the origin. Any compact set inside the petal converges uniformly to the origin under \( f^n \).

These petals do not contain all the orbits which are asymptotic to the origin because that set does not have an analytic boundary. In the rational case, that set is part of the Julia set, and the Julia set does not have tangents. The
following figures illustrate the relationship between the Julia set and these petals. The computer points are the Julia set and these petals are hand-drawn and shaded. In Sections 4 and 5, we shall see that the Julia set must contain a sequence of repelling periodic points which, in this case, converge to the origin. These points approach the origin between the petals. Figures (3.13) - (3.15) relate exactly to the pictures in Figure (3.11).

(3.13) FIGURE. The Julia set of \( z + z + z^2 \) and the associated petal. In this case the petal is cardioid shaped.

(3.14) FIGURE. The Julia set of \( z - z + z^2 \) and the petals of the origin.

(3.15) FIGURE. Part of the Julia set of \( z + e^{2\pi i/3} z + z^2 \) and its petals. The computer program was not run long enough to show the Julia set connecting to the origin. Compare this figure with Douady's rabbit (2.1).

(3.16) FIGURE. This is a picture of the Julia set of

\[ z + e^{2\pi i/20} z + z \]

Here the Julia set should approach the origin in twenty different directions (between twenty petals), yet it is hard to get a good picture of this behavior. We include the picture to inspire the reader to imagine how the Julia set of

\[ R_\lambda(z) = \lambda z + z \quad \text{where} \quad |\lambda| = 1 \]
varies with $\lambda$ as $\lambda$ approaches 1. Also the reader should be inspired to derive a good algorithm for plotting the Julia set in this case.

There were two major questions left unresolved by Fatou and Julia.
(3.17) Does (SFE) ever have a solution?
(3.18) If so, when?
In 1942, C. Siegel [Si] found a subset $\Lambda$ of the unit circle of full measure such that, whenever the eigenvalue is in $\Lambda$, then the (SFE) does have a solution

(3.19) THEOREM. Let $\lambda z_0 = e^{2\pi i \alpha}$ where $\alpha \in \mathbb{R} - \mathbb{Q}$. Suppose there exist positive constants $a$ and $b$ such that

$$|\alpha - \frac{p}{q}| > \frac{a}{q^b}$$

for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$ with $q > 1$. Then the Schroeder functional equation does have a solution. $\square$

The original proof [Si] is an extremely difficult computation involving delicate number theory. A different proof, which uses a version of Newton's method applied to function spaces, is contained in Siegel-Moser [SM], and recently a relatively simple proof for a smaller class of numbers using techniques of M. Herman has been circulated (See [dL] and [H2]).

Rather than discuss the proof, we recall some facts about diophantine approximation and continued fraction expansions which explain the number theory condition in the hypothesis. This condition loosely says "$\alpha$ is badly approximated by rational numbers". If we write $\alpha$ in its continued fraction expansion, we can make this statement precise. Let
\[
\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}}
\]

(we shall use the notation \([a_0, a_1, a_2, \ldots]\) for this expression) and define the convergents \(\frac{p_n}{q_n} \in \mathbb{Q}\) by

\[
\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}} = [a_0, \ldots, a_n].
\]

They are the best rational approximants to \(\alpha\) (see Niven [Ni]), and the accuracy of the approximation is given by the equation

\[
(3.20) \quad \frac{1}{(a_{n+1} + 2)q_n^2} < |\alpha - \frac{p_n}{q_n}| < \frac{1}{a_{n+1}q_n^2}
\]

For example, if the terms \(a_i\) stay bounded, we can verify the hypothesis of Siegel's Theorem. Using the mean value theorem, it is easy to show that any algebraic number of degree 2 has such a continued fraction expansion. Moreover, the same argument also shows that the hypothesis of (3.19) is satisfied for an algebraic number of degree \(n\) with \(b = n + 1\) (Niven [Ni, p. ]). So all algebraic numbers are in \(\Lambda\). But as mentioned above, the set \(\Lambda\) contains many other points because it is a set of full measure (Siegel and Moser [SM, p. ]).

We have seen that, if \(z_0\) is a fixed point with derivative \(\lambda_{z_0}\), where \(\lambda_{z_0} = e^{2\pi i \alpha}\), then \(\alpha \in \mathbb{Q}\) implies \(z \in J\), and \(\alpha \in \Lambda\) implies \(z \in F\). It would be nice to believe that \(z_0 \in F\) whenever \(\alpha \notin \mathbb{Q}\). However, the following example due to Cramer [C] shows that things are not this simple.
(3.21) EXAMPLE. Let

\[ p(z) = z^n + \ldots + \lambda z. \]

Suppose \( \lambda = e^{2\pi i \alpha} \) where \( \alpha \in \mathbb{R} - \mathbb{Q} \) and

\[ |\lambda^n - 1| < \left( \frac{1}{n} \right)^{d^n - 1} \]

for infinitely many natural numbers \( n \). Since the periodic points satisfy the equation \( p^n(z) = z \), we can calculate them by solving

\[ zd^n + \ldots + \lambda^n z = z \]

or

\[ zd^n - 1 + \ldots + (\lambda^n - 1) = 0 \]

if we ignore the fixed point at the origin. If \( \mu_1, \ldots, \mu_{d^n-1} \) are the \( d^n - 1 \) roots of this equation, then

\[ |\mu_1| \ldots |\mu_{d^n-1}| = |\lambda^n - 1|. \]

Let \( m = \min |\mu_1| \). We get

\[ m^{d^n - 1} < |\lambda^n - 1| < \left( \frac{1}{n} \right)^{d^n - 1} \]

for infinitely many natural numbers \( n \). Hence, there are infinitely many periodic points converging to the origin. Yet, if the origin is in the Fatou set, the (SFE) has a solution by (3.8), and the map \( p(z) \) must be an irrational rotation near the origin. Since irrational rotations do not have periodic points (except the center), we conclude that the origin is in the Julia set.

It is natural to ask if

(3.22)

\[ |\lambda^n - 1| < \left( \frac{1}{n} \right)^{d^n - 1} \]
is ever satisfied for infinitely many \( n \) if \( \alpha \notin \mathbb{Q} \). To construct such \( \lambda \), we use continued fraction expansions

\[
|\lambda^n - 1| = |e^{2\pi in\alpha} - 1|
= |e^{\pi in\alpha} - e^{-\pi in\alpha}|
= 2|\sin(\pi n\alpha)|.
\]

For each \( n \), let \( m_n \) be the integer such that

\[
|n\alpha - m_n| < \frac{1}{2}.
\]

Using the fact that \( 2x < \sin(\pi x) < \pi x < 7x \) when \( |x| < \frac{1}{2} \), we get

\[
|\lambda^n - 1| = 2|\sin(\pi n\alpha)|
= 2 \sin(\pi |n\alpha - m_n|)
\]

and

\[
4|n\alpha - m_n| < |\lambda^n - 1| < 7|n\alpha - m_n|.
\]

To find a dense set of \( \lambda \) which satisfy (3.22), we specify any initial segment \( [a_0, ..., a_k] \), and then we continue the expansion using the fact that

\[
|q_k\alpha - p_k| < \frac{1}{q_k^{a_k+1} + q_k^{-1}} < \frac{1}{a_k+1 + q_k^{-1}} < \frac{1}{a_k+1 + q_k^{-(d_k)-2}}.
\]

If we inductively chose

\[
a_k+1 > 7 q_k^{(d_k)-2},
\]

then equation (3.21) is satisfied for \( n = q_k+1, q_k+2, q_k+3, ... \).

The reader is encouraged to write out the decimal expansion of an \( \alpha \) that
satisfies this condition. This example also shows that the compliment of the set

\[ \Lambda \cup \{\text{roots of unity}\} \]

contains a dense subset of the unit circle.

Since the numbers which satisfy the hypothesis of Siegel's theorem form a set \( \Lambda \) of full measure on the unit circle, the numbers described above form a set of measure zero. But on the other hand, if we define

\[ U_i = \bigcup_{p/q \in \mathbb{Q}} \{ \alpha \in \mathbb{R} \mid |\alpha - \frac{p}{q}| < \frac{1}{i} \} , \]

then \( U_i \) is open by definition and dense because \( \mathbb{Q} \subset U_i \). The set

\[ \left( \cap_{i \in \mathbb{Z}^+} U_i \right) - \mathbb{Q} \]

is therefore a residual set. We conclude that the set \( \Lambda \) which has full measure is a meager set in terms of topology.

We cannot yet always determine if a fixed point is in either the Julia set or the Fatou set because the neutral case has not been completely resolved. The hypothesis to Siegel's theorem has been broadened somewhat \( (\quad) \), but there still are cases between Cramer's examples \((3.21)\) and the known generalizations of Siegel's theorem. We end this section by posing a few open questions.

(3.22) PROBLEMS. (1) Is membership in the Fatou set entirely determined by the eigenvalue of the fixed point?

(2) What is the dynamics in a neighborhood of a periodic orbit if its eigenvalue is not a root of unity and the (SFE) does not have a solution?
4. The Consequences of Montel's Theorem

The following theorem of Montel is fundamental to proving that the Julia set is the closure of the repelling periodic points. In this section, we use it to prove that \( J \) is a perfect set. Therefore, it is uncountable.

(4.1) **THEOREM.** Let \( F \) be a family of meromorphic functions defined on a domain \( U \). Suppose there exist three points \( a, b, c \) in \( \mathbb{C} \) such that

\[
\left( \bigcup_{f \in F} f(U) \right) \cap \{a, b, c\} = \emptyset
\]

then \( F \) is a normal family on \( U \).

It is interesting to see that a condition on the images of a family could guarantee normality. However, the result is not surprising if it is considered in light of the uniformization theorem (1.8).

**Proof.** Consider the domain

\[
V = \mathbb{C} - \{a, b, c\}.
\]

The uniformization theorem implies that its universal cover is the open unit disk \( D_1 \). For each \( f \in F \), there exists a lift \( \tilde{f} : U \to D_1 \) such that the diagram

\[
\begin{array}{ccc}
\tilde{f} & \rightarrow & D_1 \\
\downarrow & & \downarrow \\
U & \rightarrow & V \\
f & \swarrow &
\end{array}
\]

commutes.

The new family of functions \( \{\tilde{f}\} \) is a normal family because it is locally bounded. Moreover, if \( \{\tilde{f}\} \) is normal, then \( F \) is normal.

In this section we use Montel's theorem to derive a few basic properties of
the Julia set.

(4.2) COROLLARY. Let \( z \in J(R) \). If \( U \) is a neighborhood of \( z \), then the set

\[
U \cup \bigcup_{n > 0} R^n(U)
\]

contains all of \( \mathbb{C} \) except possibly one or two points. Such points are called exceptional points, and \( E_z \) will denote the set of these points. \( \square \)

(4.3) EXAMPLE. Consider a polynomial \( p(z) \). The point at \( \infty \) is very special. It is a fixed point whose only inverse image is itself. Very few fixed points have this property. In fact, if a rational map \( R \) fixes a point \( z_0 \) and \( R^{-1}(z_0) = \{z_0\} \), then \( R \) can be conjugated by a Mobius transformation which sends \( \infty \) to \( z_0 \). The result is a polynomial because it does not have any poles in \( \mathbb{C} \).

The point at infinity is always a superattractive fixed point for \( p(z) \). Hence, the Julia set of a polynomial is contained in \( \mathbb{C} \). Since

\[
\left[ \bigcup_{n > 0} p^n(\mathbb{C}) \right] = \mathbb{C}
\]

we see that \( \infty \) is indeed an exceptional point.

(4.4) REMARK. The set \( E_z \) contains at most two points. In fact, if

\[
U \cup \bigcup_{n > 0} R^n(U) = \mathbb{C} - \{\alpha, \beta\}
\]

and

\[
U \cup \bigcup_{n > 0} R^n(V) = \mathbb{C} - \{\gamma, \delta\},
\]

then, by applying the above corollary to

\[
U \cup \bigcup_{n > 0} R^n(U \cap V),
\]
we conclude that $\{\alpha, \beta\} = \{\gamma, \delta\}$. The set $E_z$ is often empty.

(4.5) THEOREM. Let $E_z$ be the set of exceptional points for some $z \in J$.

(A) If $E_z$ contains two points, then $R(z)$ is conjugate to the map

$$z + z^{\pm k}$$

where $k = \deg(R)$, and the sign is $+1$ if $E_z$ contains fixed points and $-1$ if $E_z$ consists of a periodic orbit of period two.

(B) If $E_z$ contains exactly one point, then $R(z)$ is conjugate to a polynomial. In both these cases, it follows that $E_z$ does not really depend on the choice of $z \in J$. Moreover, all exceptional points are contained in the Fatou set.

Proof. First note that $E_z$ is backwards invariant by definition. That is,

$$R^{-1}(E_z) = E_z$$

So $E_z$ either consists of one fixed point, two fixed points, or an orbit of period two.

To prove (B) we argue just as in Example (4.3). Let $M$ be a Mobius transformation which maps the point in $E_z$ to the point at infinity. Then $p(z) = M \circ R \circ M^{-1}(z)$ has no poles in $\mathbb{C}$.

To prove (A), we use a Mobius transformation which moves one point of $E_z$ to $\infty$ and the other point to 0. Then the conjugated map will be a rational function of the form

$$z + Kz^{\pm k}.$$ 

The constant $K$ can be eliminated by conjugation with an expansion or a contraction.

To prove that $E_z \cap J = \emptyset$, we observe that, in Case (A), the set $E_z$ consists of either two superattractive fixed points or a superattractive orbit of
period two. In Case(B), \( E_z \) is a superattractive fixed point. Since \( E_z \cap J = \emptyset \), we may conclude that \( E_z \) is independent of \( z \). \( \square \)

**NOTATION.** This theorem lets us simplify notation. The set of exceptional points will be denoted \( E(R) \) or simply \( E \) if the map is understood.

An easy consequence of (4.5) is the fact that \( J \) does not usually have interior.

\[ (4.6) \text{ COROLLARY. If } \text{int}(J) \neq \emptyset, \text{ then } J = \mathbb{C}. \]

**Proof.** Let \( U \) be a domain contained in the interior of \( J \). Since \( J \) is forward invariant,

\[
J \supseteq U \cap R^n(U) = \overline{U} - E
\]

Moreover, since \( J \) is closed and \( E \) contains at most two points,

\[
J = \mathbb{C}. \quad \square
\]

Another corollary is an inexpensive way to generate computer pictures of the Julia set.

\[ (4.7) \text{ COROLLARY. If } z \in \mathbb{C} - E, \text{ then the Julia set is contained in the set of accumulation points of the full backwards orbit of } z. \text{ That is,} \]

\[
J \subseteq \{ \text{accumulation points of } U \cap R^{-n}(z) \}.
\]

Consequently, if \( z \in J \), then

\[
J = \text{closure}(U \cap R^{-n}(z)). \quad \square
\]

Almost all of the pictures in the preceding sections were generated by finding
a repelling fixed point and then calculating its inverse orbit. This gives a
dense subset of $J$. However, dense subsets can be deceiving, and consequently
the more expensive methods of Mandelbrot and others often give better pictures.

PROBLEM. Design good algorithms to generate pictures of Julia sets.

In all the pictures we have seen, the Julia set appears to be infinite. We
are now able to verify that it is uncountable.

(4.8) THEOREM. The Julia set is a perfect set.

Before proving the theorem, we establish a helpful lemma.

(4.9) LEMMA. If $a \in J$, then there exists $a, b \in J$ such that $b \notin O^+(a)$ and
$a \in O^+(b)$.

Proof of lemma. If $a$ is not periodic, then $b$ can be any inverse image of $a$.
If $a$ is periodic with period $n$, consider the map $S = R^n$ and the equation

(4.10) \[ S(z) = a. \]

If $a$ is the only solution to (4.10), we may conjugate $S$ to a polynomial
as in Example (4.3). Since $a$ is in the Fatou set of a polynomial, we have
$a \in F(S)$ contradicting our assumption. Consequently, another solution $b$ to
(4.10) exists, and $b \notin O^+(a)$ because $a$ is the only solution to (4.10) in
$O^+(a)$. □

Proof of Theorem (4.8). Given $a \in J$, we prove that $a$ is an accumulation
point of $J$. Let $U$ be a neighborhood of $a$. Choose $b$ as in Lemma (4.9).
Since $b \in J$, $b \notin E$ and there exists an integer $k$ such that $b \in R^k(U)$. Let
c denote a point in $U$ such that $R^k(c) = b$. Then $c \neq a$ because $b \notin O^+(a)$,
and $c \in J$ because $J$ is $R^{-1}$ invariant. □
5. The Julia Set is the Closure of the Set of Repelling Periodic Points

Every repelling periodic point is contained in the Julia set, and since the Julia set is closed, the closure of the set of repelling periodic points is a subset of the Julia set. In this section, we show that these two sets are equal. The proof has two parts. First, we show that \( J \) is a subset of the closure of all periodic points. Then we derive a finite bound on the number of non-repelling periodic orbits.

(5.1) DEFINITION. The value \( c \) is a critical value of \( R \) if the equation

\[
R(z) = c
\]

has a solution whose multiplicity is greater than one. Such a solution \( p \) is called a critical point. The set \( C \) will denote the set of all critical points of \( R \).

(5.2) REMARK. Using local coordinates, this is equivalent to the condition \( R'(p) = 0 \). Topologically we can characterize the critical points as the points where the map \( R: \mathbb{C} \to \mathbb{C} \) is not locally injective. In other words, they are branch points of the branched cover \( R: \mathbb{C} \to \mathbb{C} \).

(5.3) DEFINITIONS. Given \( p \in \mathbb{C} \), the deficiency \( d_p \) of \( p \) is the number

\[
d - (\text{the cardinality of } R^{-1}(p))
\]

where \( d \) equals the degree of \( R \). Then \( d_p \neq 0 \) if and only if \( p \) is a critical value of \( R \). The total deficiency of the map \( R: \mathbb{C} \to \mathbb{C} \) is the sum of all the deficiencies for all values of \( R \).

(5.4) LEMMA. Let \( d \) denote the degree of \( R \). The number \( c_p \) of critical points is at most \( 2d - 2 \).
In the following proof, we actually prove that the total deficiency of $R$ is always $2d - 2$. Therefore, $d_p$ is usually 0. We use the Euler characteristic $\chi$ to prove this lemma.

**Proof.** Let $V$ be the set of critical values of $R$, $S$ be $C - V$, and $\tilde{S}$ be $C - R^{-1}(V)$. The map $R: S \rightarrow \tilde{S}$ is a $d$-fold covering space. Therefore,

$$\chi(\tilde{S}) = d \chi(S).$$

This yields

$$2 - \left( \sum_{\text{critical \ values}} (d - d_c) \right) = d(2 - c_v)$$

where $c_v$ equals the number of critical values. We get

$$d c_v - \left( \sum_{\text{critical \ values}} (d - d_p) \right) = 2d - 2$$

So

$$\sum_{\text{critical \ values}} d_c = 2d - 2.$$  

Since each critical point causes at least 1 deficiency, we conclude

$$c_p < 2d - 2.$$  

Now we can prove that the Julia set is contained in the closure of the periodic points.

(5.5) THEOREM. $J \subset \text{closure} \{ \text{periodic points} \}.$

We give one proof that works for polynomials and then we show how to modify it to include rational functions.
Proof. Let \( p: \mathcal{U} \to \mathbb{C} \) be a polynomial. Consider the subset \( K \) of \( J \) defined by

\[
K = J - \{ \text{critical values of } p \}.
\]

Since \( J \) is perfect and \( K \) differs from \( J \) by only a finite subset, we can prove the theorem by showing that \( K \subseteq \text{closure } \{ \text{periodic points} \} \).

Let \( w \in K \). There exists a neighborhood \( U \) of \( w \) such that the polynomial \( p \) has an inverse

\[
I: U \to \mathbb{C}.
\]

Form the family of meromorphic functions \( \{ g_n \} \) by

\[
g_n(z) = \frac{p^n(z) - z}{I(z) - z}.
\]

Note that the family \( \{ g_n \} \) of functions is a normal family if and only if the family \( \{ p^n \} \) is normal. Since \( w \in J \), the family \( \{ g_n \} \) cannot be normal on any open subset \( V \) of \( U \). However, if \( w \) were not in the closure of the set of periodic points, then \( \{ g_n \} \) would omit \( 0, 1, \) and \( \infty \). This contradicts Montel's Theorem(4.1).

\[\square\]

To generalize this proof to the case of rational functions we must use a slightly different family \( g_n(z) \) and a slightly smaller set \( K \). In that case we define

\[
K = J - \{ \infty, \text{critical values of } R^2, \text{poles of } R^2 \}.
\]

Then, given \( w \in K \), we define

\[
g_n(z) = \frac{(R^n - I_1)}{(R^n - I_2)} \frac{(I_3 - I_2)}{(I_3 - I_1)}
\]
where $I_1$, $I_2$, and $I_3$ are three possible inverses to $R^2$ in a neighborhood $U$ of $w$. Since

$$R^n = I_2 + Q \left( \frac{I_2 - I_1}{g_n - Q} \right)$$

where

$$Q = \frac{I_3 - I_2}{I_3 - I_1},$$

we see that $\{g_n\}$ is a normal family if and only if $\{R^n\}$ is a normal family.

To finish proving that the Julia set is the closure of the repelling periodic points, we use the theorem just proved along with the fact that the number of non-repelling periodic points is finite. Since the Julia set is perfect(4.8), every point in it will be the accumulation point of some sequence of repelling periodic points.

(5.6) DEFINITION. Let $p$ be an attractive fixed point of $R$. Then the stable set of $p$ is the set

$$W^s(p) = \{ z \mid R(z) + p \text{ as } n \to \infty \}.$$

The immediate stable set $A(p)$ of $p$ is the maximal domain containing $p$ on which the family $\{R^n\}$ is normal.

(5.7) PROPOSITION. The set $A(p)$ is the component of $W^s(p)$ containing $p$. Moreover, the frontier of $A(p)$ is contained in $J$ and the frontier of $W^s(p)$ is $J$.

(5.8) THEOREM. The immediate attractive set $A(p)$ contains at least one critical value.

To prove (5.8), we use the following lemma which is an easy consequence of the
monodromy theorem.

(5.9) LEMMA. Suppose $S$ is a rational function and $D$ is a simply connected domain which does not contain any of the critical values of $S$. Given any $d \in D$ and any $c \in S^{-1}(d)$, there exists a unique analytic inverse $I$ to $S$ defined on $D$ such that $I(d) = c$.

Proof of Theorem (5.8). Suppose $A(p)$ does not contain any critical value. Take a simply connected open neighborhood $U$ of $p$ contained in $A(p)$. Apply the lemma to $R|U$ to get an inverse map $S^1_1$ such that $S_1(p) = p$. Then $S_1(U)$ is a subset of $A(p)$, and it is simply connected (note that $R$ and $S_1$ are analytic inverse homeomorphisms between $U$ and $S_1(U)$). Therefore, this process can be repeated infinitely often to get a family of functions $\{S_k\}$ on the set $U$. The family is a normal family by Montel's theorem because $S_k(U) \subset A(p)$ for all $k$. This contradicts the fact that $p$ is a repelling fixed point for all of the maps $S_k$.

Of course, we may conclude that the number of attracting fixed points is at most $2d - 2$. Actually, using this reasoning, we can prove the following corollary.

(5.10) COROLLARY. The number of attracting periodic orbits is at most $2d - 2$.

To prove this corollary, one applies the same argument as above with the following definition.

(5.11) DEFINITION. Let $p$ be an attracting periodic point of period $n$. Then the immediate attractive set $A(p)$ is the set

$$
\bigcup_{k=0}^{n-1} A(R^k(p), R^n)
$$

where $A(x, S)$ represents the immediate attractive set of the attractive fixed point $x$ of the map $S$. 
How do the neutral points relate to the attractive points? We get a bound on the number of neutral orbits by considering the following one-parameter family of rational functions. Let

\[ R(z,w) = (1 - w)R(z) + wz^2. \]

(5.12) THEOREM. Suppose \( R \) has \( N \) neutral periodic points. There exists an \( \epsilon > 0 \) and a direction \( \theta \) in the w plane such that, if \( 0 < \rho < \epsilon \), the rational map

\[ R_\rho(z) = R(z, \rho e^{2\pi i \theta}) \]

has at least \( N/2 \) attracting periodic points which are continuations of neutral orbits of \( R \).

Proof. For each periodic point \( z_i \) we let \( n_i \) denote its period and \( s_i \) denote its eigenvalue. The pair \( (z_i, 0) \) satisfies the equation

\[ F_i(z, w) = R_i^{n_i}(z, w) - z = 0. \]

Since \( F_i(z, w) \) is a rational function of \( z \) and \( w \), there is an algebraic equation

\[ A_i(z, w) = p_0(w)z^n + p_1(w)z^{n-1} + \ldots + p_n(w) = 0 \]

(where the \( p_i(w) \) are polynomials) which is equivalent to \( F_i(z, w) = 0 \). Applying the theory of algebraic functions to each equation \( A_i(z, w) = 0 \), we can introduce new variables \( w_i \) and analytic functions \( z_i(w_i) \) such that

\[ w = w_i^{m_i}, \quad z = z_i(w_i) \text{ where } m_i \text{ is a positive integer, and} \]

\[ p_i(z_i(w_i), w_i^{m_i}) = F(z_i(w_i), w_i^{m_i}) = 0. \]
(see [N, pp. 47-55] for more details). Note that, when \( s_i \neq 1 \), this is the usual implicit function theorem. We can define derivative functions by

\[
s_i(z_i) = \frac{\partial^n}{\partial z^n} \left( z_i(w_i, w_1^{m_1}) \right).
\]

Let \( m \) be the least common multiple of all the \( m_i \) corresponding to the \( N \) neutral orbits. We introduce a new variable \( v \) such that \( v^m = w \). Then

\[
\left( \frac{m}{m_i} \right) u = w_i.
\]

The angle \( \theta \) is obtained by calculating an angle \( \theta_1 \) in the \( u \) plane.

To find \( \theta_1 \), write

\[
s_i(u) = s_i + a_i u^i + \ldots \quad \text{where} \quad a_i \neq 0.
\]

A non-zero higher order term exists because \( z^2 \) does not have any neutral periodic orbits. Let

\[
\tilde{s}_i(u) = s_i + a_i u^i.
\]

(5.13) claim: If we find \( \varepsilon_1 > 0 \) and \( \theta_1 \) such that at least half of the values satisfy

\[
|s_i(\rho e^{i\theta_1})| < 1
\]

then we can use \( \theta_1 \) as our appropriate direction which will yield the same result for \( s_i \).

We shall verify the hypothesis of the claim and leave the rest to the reader.

Let \( 2^k \) be the highest power of two that divides any of the \( m_i \). For each \( s_i \) on the unit circle, consider the tangent direction and the directions which differ from the tangent by an angle which is a multiple of \( \pi/2^k \). The following diagram illustrates the case where \( k = 1 \) and there are three neutral
orbits.

By considering all such $s_i$, we obtain a finite set of directions (at most $N$ times $2^\ell+2$).

Choose any angle $\theta_2$ which is not one of these directions. The angle $\theta_1$ that we seek will be $\theta_1 = \theta_2 + \theta_3$ where $\theta_3$ is determined in an inductive fashion.

The angle

$$\theta_3 = \pi(b_0 + \frac{b_1}{2} + \ldots + \frac{b_\ell}{2^\ell})$$

where $b_i \in \{0,1\}$. We determine $b_{i-1}$ after we have calculated $b_i, \ldots, b_\ell$.

Consider all periodic points $z_i$ whose periods $n_i$ are divisible by $2^\ell$.

Then

$$s_i(\rho e^{i\theta_1}) = s_i + \rho e^{i(2\pi b_\ell + \pi \theta_2)}$$

Since $k_i \theta_2$ is a direction which is not tangent to the unit circle, we choose $b_\ell$ such that at least half of the $s_i(v)$ go inside the unit circle.
To determine $b_{k-1}$, repeat this calculation on all orbits $z_i$ for which $k_i$ is divisible by $2^{k-1}$ but not by $2^k$. Then

$$s_i(\rho e^{2\pi i(\theta_2 + \theta_3)}) = s_i + \rho e^{2\pi i(k_i\theta_i + \pi b_{k-1} + (q/2)\pi b_\lambda)}$$

where $q$ is some odd number.

Since $k_i\theta_i + (q/2)\pi b_{k-1}$ is not tangent to the unit circle, we choose $b_{k-1}$ so that at least half of these orbits go inside the circle. We just continue this induction until the angle $\theta_1$ is determined.

When a rational map with (super)attractive periodic orbits is varied along a one-parameter family, these orbits will remain for, at least, some small interval. Therefore, we can combine (5.10) with (5.12) to get the classical bound stated below.

(5.14) COROLLARY. The number of attracting periodic orbits plus half the number of neutral periodic orbits is at most $2d - 2$.

These estimates are not the whole story. Douady[D] has improved them in many cases (see (9.1)). But (5.14), when combined with (5.5), is enough to prove the fundamental decomposition theorem.

(5.15) THEOREM. The Julia set equals the closure of the repelling periodic points.

One immediate consequence of (5.15) is the property that neighborhoods of points in $J$ are eventually surjective (i.e., $J$ is "locally, eventually onto = leo").
(5.16) COROLLARY. Let $A$ be a closed subset of $\mathcal{T}$ such that $A \cap E = \emptyset$.

Given a neighborhood $U$ of a point $p \in J$, there exists an integer $N$ such that $A \subset R^N(U)$. Therefore, if $D$ is a domain such that $D \cap J \neq \emptyset$, then there exists an $N$ such that

$$R^N(D \cap J) = J.$$ 

Proof. Let $q \in U$ be a repelling periodic point of period $n$. Choose a neighborhood $V$ of $q$ such that $V \subset U$ and $R^n(V) \supset V$. Since $A \cap E = \emptyset$ and $q \in J$, 

$$A \subset \bigcup_{k=1}^{\infty} R^{kn}(V).$$ 

By construction,

$$R(V) \subset R^{2n}(V) \subset R^{3n}(V) \subset \ldots.$$ 

Since $A$ is compact,

$$A \subset R^N(V) \subset R^N(U)$$

for some $N = kn$.

In the classical papers, this result is often referred to as local homogeneity. Usually, the Julia set "looks the same" in small as well as the large.
6. Classical Results Concerning the Fatou Set

The papers of Fatou and Julia contain many results beyond the fundamental decomposition theorem. In this section, we mention two of their theorems regarding the components of the Fatou set.

(6.1) PROPOSITION. Let $D$ be a simply connected, completely invariant component of the Fatou set. Then the total deficiency (see (5.3)) of the map $R:D \to D$ is $d - 1$.

Proof. This proposition is proven using the same covering space theory as in the proof of Lemma(5.4). Viewing $R:D \to D$ as $d$-fold branched cover, we get

$$1 - (\sum_{\text{critical values } c} (d - d_c)) = d(1 - c_v)$$

and conclude that

$$\left(\sum_{\text{critical values } c \in D} d_c\right) = d - 1.$$

(6.2) COROLLARY. The Fatou set cannot contain more than two different, completely invariant, simply connected components.

With this observation, we can count the number of components in the Fatou set.

(6.3) LEMMA. The frontier of any completely invariant component of the Fatou set is the Julia set.

Proof. The frontier is a non-empty, closed, completely invariant subset of $J$. 
Therefore, it must be all of $J$.

(6.4) DEFINITIONS. Let $D_i$ be a component of the Fatou set. If there exists an integer $n_i$ such that $R^{n_i}(D_i) = D_i$, then $D_i$ is periodic. Another component $D_j$ is eventually periodic if there exists $n_j$ such that $R^{n_j}(D_j)$ is periodic. The domain $D_j$ is preperiodic if it is eventually periodic but not periodic.

(6.5) REMARK. Since $R$ must be surjective, the existence of one preperiodic component in the Fatou set implies the existence of infinitely many preperiodic components.

(6.6) THEOREM. If the number of components of $F$ is finite, then there are at most two.

Proof. Assume that the number of components is finite but greater than two. Using (6.5), we obtain an integer $n$ such that the rational map $S = R^n$ setwise fixes each component of $F$. Therefore, each component is completely invariant, and one (denoted $D$) is not simply connected. The space $\mathbb{C} - D$ has at least two completely invariant components. Using (6.3), we see that each of these disjoint components must contain $J$.

In Section 2 we saw examples of all possibilities except $F = \phi$. Figure (2.3) illustrates a Fatou set which is equal to $A(\infty)$. The Julia set in Figure (2.2) divides $\mathbb{C}$ into two simply connected, completely invariant domains, and the Fatou set of Douady's Rabbit(2.1) contains preperiodic domains.

In Section 3, the Weierstrass $p$ function was used to construct an example where $F = \phi$, and in the next section Sullivan's classification theorem will be used to find more examples where $F = \phi$ (see (7.9)).
The immediate attractive set $A(\infty)$ in Figure 2.3 is of infinite connectivity, and $A(0)$ in Figure 2.2 is simply connected. This is all that can happen.

(6.7) THEOREM. Let $p$ be an attractive fixed point. Then $A(p)$ is either simply connected or of infinite connectivity.

Proof. Choose a simply connected, closed disk $D \subset A(p)$ such that $p \in D$, $R(D) \subset \text{int}(D)$, and

$$(\partial D) \cap \left( \bigcup_{c} \text{O}^+(c) \right) = \emptyset.$$  

Inductively define $E_{i}$ by $E_{0} = D$ and $E_{n} = R^{-1}(E_{n-1}) \cap A(p)$. Then $E_{0} \subset E_{1} \subset \ldots$ and

$$A(p) = \bigcup_{n=0}^{\infty} E_{n}.$$  

If $A(p)$ is not simply connected, then there exists an integer $N$ such that $E_{N}$ is not simply connected. Choose $N$ to be the first integer such that $E_{N}$ is not simply connected. Then $E_{N}$ is an orientable, two-dimensional manifold with boundary, and $\partial E_{N}$ has at least two components.

(6.8) FIGURE. A typical $E_{N}$. 

\[ \text{Diagram of } E_{N} \text{ and } \partial E_{N} \]
The boundary of $E_N$ separates $\mathcal{C} - E_N$ into at least two components. 
Repeatedly consider the branched covers

$$R : E_N + k + 1 + E_N + k$$

$k = 0, 1, 2, \ldots$.

They are actually covering spaces if $(E_N + k + 1 - E_N + k)$ does not contain any critical points, but nonetheless the number of boundary curves of $E_N + k$ is at least $2^{k+1}$.

(6.9) FIGURE. A typical $E_{N+2}$.

Consequently, $\partial A(\infty)$ will have infinitely many components and $A(\infty)$ is of infinite connectivity.
7. Sullivan's Classification of the Fatou Set

Recently Sullivan has completed a classification of the Fatou set ([S1] and [S2]). His theorems combine results of Fatou, Julia, Siegel ([S1] and [SM]), Arnold [A], Moser [ ], and Herman [H1] on complex analytic dynamics with the theory of quasi-conformal homeomorphisms (see Ahlfors [A3] and Lehto-Virtanen [LV]) to yield a simple picture of $R|F$. The classification can be summarized in two theorems (see (6.4) for relevant definitions).

(7.1) **THEOREM.** Every component of the Fatou set is eventually periodic.

The Fatou set often contains preperiodic components in addition to the periodic ones (see (6.5) and (6.6)). However, we usually focus our attention on the periodic components. To make this distinction easier to discuss, we introduce the following terminology.

(7.2) **DEFINITION.** A **Sullivan domain** of $R: \mathbb{C} \to \mathbb{C}$ is a periodic component of the Fatou set.

So we need only consider the dynamics in the Sullivan domains in order to understand the dynamics of $R|F$.

Sullivan's second theorem classifies the dynamics of $R$ restricted to a Sullivan domain. Five kinds of dynamics are possible.

(7.3) **DEFINITIONS.** Let $D$ be a Sullivan domain of period $n$ and let $S = R^n$.

1. The domain $D$ is an **attracting domain** if $D$ contains a periodic point $p$ such that $0 < |S'(p)| < 1$ and $D = A(p,S)$.
2. The domain $D$ is a **superattracting domain** if $D$ contains a periodic point $p$ such that $p$ is a critical point of $S$ (i.e. $S'(p) = 0$) and $D = A(p,S)$.
3. The domain $D$ is a **parabolic domain** if there exists a periodic point $p$
in a domain whose period divides $n$ and $S^k(z) + p$ as $k \to \infty$ for all $z \in D$.

(4) The domain $D$ is a **Siegel disk** if $D$ is simply connected and if $S|D$ is analytically conjugate to a rotation. In other words, there exists an analytic homeomorphism $h:D \to D_r$ such that the diagram

$$
\begin{array}{ccc}
D & \xrightarrow{S} & D \\
\downarrow h & & \downarrow h \\
D_r & \xrightarrow{z + e^{i\theta}z} & D_r
\end{array}
$$

commutes.

(5) The domain $D$ is a **Herman ring** if $D$ is conformally equivalent to an annulus $A = \{ z \in \mathbb{C} \mid r_1 < |z| < r_2 \}$ (where $r_1, r_2 \in \mathbb{R}$, $r_1 > 0$, $r_2 > 0$) and the map $S|D$ is analytically conjugate to a rigid rotation of the annulus.

Siegel disks and Herman rings are often referred to as **rotation domains**.

(7.4) REMARKS. (1) In the definitions of a Siegel disk and a Herman ring, it is enough to specify that the maps be topologically conjugate to rotations.

Topological conjugacy implies analytic conjugacy.

(2) Since $\deg(R) > 2$ and $R$ is analytic, $S|D$ can never be conjugate to a rotation $z + e^{i\theta}z$ where $\theta$ is rational (see (3.9)).

The Sullivan domains in Figure(2.1) are superattracting, and the finite Sullivan domain in Figure(2.2) is a typical attracting domain. Parabolic domains are intimately connected with the Flower Theorem (3.12) (see Figures (3.13) - (3.16)). Because the rotation domains exist in maps which satisfy delicate number theory conditions, we have not made any computer pictures of these domains. The following diagrams give some idea of how these domains should look.
$x = \text{points in forward orbits of the critical points}$

(7.5) FIGURE. A Siegel Disk. The disk has a "foliation" by invariant curves. The boundary of the disk is contained in the closure of the forward orbits of critical points.

(7.6) FIGURE. A Herman Ring. There are many properties of Herman rings which are analogous to those of Siegel disks. The domain is foliated by invariant curves and the boundary of the ring is contained in the closure of the forward orbits of the critical points.
Sullivan's second theorem combines his work with the results referred to in
the first paragraph of this section to finish the classification of the Fatou
set.

(7.7) THEOREM. Every Sullivan domain is either attracting, superattracting,
parabolic, a Siegel disk, or a Herman ring. Furthermore, there are finitely many
such domains. In the parabolic case, $S'(p) = 1$. The attracting and parabolic
domains both contain infinite forward orbits of critical points, and the boun-
daries of rotation domains are contained in the closure of the forward orbits of
the critical points.

One very important problem concerning the number of Sullivan domains
remains.

(7.8) PROBLEM. Given a rational map of degree $d$, can it have more than $2d - 2$
orbits of Sullivan domains?

The classification has many interesting consequences.

(7.9) COROLLARY. Suppose every critical point of $R$ is preperiodic. Then
$J(R) = \mathcal{T}$.

Proof. Since every Sullivan domain except the superattracting ones requires cri-
tical points with infinite orbits and since superattracting domains contain
periodic critical points, $F(R)$ does not contain any Sullivan domains.

EXAMPLE. The rational function

$$R(z) = \frac{(z - 2)^2}{z^2}$$

satisfies the hypothesis of (7.9).

In the polynomial case, this same argument can be used to show that $J$
is a "tree" (i.e. that \( F \) consists of exactly one component - \( A(\infty) \)).

(7.10) COROLLARY. Let \( p(z) \) be a polynomial and suppose all critical points except \( \infty \) are preperiodic. Then \( \mathcal{C} = J \cup A(\infty) \).

**Proof.** The argument in (7.8) proves that \( \mathcal{C} = J \cup W^s(\infty) \). But since \( W^s(\infty) = A(\infty) \) for polynomials, the result follows. \( \square \)

(7.11) DEFINITION. When the conclusion of the above corollary holds, we say that \( J \) is a **dendrite**.

(7.12) EXAMPLES. (1) The polynomial \( z + z^2 - 2 \) satisfies the hypothesis of this corollary and \( J = \{ z \mid z \in \mathbb{R} \text{ and } -2 < z < 2 \} \).

(2) The Julia set of the polynomial \( z + z^2 + i \) is also a dendrite. See Figure (7.13).

(7.13) FIGURE. The Julia set of \( z^2 + i \).

(7.14) EXAMPLE. Let \( p(z) = \lambda z + z^2 \) where \( \lambda = e^{2\pi i \theta} \) where \( \theta \notin \mathbb{Q} \) but is very nearly approximated by rationals (the exact diophantine condition is given in (3.21)), then \( J(p) \) is also a dendrite. The proof needs results of Douady[D] as well as the classification. See (9.4) for more details.
8. A Condition for Expansion on the Julia Set

Given the success of the last twenty years in the study of expanding and hyperbolic systems, it would be useful to know if $R|J$ is expanding. In our context, there is a relevant classical result. We shall need it in the next section, and it is also related to the structural stability results of Mané, Sad, and Sullivan [MSS].

(8.1) THEOREM. Let $\overline{O^+(C)}$ denote the closure of the forward orbits of the critical points. If $\overline{O^+(C)} \cap J = \emptyset$, then, given $K > 1$, there exists an integer $N$ such that

$$|\left(\overline{R^n}\right)'(z)| > K$$

if $n > N$ and $z \in J$.

To prove this theorem, we consider sequences of inverse functions of the family $\{R^n\}$. Recall (5.9) that, if $D$ is a simply connected domain which does not contain a critical value of $R$, then there exist at least two inverse functions to $R$ defined on $D$. Suppose that in addition, $D$ is disjoint from the forward orbits $O^+(C)$ of all the critical points of $R$. Then all the iterates $R^n$ can be inverted on $D$. In the proof of the theorem, we use this observation and consider sequences $\{I_j : D \to \mathbb{C}\}$ of inverse functions defined on simply connected domains $D$ which are disjoint from $O^+(C)$ and which satisfy $I_0 = \text{Id}$ and $R \circ I_j = I_{j-1}$.

(8.2) LEMMA. Let $D \cap \overline{O^+(C)} = \emptyset$ and $\{I_j\}$ be a sequence of inverse functions as defined above. If $J \cap \overline{O^+(C)} = \emptyset$, then the sequence $\{I_j\}$ converges to a constant function.

Proof of (8.2). First we prove that the sequence $\{I_j\}$ is a normal family on $D$. 
Choose a repelling periodic orbit $O^+(p)$. Since $p$ is not an exceptional point, there are two distinct points $p_1$ and $p_2$ which are also distinct from $p$ such that $R(p_1) = p$ and $R(p_2) = p_1$. To prove that $\{I_j\}$ is a normal family on $D_p = D - O^+(p)$, note that $I_j|_{D_p}$ must omit $\{p, p_1, p_2\}$ if $j > 2$. That is, if $I_j(z) = p$, then $z = R^j(I_j(z)) = R^j(p) \in O^+(p)$. To prove normality on all of $D$, choose a different repelling periodic orbit $O^+(q)$ and prove normality on $D_q = D - O^+(q)$. Since $D = D_p \cup D_q$, the family $\{I_j\}$ must be normal on $D$.

The assumption $J \cap O^+(C) = \emptyset$ implies that $F \neq \emptyset$ and that the Sullivan domains are either attracting or superattracting domains. Consequently, the forward orbits of all points in $D \cap F$ accumulate on a finite set of periodic points which are contained in $O^+(C)$.

Now, let $D_{-k}$ denote $I_k(D)$. We show that $D_{-k}$ converges uniformly to $J$ as $k \to \infty$. If not, there would exist two sequences $k_i$ and $z_i$ such that $R^{k_i}(z_i) \in D$, yet all of the $z_i$ are outside some neighborhood of $J$. Let $z^*$ be an accumulation point of the $z_i$. Then $\{R^i\}$ is a normal family on some neighborhood of $z^*$. Choose a convergent subsequence $\{R^{k_i}\}$ of the sequence $\{R^i\}$ and consider the images $R^{k_i}(z^*)$. Since the sequence converges uniformly, we can conclude that $R^{k_i}(z^*)$ converges to some point in $D$. But this contradicts the fact that

$D \cap \{\text{attracting and superattracting periodic orbits}\} = \emptyset$.

We have a normal family $\{I_j\}$ whose images $D_j$ are converging uniformly to the Julia set. However, the Julia set does not contain interior. Therefore, any limit function has to be a constant function since its image does not contain interior.

Given this lemma, it is not difficult to prove the theorem.

**Proof of Theorem.** Cover the Julia set $J$ with a finite number of simply con-
connected domains $D_i$ whose closures are disjoint from $0^+(C)$. By the lemma, any sequence of inverse functions $I_j: D_i \rightarrow \mathbb{C}$ must limit to a constant function. Therefore, the derivatives $I_j$ limit to zero. From this the result follows because we can obtain an $N$ such that

$$\frac{1}{|(R^n)'(z)|} < \frac{1}{K}$$

for all $n > N$ and all $z \in D_i$. 

The proof of the lemma is useful in the actual classification theorem. We stated that the boundary of a rotation domain $D$ is contained in the closure of the forward orbits of the critical points. We now briefly show why the proof of this fact is essentially the same as the argument in the proof of Lemma(8.2).

Proof of part of Theorem(7.7). Suppose $z_0$ is a point in $\partial D$ but not in the closure of the forward orbits of the critical points. Using the leo property (5.14) of the Julia set and the fact that rotation domains have preimages disjoint from themselves, we can find a path $\lambda$ from inside the rotation domain to a point $z_1$ in a preperiodic component of the Fatou set such that

$$\lambda \cap 0^+(C) = \emptyset.$$
(8.3) FIGURE. The Rotation Domain $D$ and the Path $\ell$. The point $z_1$ is in a preperiodic component of $F$; the point $z_0$ is in the boundary of $D$; and the point $z_2$ is an element of the rotation domain.

Using $\ell$, find a simply connected domain $D_1$ which contains $\ell$ but is disjoint from $O^+(C)$. We define a sequence of inverse functions $I_j:D_1 \to \mathbb{C}$ such that $I_j(z_2) \in D$ for all $j$.

A contradiction is derived using the limit functions of the $I_j$. Consider $I_j:D_2 \to \mathbb{C}$ and $I_j:D_3 \to \mathbb{C}$ where $D_2$ and $D_3$ are subsets of $D_1$ but $D_2 \subset D$ and $D_3 \cap D = \phi$.

(8.4) FIGURE. The disks $D_1$, $D_2$, and $D_3$ in a subset of the region shown in (8.3).

The proof and, therefore, the conclusion of Lemma(8.2) apply to $I_j:D_3 \to \mathbb{C}$. 
Consequently, the functions limit to constant functions. However, \( I_j : D_2 \to \mathbb{C} \) is determined by an irrational rotation and therefore the limits are never constants. Since a constant function can never be analytically continued to a non-constant one, we derive a contradiction. \[ \square \]

In Section 5, we referred to the le{\oe} property as a justification of the statement "Julia sets are fractals". This statement is not precisely true all the time, but Sullivan has a simple description of the self-similarity when \( R|J \) is expanding.

(8.5) DEFINITIONS. A function \( f : X \to X \) of a metric space \( X \) with metric \( d \) is a K-quasi-isometry if
\[
\frac{1}{K} d(x,y) < d(f(x),f(y)) < Kd(x,y).
\]
The Julia set \( J(R) \) is quasi-self-similar if there exists a \( K \) and \( r_0 \) such that
\[
\phi_r(J \cap D_r(x))
\]
maps into \( J \) by a K-quasi-isometry for all \( r < r_0 \) and all \( x \in J \) (here \( \phi_r \) is multiplication by \( \frac{1}{r} \)).

In other words, a set is quasi-self-similar if there exists a \( K \) such that every small piece can be expanded to full size and then placed onto \( J \) by a K-quasi-isometry.

(8.6) THEOREM (Sullivan [S3][S4]). If \( J \cap O^+(C) = \phi \), then \( J \) is quasi-self-similar. \[ \square \]
9. The Dynamics of Polynomials

Some of the special characteristics of the dynamics of polynomials have already been described, but since they form a particularly important class of systems, they are worthy of more attention. In this section, we discuss results which apply to all polynomials regardless of their degree.

Let \( p(z) \) be a polynomial of degree \( d \). As we have frequently mentioned, the point at infinity plays a distinguished role. It is both an exceptional point and a superattractive fixed point whose deficiency (see (5.4)) is \( d - 1 \). Consequently, \( F(p) \neq \emptyset \) always; \( J(p) \) is contained in a bounded subset of the complex plane; and the immediate stable set \( A(\infty) \) equals the entire stable set \( \mathcal{W}^s(\infty) \). Moreover, there is a neighborhood \( U \) of infinity and a real number \( r > 1 \) such that \( p|_U \) is analytically conjugate to the map \( z \mapsto z^d \) restricted to the set \( \mathcal{C} - D_r \).

Since infinity is a critical value whose deficiency is \( d - 1 \), the proof of Lemma (5.4) tells us that there are at most \( d - 1 \) finite critical points. The behavior of these critical points often determines a great deal about the dynamics. As a result, it is handy to distinguish between the set of all critical points \( \mathcal{C} \) and the set of finite critical points \( \mathcal{C}' \). Because there are at most \( d - 1 \) points in \( \mathcal{C}' \), we can conclude (using (5.8) and (5.14)) that the number of finite attracting periodic orbits plus half of the number of neutral periodic orbits is at most \( d - 1 \).

(9.1) REMARK. Douady [D], using the theory of polynomial-like functions, has sharpened this count of non-repelling orbits. There are at most \( d - 1 \) such orbits.

The Fatou set of a polynomial is somewhat simpler than the general case.

(9.2) THEOREM. The Sullivan domains of a polynomial are never Herman rings.
**Proof.** We prove the theorem by applying the maximum principle to the family of functions \( \{ p^n \} \). Suppose one of the Sullivan domains \( A \) was a Herman ring. Let \( \ell \) denote any one of the invariant Jordan curves in \( A \).

![Diagram showing the possible Herman ring](image)

(9.3) **FIGURE.** The possible Herman ring.

Let \( U \) be the open disk which is bounded by \( \ell \) and which does not contain infinity. Since \( U \cap J(p) \neq \emptyset \) and since we are not dealing with the case where the cardinality of \( E(p) \) is two, we conclude that \( \bigcup p^n(U) = \mathbb{C} \). However, this conclusion contradicts the maximum principle applied to the functions \( p^n|U \) because \( p^n(\ell) = \ell \).

(9.4) **REMARK.** Douady's improved bound of the number of non-repelling periodic orbits (9.1) is the missing ingredient necessary to prove (7.14). If \( f_\lambda(z) = \lambda z + z^2 \) and \( |\lambda| = 1 \) but \( \lambda \) is not a root of unity, then either \( f_\lambda \) is linearizable in a neighborhood of 0 (i.e. the (SFE) has a solution) or \( J(f_\lambda) \) is a dentrite. Like (7.10), we prove this by analyzing the possible Sullivan Domains. Since 0 is non-repelling, (9.1) implies that there are no (super)attracting domains except the one containing infinity and that there are no parabolic domains. The only available candidate for a finite Sullivan domain is a Siegel disk. Then \( f_\lambda \) would be linearizable around 0 (see (3.20)).

In the beginning of this section, we mentioned that there is a conjugacy
between \( p(z) \) and \( z + z^d \) in some neighborhood of infinity. A great deal of information can be derived from attempting to extend this conjugacy over as large a region as possible. In fact, the rest of this section is essentially a treatment of this question.

(9.5) THEOREM. The following four statements are equivalent.

1. The map \( p|A(\infty) \) is analytically conjugate to the map \( z + z^d \) restricted to the exterior of the unit circle.

2. The set \( A(\infty) \) is simply connected.

3. The Julia set \( J(p) \) is connected.

4. The sets \( A(\infty) \) and \( \mathbb{C}' \) are disjoint.

Proof. 1 \( \rightarrow \) 2. The conjugacy is a homeomorphism between a disk and \( A(\infty) \).

2 \( \rightarrow \) 3. A plane set is simply connected if and only if its frontier is connected (see Newman [Ne]). The implication follows since \( \partial A(\infty) = \partial W^S(\infty) = J(p) \).

2 \( \rightarrow \) 3. We prove this implication by showing that \( A(\infty) \cap \mathbb{C}' \neq \emptyset \) implies \( J(p) \) is disconnected. Let \( h: U + D_r \) denote the conjugacy between \( p(z) \) and \( z + z^d \) in a neighborhood of \( \infty \), and let \( A_0 \) be the annulus \( h^{-1}(D_r \mathbb{C} - D_r) \). Using the conjugacy, we construct an orthogonal coordinate system on \( U \) where circles

(9.6) FIGURE. The orthogonal coordinate system on \( U \) and the annulus \( A_0 \).
concentric to \( \mathcal{A} \) are wrapped by \( p(z) \) onto other such circles in a \( d \)-to-\( 1 \) fashion. Radii are mapped to other radii.

We try to extend the conjugacy by taking inverse images by \( p(z) \). Let \( A_{-1} = p^{-1}(A_0) \). If \( A_0 \cap p(C') = \emptyset \), then \( p: A_{-1} \to A_0 \) is a \( d \)-fold covering space and \( A_{-1} \) is an annulus. Hence, we can pull back the coordinate system and define a conjugacy \( h: A_{-1} \to D_s \) where \( s = \sqrt{d} \). We repeat this process inductively by defining \( A_{-k} = p^{-1}(A_{-k+1}) \) as long as \( A_{-k} \cap p(C') = \emptyset \).

Our assumption implies the existence of an integer \( N \) such that \( p(c) \in A_N \) for some \( c \in C' \). In this case, the map \( p: p^{-1}(A_N) \to A_N \) is not a covering space but is a branched cover.

(9.7) FIGURE. The sets \( p^{-1}(A_N) \) and \( A_N \).

The inverse image of the Jordan curve \( \mathcal{A} \) through \( p(c) \) is a pinched curve which bounds at least two finite, open sets, and these sets disconnect \( J(p) \).

4 \( +1 \). This implication is proved using exactly the same observations as in the proof of 3 \( +4 \). We are able to use the same notation for the annuli \( A_{-k} \), and our assumption implies that \( p: A_{-k} \to A_{-k+1} \) is always a \( d \)-fold covering space.
map. Since
\[ A(\infty) = U \cup \left( \bigcup_{n=1}^{\infty} A_{-n} \right), \]
the conclusion follows immediately. \( \square \)

The other extreme - all the finite critical points have orbits asymptotic to infinity - has behavior which is quite different from the type described above. The Julia set is totally disconnected. In fact, we can give a complete topological description of the dynamics on the Julia set.

(9.8) DEFINITION. Let
\[ \Sigma_n = \prod_{k=0}^{\infty} \{1, \ldots, n\}. \]
The shift map \( \sigma: \Sigma_n \to \Sigma_n \) is defined by
\[ \sigma([s_i]_0^\infty)_i = S_{i+1}. \]
The dynamical system \( \sigma: \Sigma_n \to \Sigma_n \) is called a one-sided shift on \( n \) symbols.

If one has not encountered this system before, one should note that it has rather complicated dynamics. It has lots of periodic orbits, and there are also dense orbits.

(9.9) THEOREM. Suppose \( C' \subset A(\infty) \). Then \( J \) is totally disconnected and \( p|J \) is a one-sided shift on \( d \) symbols.

Figure(2.3) is a picture of the Julia set of a quadratic polynomial whose finite critical point (namely \( 0 \) in this case) has an orbit asymptotic to infinity.

The proof of this theorem is based on the same idea as the last proof - attempting to extend the conjugacy near infinity. Finite critical values cause pinching of the coordinate system. Actually there is so much pinching that \( J \)
must be totally disconnected.

Proof. We could simply give a proof that works in the general case, but to illustrate the above observations concerning extending conjugacies, we give a proof that works in the quadratic case before we do the general case.

Suppose \( \text{deg}(p) = 2 \). Just as in the last proof, we start with a conjugacy \( h: U + D_r \) between \( p(z) \) and \( z + z^2 \) in a neighborhood of infinity, and we consider the annulus \( A_0 = h^{-1}(D_r - D_r) \). We can inductively define annuli \( A_{-k} = p^{-1}(A_{-k+1}) \) as long as \( A_{-k+1} \) does not contain the finite critical value of \( p(z) \). If \( p(c) \in A_N \) for \( c \in C' \), then \( p^{-1}(A_N) \) is a disk minus two holes.

(9.10) FIGURE. The pinched foliation. Infinity is not in the picture.
Another way to view this is to regard the coordinate system as a height function. Then we have the following picture.

(9.11) FIGURE. A pair of pants with a cap.

The two holes in \( p^{-1}(A_N) \) yield two inverse functions defined on

\[
B = \overline{\mathbb{C}} - [U \cup \bigcup_{k=N}^{-1} A_k].
\]

(9.12) FIGURE. The disk \( B \) with the two disks \( D_1 \) and \( D_2 \) contained in it.

The inverse function \( I_i : B \rightarrow D_i \) is an analytic homeomorphism. Consequently, \( I_i(p^{-1}(A_N)) \) is also a disk minus two holes (\( i = 1 \) and \( 2 \)).
Given any element \( \{s_i\} \in \sum_2 \) we can define \( \phi(\{s_i\}) \) to be the constant in the constant limit function of the normal family

\[ \{I_{s_1}, I_{s_2} \circ I_{s_1}, I_{s_3} \circ I_{s_2} \circ I_{s_1}, \ldots \} \]

This family limits to a constant function because the hypothesis to Lemma(8.2) is satisfied. We leave it to the reader to verify that \( \phi \) is a topological conjugacy between \( p|J \) and \( \sigma|\sum_2 \) i.e. \( \phi \) is a homeomorphism and

\[ \sigma \circ \phi = \phi \circ p. \]

In the general case, there can be more than one critical point in \( C' \). Connect \( p(C') \) to infinity by disjoint paths - one for each element of \( p(C') \) - with the property that, if \( B = \overline{\Sigma} - \{\text{paths}\} \), then \( p^{-1}(B) \subseteq B \). The set \( B \) is simply connected and \( d \) inverse functions \( I_i : B \to B \) (\( i = 1, \ldots, n \)) can be defined. A conjugacy \( \phi : J \to \sum_d \) is defined just as above.
10. The Mandelbrot Set and the Work of Douady and Hubbard

Even a rational function as simple as a quadratic polynomial can be the source of complicated and intriguing dynamics, and there remain important unresolved questions regarding their dynamics. In this section, we sketch the recent computer work of Mandelbrot and the results of Douady and Hubbard. These surprising results have been a major element in the explosion of interest in the subject during the last two years.

In Example (1.4), we showed how every quadratic polynomial is analytically conjugate to one of the form

\[(10.1) \quad p_c(z) = z^2 + c.\]

Hence, the family of quadratics is really a one complex-dimensional family of dynamical systems. This normal form (10.1) is particularly handy because it permits accurate and complete computer studies relying on computer graphics. Another useful representation of the family of quadratics is

\[(10.2) \quad f_\lambda(z) = \lambda z + z^2.\]

Whereas the quadratic (10.1) always has the finite critical point located at the origin (so (10.1) is the best representation of the family to use if one is investigating questions concerning the forward orbits of critical points), the function (10.2) always has a fixed point at the origin whose eigenvalue is \(\lambda\). As a result, (10.2) is useful when studying bifurcation questions when \(|\lambda|\) is near 1. The reader should review the relationship between (10.1) and (10.2).

Theorems (9.5) and (9.9) yield a useful dichotomy between \(c\) values for the family (10.1). If \(0 \in A(\infty)\) or, equivalently, if the sequence \(0^+(0)\)

\[
0, c, c^2 + c, (c^2 + c)^2 + c, \ldots
\]
converges to infinity, then \( J(p_c) \) is totally disconnected and \( p_c \mid J \) is a one-sided shift on two symbols. On the other hand, if \( 0 \notin A(\omega) \), then \( A(\omega) \) is simply connected and \( J \) is connected. Mandelbrot [M2] investigated this dichotomy and found that it resulted in an unusual fractal.

(10.3) DEFINITION. The **Mandelbrot Set** \( \mathbb{M} \) is the subset of \( \mathbb{C} \) defined as

\[
\{ c \in \mathbb{C} \mid J(p_c) \text{ is connected} \}.
\]

The reader should note that this is the first set we have defined in a parameter space. Each point represents a different dynamical system. The following picture is the author's humble attempt to reproduce Mandelbrot's spectacular pictures in [M1].

See Supplement

(10.4) FIGURE. The Mandelbrot Set.

The reader may find it interesting to go back to the previous figures of Julia sets of quadratics and locate their \( c \) values in (10.4). Many important open questions regarding quadratics are best phased in terms of the Mandelbrot set, and some of these will be discussed in this section.

Computing pictures of the Mandelbrot set \( \mathbb{M} \) is tricky business. The initial work [M2] indicated that \( \mathbb{M} \) had more than one "main body" and that it may be disconnected. In fact, it is connected.

(10.5) THEOREM. (Douady and Hubbard [DH]). The **Mandelbrot Set** is connected. \( \square \)

Douady and Hubbard prove (10.5) by constructing, using dynamics, a conformal automorphism

\[
\psi: \overline{\mathbb{C}} - D_1 + \overline{\mathbb{C}} - \mathbb{M}.
\]
Hence, the compliment of \( M \) in the Riemann sphere is conformally equivalent to a disk in a dynamically natural way.

Douady and Hubbard have also analyzed the components of the interior of \( M \). Let

\[
H = \{ c \mid p_c \text{ has a finite (super)attractive periodic orbit}\}.
\]

Theorems (5.8) and (8.1) imply that, if \( c \in H \), then \( p_c \mid J \) is expanding. It is routine to prove that \( H \subset \text{int}(M) \) and that \( H \) is open. The following conjecture has been the subject of much work in the last fifteen years.

(10.6) CONJECTURE. The set \( H \) equals the interior of \( M \).

Most workers believe that (10.6) is true. However, until the problem is resolved, we must use some annoying terminology.

(10.7) DEFINITION. A component \( K \) of the interior of \( M \) is hyperbolic if \( K \subset H \).

Douady and Hubbard [DH] have also constructed a conformal representation of the hyperbolic components.

(10.8) THEOREM. Let \( K \) be a hyperbolic component of the interior of \( M \). Then \( K \) is conformally equivalent to the disk \( D_1 \), and the equivalence may be given by the map \( p:K \to D_1 \) where

\[
p_K(c) = \lambda_c
\]

and \( \lambda_c \) is the eigenvalue of the unique finite (super)attracting periodic orbit of \( p_c \).

The existence of such a simple conformal equivalence is quite surprising. It
determines a center $\rho_K^{-1}(0)$ of each $K$, and we shall also talk about the root of $K$ which is the point

$$\lim_{t \in \mathbb{R}^+} \rho_K^{-1}(t)$$

(Douady and Hubbard prove that such a limit exists).

Now we take a tour through $M$, and at each stop we describe the Julia set and its relationship with the Julia sets of other points in $M$. The tour starts at the origin which is contained in a hyperbolic component which is actually the interior of a cardioid. Since $p_0(z) = z^2$, we are starting with the simplest Julia set in $M$. For $\epsilon \neq 0$ but small, Sullivan [S4] proved that $J$ is still a Jordan curve, but its Hausdorff dimension must be greater than one. As we mentioned previously, Ruelle [R] showed that this dimension varies real analytically in the norm of the parameter. Figure (2.2) illustrates a typical Julia set in this component. Sullivan also established that $J$ is a quasi-circle.

We now move along the negative real axis until we stop at $c = -(3/4)$. We are at a boundary point between two hyperbolic components. This point is the root of the hyperbolic component containing $c = -1$. Figure (3.13) shows the Julia set of $p_{-(3/4)}$. This map does not have any (super)attracting periodic orbits. Oddly enough, it is the only quadratic polynomial of the form (10.1) with no periodic orbits of period two. The finite Sullivan domains are the two parabolic domains whose boundaries contain the neutral fixed point. They form a cycle of period two.

At $c = -1$, we are at the center of this hyperbolic component. Its Julia set is pictured in Figure (3.6). Since the centers of hyperbolic components are the $c$ values for which $p_c$ has a finite superattractive periodic orbit, the origin in $p_{-1}$ is part of a superattractive periodic orbit. The one-parameter
family that we have described so far (i.e. $t + p_t$ with $t$ varying from 0 to -1) contains exactly one \textit{period doubling bifurcation}. In other words, the period of the finite (super)attractive cycle has doubled.

If we continue along the negative real axis, we encounter an infinite sequence of period doubling bifurcations - The Feigenbaum bifurcations. The reader should consult the book by Collet and Eckmann [CE] to learn more about these bifurcations.

Now we backtrack for a moment. Suppose that, instead of leaving the origin along the negative real axis, we approached the boundary of the cardioid at the $c$ value

\begin{equation}
(10.8) \quad c = \frac{\lambda}{2} - \frac{\lambda^2}{4}
\end{equation}

where $\lambda = e^{2\pi i/3}$ so that $p_c$ is conjugate to $f_{\lambda}$. Figure (3.14) is a picture of the Julia set of $f_{\lambda}$.

See Supplement 1

\begin{equation}
(10.9) \quad \text{FIGURE. The Julia set of } p_\alpha \text{ where } \alpha = .9c \text{ and } c \text{ is defined by the equation(10.8). This is the Julia set just before a period tripling bifurcation. We can see how the rabbit-like features of (2.1) evolve continuously from the Jordan curves of the main cardioid.}
\end{equation}

Our journey so far has been entirely in the \textit{primary region} of $M$. This primary region is inductively formed by starting with the cardioid $M_0$. At each successive step, an infinite number of hyperbolic components and their roots are added to $M_i$ to form $M_{i+1}$. A component $K$ is added to $M_i$ if its root is in the boundary of $M_i$. The primary region is the union

$$\bigcup_{i=0}^{\infty} M_i.$$
We do not get all of \( M \) (by any means!) with this method. Suppose we had continued along the negative real axis. The corresponding systems would have undergone an infinite number of period doubling bifurcations. Yet, we would still be far from the end of \( M \cap \mathbb{R} = [-2, \frac{1}{4}] \). Values of \( c \) intervene where \( p_c \mid \mathcal{J} \) has "aperiodic" behavior. Then new secondary regions develop. Each of these has a main body which is something like a cardioid, and its structure is very similar to the primary body. The following figures locate one such secondary body and show it in more detail.

(10.10) FIGURE. A secondary body in the Mandelbrot set.

(10.11) FIGURE. An enlargement of the secondary body in (10.10).

This computer evidence strongly suggests that the Mandelbrot set is also a fractal and that there is a "universality" present. Douady and Hubbard have results for polynomial-like mappings which support these ideas.

If \( c \in M \), then \( A(\infty) \) is simply connected and there exists a conjugacy

\[
\phi_c : \mathbb{C} - D_1 \to A(\infty)
\]

such that \( \phi_c(\infty) = \infty \), \( \phi_c'(\infty) = 1 \), and \( \phi_c(z^2) = p_c(\phi_c(z)) \) for all \( z \in \mathbb{C} - D_1 \) (see (9.5)). Much information can be gained by extending \( \phi \) to the unit circle \( S^1 \). Extending Riemann maps is a classical aspect of complex analysis, and Caratheodory [Cd] obtained the relevant result.

(10.12) DEFINITION. A subset \( S \subset \mathbb{C} \) is **locally connected at** \( p \in S \) if, given any \( \varepsilon > 0 \), there exists an \( \varepsilon' \) such that \( 0 < \varepsilon' < \varepsilon \) and \( D_{\varepsilon'}(p) \cap S \) is a connected set.
(10.13) THEOREM (Caratheodory). Let $D$ be a simply connected domain which is conformally equivalent to the disk $D_1$. The boundary of $D$ is locally connected if and only if the conformal equivalence extends to a continuous map from $D_1$ to $D$.

Often the Julia set of a polynomial is locally connected and the map $\phi$ extends ([D], [DH], [T1]). For example, if $c \in H$, then $J(p_c)$ is locally connected. Also, if $c$ is preperiodic, then $J(p_c)$ is locally connected and a dendrite (see (7.10)). So there is a continuous map $\phi_c : S^1 \to J(p_c)$ such that the diagram

\[
\begin{array}{ccc}
S^1 & \xrightarrow{z + z^2} & S^1 \\
\phi_c & \downarrow & \phi_c \\
J(p_c) & \to & J(p_c)
\end{array}
\]

commutes. Since the map $\phi_c$ is not injective, it is not a conjugacy. Usually, it is called a semiconjugacy. Describing the map on the Julia set amounts to establishing the identifications made by the map $\phi_c$.

However, the Julia set is not always locally connected.

(10.14) THEOREM (Douady and Sullivan [S4]). If $p_c$ has a neutral fixed point whose eigenvalue is not a root of unity and $p_c$ is not locally conjugate to its derivative, then $J(p_c)$ is not locally connected.

The proof of (10.14) requires a local analysis of "radial limits" to the fixed point. The arguments are similar to those Sullivan uses to show that orbits in parabolic domains are asymptotic to periodic points whose derivatives are roots of unity. The interested reader should consult [S4] and [S2].

Since the exterior of $M$ in $\mathcal{C}$ is also simply connected, the same
questions apply to \( \mathbb{C} - M \). Douady and Hubbard [DH] have partial results regarding the Riemann map on the exterior of the Mandelbrot set.

(10.15) **THEOREM.** For all \( \theta \in \mathbb{Q} \), the radial arc \( r + \psi(re^{2\pi i \theta}) \) with \( r > 1 \) has a limit as \( r \to 1 \).

Given a map \( p_c \) which is such a limit, Douady and Hubbard [DH] also construct an associated tree \( H_c \) and a map on the tree. They give a combinatorial algorithm for determining \( \theta \) from the map on \( H_c \).

We conclude with one of the basic problems in the dynamical theory of quadratics.

(10.16) **PROBLEM.** Is \( M \) locally connected?
REFERENCES


[H1] Herman, M.: 

[H2] Herman, M.: Lecture notes on an easy proof of Siegel's theorem.


[T2] Thurston, W.: Lecture notes--CBMS Conference--Univ. of Minnesota--Duluth, Minnesota.
SUPPLEMENT

The following pictures are micro-computer versions of the actual ones that will be in the paper. They are far from the best available and are included here simply to give a rough illustration of the final ones.

Figure (2.1) - Dauady's Rabbit.
Figure (2.2) - A Quasicircle.
Figure (2.3) - A Cantor set.
Figure (2.4) - An enlargement of (2.3).
Figure (2.5) - The Julia set of Newton's method.
Figure (3.7) - The Julia set of $z + z^2 - 1$. 
Figure (3.13) - The Julia set of $z + z + z^2$ and one petal.
Figure (7.13) - The Julia set of $z + z^2 + i$. 