THE YANG-BAXTER EQUATION FOR INTEGRABLE SYSTEMS

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Pu Fu-cho

and

D.H. Sattinger

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Pu Fu-cho[1], *International Centre for Theoretical Physics, Miramare, 34100, Trieste, Italy*

D.H. Sattinger [2], *School of Mathematics, University of Minnesota Minneapolis, Minnesota*

Abstract: The computation of Poisson brackets of the scattering data for classical integrable systems and the corresponding commutation relations for the quantized versions for both the Fermion and Boson case is given a unified treatment by systematic use of certain differential identities. A parallelism between the classical and quantum cases is developed. A derivation of the Yang-Baxter equation based on these identities is given; and it is shown that the Yang-Baxter equation is equivalent to a differential identity.

1. Introduction.

The calculation of the Poisson brackets for the scattering data for the KdV equation was first carried out by Faddeev and Zakharov [2]. The calculations depended on a set of differential identities and on the evaluation of certain limits in the sense of distributions. More recently, Fadeev and Tahktajan have calculated the Poisson brackets for the nonlinear Schrödinger equation in a formalism based on the so-called Yang-Baxter equation, which plays a fundamental role in solvable statistical models. There is a close relationship between the differential identities and the Yang-Baxter equation; and in fact, we present in this paper a simple derivation of the Yang-Baxter equation through a systematic use of some standard differential identities, similar to the generalized

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[1] Permanent address: Institute of Physics, Academica Sinica, Beijing, P.R. China

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Wronskian identities used by Calogero and Degasparis. These results hold not only for the classical case, but for the quantum case as well, both for Bosonic and Fermionic fields.

2. Poisson brackets for classical systems.

We consider a class of isospectral operators with skew-Hermitian potentials, namely

$$L = \frac{d}{dx} - U(x,\lambda),$$

where $U(x,\lambda) = \lambda J + Q(x)$, with $J^* = -J$ and $Q^* = -Q$. We consider at first the nxn case, where J and Q are nxn matrices. Later, we shall specialize to the case of the nonlinear Schrödinger equations, where $J = i\lambda\sigma_3/2$, σ_3 being the usual Pauli spin matrix, and $Q(x) = u(x)\sigma_+ - u^*\sigma_-$. Associated with L is the so-called "monodromy operator" $M(x,y,\lambda)$ which satisfies the forward and backward equations:

$$\frac{dM(x,y,\lambda)}{dx} = U(x,\lambda)M \qquad \frac{dM(x,y,\lambda)}{dy} = -M(x,y,\lambda)\, U(y,\lambda)$$

$$M(x,x,\lambda) = I$$

Note that the backward equation is satisfied by the inverse of any matrix solution of the forward equation. In the quantum case, the elements of the matrices M are operators on Fock space, and the notion of the inverse of an operator with operator valued entries is ambiguous. On the other hand, the backward equation makes sense in the quantum case as well.

The Jost functions are obtained by taking the limits

$$M^-(x,\lambda) = \lim_{y\to-\infty} M(x,y,\lambda)E(y,\lambda) \qquad M^+(y,\lambda) = \lim_{x\to\infty} E(-x,\lambda)M(x,y,\lambda)$$

where $E(x,\lambda) = \exp\{x\lambda J\}$. These functions satisfy the differential equations

$$\frac{dM^-(x,\lambda)}{dx} = U(x,\lambda)M^-(x,\lambda) \qquad \frac{dM^+(x,\lambda)}{dx} = -M^+(x,\lambda)U(x,\lambda)$$

together with the asymptotic behavior

$$I = \lim_{x\to -\infty} E(-x,\lambda)M^-(x,\lambda) = \lim_{x\to \infty} M^+(x,\lambda)E(x,\lambda).$$

The Jost scattering data is obtained by taking the limit

$$S(\lambda) = \lim_{x\to\infty} E(-x,\lambda)M^-(x,\lambda) = \lim_{x\to -\infty} M^+(x,\lambda)E(x,\lambda)$$

It is easily seen that $M^+(x,\lambda)M^-(x,\lambda)$ is independent of x. In fact, using the differential equations satisfied by these matrices, we get $(d/dx)M^+M^- = (dM^+/dx)M^- + M^+(dM^-/dx) = M^+(-U + U)M^- = 0$. Moreover, $M^+(x,\lambda)M^-(x,\lambda) = M^+(x,\lambda)E(x,\lambda)E(-x,\lambda)M^-(x,\lambda) \to S(\lambda)$ as $x \to \pm\infty$, so $S(\lambda) = M^+(x,\lambda)M^-(x,\lambda)$.

We next turn to the calculation of gradients of functionals. Let **H** be the Hilbert space of matrix valued functions with entries in $L_2(R)$ with inner product given by

$$(A,B) = \int_{-\infty}^{\infty} \mathrm{Tr}\, A(x)B(x)\, dx$$

Suppose F is a functional of $Q \varepsilon$ **H**. The gradient of F with respect to Q is defined by the relation

$$\frac{dF(Q(t))}{dt} = \int_{-\infty}^{\infty} \mathrm{Tr}\{ (\frac{\delta F}{\delta Q}) \frac{dQ}{dt} \}\, dx$$

where Q(t) is a one parameter family of potentials. For the nxn case, we may write $Q(x) = \Sigma_\alpha u_\alpha(x)e_\alpha$, where e_α are the root vectors for sl(n,C). Since $\mathrm{Tr}\, e_\alpha e_\beta{}^* = \delta_{\alpha\beta}$,

$$\frac{\delta F}{\delta Q} = \sum_{\alpha} \frac{\delta F}{\delta u_{\alpha}} e_{\alpha}{}^{*}$$

In particular, the *value* of the field Q at x can be regarded as a functional of Q, and it is useful, at least formally, to consider the gradient. For smooth potentials, say Q in Schwartz class, this leads to the relations

$$\frac{\delta u_{\alpha}(x)}{\delta Q}(z) = \delta(x - z) e_{\alpha}{}^{*}$$

hence

$$\frac{\delta u_{\alpha}(x)}{\delta u_{\beta}}(z) = \delta(x-z)\delta_{\alpha\beta} \qquad \frac{\delta u_{\alpha}(x)}{\delta u_{\beta}{}^{*}} = 0$$

Similarly, the monodromy operator $M(x,y,\lambda)$ can be regarded as a functional of Q. Let δM denote the variation in M due to a variation in Q. Taking the variation of the equation for M we get

$$\frac{d}{dx}\delta M = \delta Q\, M + Q\, \delta M$$

The following identity is fundamental to the computations we shall be making and is an immediate consequence of the differential equations for M and δM:

$$\frac{d}{dz} M(x,z,\lambda)\delta M(z,y,\lambda) = M(x,z,\lambda)\delta Q(z,\lambda)M(z,y,\lambda)$$

Integrating this identity over the interval $[y,x]$, where $y<x$, we obtain

$$\delta M(x,y,\lambda) = \int_{y}^{x} M(x,z,\lambda)\, \delta Q(z,\lambda) M(z,y,\lambda) dz \qquad (1.1)$$

Here we have used the initial condition $M(x,x,\lambda)=I$, and hence $\delta M(y,y,\lambda)=0$.

Using this identity we may compute the variations of M due to variations in u_α. We have

$$\frac{\delta M(x,y,\lambda)}{\delta u_\alpha}(z) = \int_y^x M(x,s,\lambda)\, \frac{\delta Q(s,\lambda)}{\delta u_\alpha}(z)\, M(s,y,\lambda)\, ds$$

$$= \int_y^x M(x,s,\lambda) e_\alpha \delta(s-z) M(s,y,\lambda) ds$$

$$= \begin{cases} 0 & z<y \text{ or } z>x \\ (1/2) e_\alpha M(x,y,\lambda) & z=x \\ M(x,z,\lambda) e_\alpha M(z,y,\lambda) & y < z < x \\ (1/2)\, M(x,y,\lambda) e_\alpha & z = y \end{cases} \qquad (1.2)$$

In evaluating the integral for z=x or z=y we assumed that the delta function is the distributional limit of a family of even functions, in which case

$$\int_0^\infty \delta(x) f(x) dx = \int_{-\infty}^0 \delta(x) f(x)\, dx = \frac{1}{2} f(0).$$

From the gradients of the monodromy matrix we can obtain the gradients of the Jost functions by multiplying by the appropriate exponential factor and taking limits. For example, multiplying the first relation in (1.2) on the right by $E(y,\lambda)$ and taking the limit as $y \to -\infty$, we get

$$\frac{\delta M^-(x,\lambda)}{\delta u_\alpha}(z) = 0 \qquad \text{if } x < z.$$

By similar arguments applied to the second and fourth relations we get

$$\frac{\delta M^-(x,\lambda)}{\delta u_\alpha}(x) = \frac{1}{2} e_\alpha M^-(x,\lambda) \qquad \frac{\delta M^+(y,\lambda)}{\delta u_\alpha}(y) = \frac{1}{2} M^+(y,\lambda) e_\alpha \tag{1.3}$$

From the relation $S(\lambda)=M^+M^-$, and the Leibnitz rule for gradients of functionals, we can immediately derive the gradients of S with respect to the u_α:

$$\frac{\delta S(\lambda)}{\delta u_\alpha}(x) = \frac{\delta}{\delta u_\alpha} M^+M^- = M^+\frac{\delta M^-}{\delta u_\alpha} + \frac{\delta M^+}{\delta u_\alpha} M^- = M^+\frac{e_\alpha}{2} M^- + M^+\frac{e_\alpha}{2} M^-$$

$$= M^+(x,\lambda) e_\alpha M^-(x,\lambda) \tag{1.4}$$

Using the relationship $u_\alpha^* = -u_{-\alpha}$, we get

$$\frac{\delta S}{\delta u_\alpha^*} = -M^+(x,\lambda) e_{-\alpha} M^-(x,\lambda) \tag{1.4'}$$

The gradients of $S(\lambda)$ can also be obtained by multiplying (1.1) on the left by $E(-x,\lambda)$ and on the right by $E(y,\lambda)$ and taking limits as $x \to \infty$ or as $y \to -\infty$. We obtain

$$\delta S(\lambda) = \int_{-\infty}^{\infty} M^+(z,\lambda)\delta Q(z,\lambda) M^-(z,\lambda)\, dz \tag{1.5}$$

We now turn to the computation of the Poisson brackets of these functionals. The Poisson brackets of two functionals of Q are given by

$$\{F, G\} = \int_{-\infty}^{\infty} \mathrm{Tr}\, [J, \frac{\delta F}{\delta Q}] \frac{\delta G}{\delta Q}\, dx$$

These Poisson brackets are skew symmetric and satisfy the Jacobi identity [ ]. For the general sl(n) case, let $[J, e_\alpha] = \alpha(J)e_\alpha$, where α are the roots of J and e_α are the root vectors. Then the Poisson brackets work out to

$$\{F,G\} = \int_{-\infty}^{\infty} \mathrm{Tr}\,[J, \sum_\alpha \frac{\delta F}{\delta u_\alpha} e_\alpha{}^*] \sum_\beta \frac{\delta G}{\delta u_\beta} e_\beta{}^*\, dx$$

$$= \sum_{\alpha,\beta} \mathrm{Tr}\,[J, e_\alpha{}^*] e_\beta{}^* \int_{-\infty}^{\infty} \frac{\delta F}{\delta u_\alpha} \frac{\delta G}{\delta u_\beta}\, dx = \sum_\alpha \alpha(J) \int_{-\infty}^{\infty} \frac{\delta F}{\delta u_\alpha} \frac{\delta G}{\delta u_\alpha{}^*}\, dx$$

$$= \sum_{\alpha>0} \alpha(J) \int_{-\infty}^{\infty} \frac{\delta F}{\delta u_\alpha} \frac{\delta G}{\delta u_\alpha{}^*} - \frac{\delta F}{\delta u_\alpha{}^*} \frac{\delta G}{\delta u_\alpha}\, dx$$

In this calculation we used the facts that $[J, e_\alpha{}^*] = -\alpha(J)\, e_\alpha{}^*$, and $u_{-\alpha} = -u_\alpha{}^*$; note that the last sum goes only over the positive roots. For the 2x2 case, $J=(i/2)\sigma_3$, $e_\alpha = \sigma_+$, and $e_\alpha{}^* = \sigma_-$; and the one positive root α is $\alpha(i\sigma_3) = i$, so the Poisson brackets for sl(2,C) are

$$\{F,G\} = i \int_{-\infty}^{\infty} \frac{\delta F}{\delta u} \frac{\delta G}{\delta u^*} - \frac{\delta F}{\delta u^*} \frac{\delta G}{\delta u}\, dx$$

From the gradients of the functionals $u_\alpha(x)$ we easily compute the Poisson brackets of a general functional with $u_\alpha(x)$:

$$\{u_\alpha(x), F\} = \sum_{\gamma>0} \gamma(J) \int_{-\infty}^{\infty} \frac{\delta u_\alpha(x)}{\delta u_\gamma} \frac{\delta F}{\delta u_\gamma{}^*} - \frac{\delta u_\alpha}{\delta u_\gamma{}^*} \frac{\delta F}{\delta u_\gamma}\, dz$$

We get

$$\{u_\alpha(x), F\} = \alpha(J)\,\frac{\delta F}{\delta u_\alpha{}^*}\,(x) \qquad \{u_\alpha{}^*(x)\,,F\} = -\alpha(J)\,\frac{\delta F}{\delta u_\alpha}\,(x) \tag{1.6}$$

In particular, taking F to be the functional $u_\beta{}^*(y)$, we get , for α and β positive:

$$\{u_\alpha(x)\,,u_\beta{}^*(y)\,\} = \alpha(J)\,\frac{\delta u_\beta{}^*(y)}{\delta u_\alpha{}^*}\,(x) = \alpha(J)\,\delta_{\alpha\beta}\delta(x-y) \tag{1.7}$$

For the 2x2 case, $J=i\sigma_3/2$, $Q(x) = i\sqrt{c}(u(x)\,\sigma_+-u^*(x)\sigma_-)$, and

$\{u(x)\,,u^*(y)\,\} = i\,\delta(x-y)$.

Writing the gradients in terms of Poisson brackets is useful in comparing the classical with the quantum case. In most cases, the commutators of the operators in the quantum case are identical to the Poisson brackets for the classical case; but in certain cases there are fundamental "quantum corrections" which arise due to non-commutivity. From (1.1) and (1.3) we get

Theorem 1.1 *The Poisson brackets of the monodromy matrix with the field variables for the classical case are given by*

$$\{\,M(x,y,\lambda)\,,u_\alpha{}^*(z)\,\} = \begin{cases} 0 & x<z \\ (\alpha(J)/2)e_\alpha M(x,y,\lambda) & z=x \\ \alpha(J)\,M(x,z,\lambda e_\alpha M(z,y,\lambda) & y<z<x \\ (\alpha(J)/2)M(x,y,\lambda)\,e_\alpha & z=y. \end{cases} \tag{1.8}$$

From (1.4) and (1.6) we get

$$\{u_\alpha{}^*(x)\,,S(\lambda)\,\} = -\alpha(J)\,M^+(x,\lambda)e_\alpha M^-(x,\lambda)$$

or

$$\{u_\alpha(x)\,,S(\lambda)\,\} = -\alpha(J)\,M^+(x,\lambda)e_{-\alpha}M^-(x,\lambda)$$

Next we wish to calculate the Poisson brackets between the entries of the monodromy matrices $M(x,y,\lambda)$ and $M(x,y,\mu)$, regarded as functionals of the field variables. In the process we shall solve the Yang-Baxter equation. This may be done in a compact way by using the tensor product of matrices. Let the matrices A and B act on a vector space V. Their tensor product operating on $V\otimes V$ is defined by $(A\otimes B)(\xi\otimes\eta) = A\xi\otimes B\eta$. If $\{e_j\}$ form a basis for V then a basis for $V\otimes V$ is given by $\{e_j \otimes e_k\}$. Relative to this basis the entries of $A\otimes B$ are $(A\otimes B)_{jk,lm} = A_{jl}B_{km}$. Given nxn matrices of functionals A and B we define $\{A\otimes,B\}_{jk,lm} = \{A_{jl}, B_{km}\}$. Then the Poisson brackets $\{M(x,y,\lambda)\otimes, M(x,y,\mu)\}$ are given by

$$\{M(x,y,\lambda)\otimes, M(x,y,\mu)\} = \int_y^x \sum_{\alpha>0} \alpha(J)\Big\{\frac{\delta M(x,y,\lambda)}{\delta u_\alpha} \otimes \frac{\delta M(x,y,\mu)}{\delta u_\alpha^*} - \frac{\delta M(x,y,\lambda)}{\delta u_\alpha^*} \otimes \frac{\delta M(x,y,\mu)}{\delta u_\alpha}\Big\}\,dz$$

Using the relations (1.2) we get

$$= \int_y^x \sum_{\alpha>0} \alpha(J)\{M(x,z,\lambda)e_\alpha M(z,y,\lambda)\otimes M(x,z,\mu)e_\alpha^* M(z,y,\mu) - M(x,z,\lambda)e_\alpha^* M(z,y,\lambda)\otimes M(x,z,\mu)e_\alpha M(z,y,\mu)\}dz$$

This integral can be evaluated explicitly by using the following identity:

Theorem 1.2

$$\frac{1}{\lambda-\mu}\frac{d}{dz}(M(x,z,\lambda)\otimes M(x,z,\mu)\, P\, M(z,y,\lambda)\otimes M(z,y,\mu)$$

$$= -\sum_\alpha \alpha(J)\,[M(x,z,\lambda)e_\alpha M(z,y,\lambda)]\otimes[M(x,z,\mu)e_{-\alpha}M(z,y,\mu)]$$

Here P is the permutation matrix acting on $V\otimes V$ defined by $\xi\otimes\eta P = \eta\otimes\xi$. P acts on matrices by

$PA\otimes B = B\otimes AP$. P is given by

$$P = \sum_{jk} e_{jk} \otimes e_{kj}$$

where e_{jk} are the unit matrices: $(e_{jk})_{lm} = \delta_{jl}\delta_{km}$.

Proof: Note that the sum is over *all* roots. The proof is simply a direct computation. Carrying out the differentiation on the left we obtain

$$\frac{1}{\lambda - \mu} [M(x,z,\lambda)\otimes M(x,z,\mu)] [(-U(z,\lambda)\otimes 1 - 1\otimes U(z,\mu))P$$

$$+P(U(z,\lambda)\otimes 1 + 1\otimes U(z,\mu)] [M(z,y,\lambda) \otimes M(z,y,\mu)]$$

Since $U(z,\lambda) = \lambda J + Q(z)$, the matrix in the center is

$$-(+\lambda J\otimes 1 + \mu 1\otimes J)P + P(\lambda J\otimes 1 + \mu 1\otimes J)$$

$$(-Q(z)\otimes 1 - 1\otimes Q(z))\, P + P\, (Q(z)\otimes 1 + 1\otimes Q(z))$$

$$=(\lambda - \mu)[1\otimes J , P].$$

Now

$$[1\otimes J,P] = \sum_{jk} [1\otimes J , e_{jk}\otimes e_{kl}] = \sum_{jk} e_{jk}\otimes [J, e_{kj}]$$

$$= -\sum_{\alpha} \alpha(J)\, e_{\alpha} \otimes e_{-\alpha}$$

Note here that $[J,e_{jk}] = (\lambda_j - \lambda_k)e_{jk}$, where λ_j are the eigenvalues of J; the roots are $\alpha_{jk}(J) = (\lambda_j - \lambda_k)$. Using this, the derivative becomes

$$-\sum_{\alpha}\alpha(J)\,[M(x,z,\lambda)e_{\alpha}M(z,y,\lambda)]\otimes[M(x,z,\mu)\,e_{-\alpha}M(z,y,\mu)]$$

Using Theorem 1 we can evaluate the Poisson brackets $\{M(x,y,\lambda)\otimes,M(x,y,\mu)\}$ directly:

$$\{M(x,y,\lambda)\otimes,M(x,y,\mu)\} = -\frac{1}{\lambda-\mu}\int_y^x \frac{d}{dz}\,M(x,z,\lambda)\otimes M(x,z,\mu)\;P\;M(z,y,\lambda)\otimes M(z,y,\mu)\,dz$$

$$= [\,M(x,y,\lambda)\otimes M(x,y,\mu)\,,\,R(\lambda-\mu)]$$

where

$$R(\lambda) = \frac{P}{\lambda}$$

and P is the permutation matrix described above.

The above relation is the Yang-Baxter equation for the monodromy matrix. To find the Poisson brackets for the entries of the scattering matrix $S(\lambda)$ we need to take limits as $x\to\infty$ and $y\to-\infty$. Let $E(x,\lambda,\mu) = E(x,\lambda)\otimes E(x,\mu)$. Now

$$\lim_{x\to\infty,\,y\to-\infty} E(-x,\lambda,\mu)\,M(x,y,\lambda)\otimes M(x,y,\mu)\,E(y,\lambda,\mu)$$

$$= \lim_{x\to\infty,\,y\to-\infty} E(-x,\lambda)M(x,y,\lambda)E(y,\lambda)\otimes E(-x,\mu)M(x,y,\mu)E(y,\mu)$$

$$= S(\lambda)\otimes S(\mu).$$

Since $E(x,\lambda,\mu)$ is a constant functional, independent of $Q(x)$, it commutes through the Poisson brackets; hence

$$E(-x,\lambda,\mu)\{M(x,y.\lambda)\otimes,M(x,y,\mu)\}E(y,\lambda,\mu) =$$
$$\{\,E(-x,\lambda,\mu)M(x,y.\lambda)\otimes,M(x,y,\mu)\,E(y,\lambda,\mu)\}.$$

Multiplying ( ) on the left by $E(-x,\lambda,\mu)$ and on the right by $E(y,\lambda,\mu)$ and taking limits as $x \to \infty$ and $y \to -\infty$, we get

$$\{S(\lambda) \otimes, S(\mu)\} = S(\lambda)\otimes, S(\mu)\, R_+(\lambda-\mu) \;-R_-(\lambda-\mu)\; S(\lambda)\otimes, S(\mu),$$

where

$$R_\pm(\lambda-\mu) = \lim_{x\to\pm\infty} E(x,\lambda,\mu)R(\lambda-\mu)E(-x,\lambda,\mu).$$

To evaluate these limits we use the representation for the permutation matrix P given by

$$P=\sum_{jk} e_{jk}\otimes e_{kj} = E_0 + \sum_\alpha e_\alpha \otimes e_{-\alpha}$$

where

$$E_0=\sum_{j=1}^{n} e_{jj}\otimes e_{jj}$$

and α denotes the roots of sl(n,C). We have

$$E(x,\lambda,\mu)R(\lambda-\mu)E(-x,\lambda,\mu) = \frac{E_0}{\lambda-\mu} + \sum_\alpha \frac{[E(x,\lambda)e_\alpha E(-x,\lambda)]\otimes[E(x,\mu)e_{-\alpha}E(-x,\mu)]}{\lambda-\mu}$$

$$= \frac{E_0}{\lambda-\mu} + \sum_\alpha \frac{e^{\alpha\lambda x}e_\alpha \otimes e^{-\alpha\mu x}e_{-\alpha}}{\lambda-\mu}$$

$$= \frac{E_0}{\lambda-\mu} + \sum_\alpha \frac{e^{\alpha(\lambda-\mu)x}}{\lambda-\mu}\, e_\alpha \otimes e_{-\alpha}$$

To evaluate the limits as $x \to \pm\infty$, the following generalized limits (in the sense of distributions) a used:

$$\lim_{x\to\pm\infty} P\frac{e^{i\lambda x}}{\lambda} = \pm\pi\, i\, \delta(\lambda)$$

where P here is the Cauchy principal value of the distribution $1/\lambda$. Recall that the roots of sl(n,C) acting on the matrix $J = \text{diag}\ i\lambda_1, \ldots i\lambda_n)$ are purely imaginary: $\alpha_{jk}(J)=i(\lambda_j-\lambda_k)$. Hence

$$R_{\pm}(\lambda-\mu) = \frac{E_0}{\lambda-\mu} \pm \pi i \sum_{\alpha} e_{\alpha} \otimes e_{-\alpha}\ \delta(\lambda-\mu)\alpha(J)$$

$$= \frac{E_0}{\lambda-\mu} + \pi \sum_{j<k} e_{jk} \otimes e_{kj} \delta(\lambda-\mu)(\lambda_j-\lambda_k)$$

3. The Boson case.

In §2 we showed that the Yang-Baxter equations for the monodromy matrix and the scattering matrix can be obtained directly from the differential equation for the wave functions, by use of certain differential identities. This is also true for the quantum case. We shall demonstrate this for a class of systems that includes the nonlinear Schrödinger equation for bosons. The Hamiltonian is

$$H = \int_{-\infty}^{\infty} \sum_{j=1}^{n} \frac{\partial u_j^*}{\partial x} \frac{\partial u_j}{\partial x} + \sum_{j,k=1}^{n} c : u_j^* u_j u_k^* u_k : dx \tag{3.1}$$

where c>0 for the repulsive case. The notation : A : denotes that the operator A is to be written in "normal order"; this means that the destruction operators u_j are to appear all the way to the right of the expression. For example, $u_j^* u_k$ is in normal order, while $u_k u_j^*$ is not. The reasons for writing expressions in normal order are somewhat subtle. In particular, the order in which the operators appear in an expression can have an effect on the asymptotic behavior as $x \to \pm\infty$. It is assumed that u_j and u_k^* act on Fock space and satisfy the commutation relations (compare with the corresponding Poisson brackets in the classical case.)

$$[u_j(x), u_k^*(y)] = \delta_{jk}\, \delta(x-y), \quad [u_j(x), u_k(y)] = [u_j^*(x), u_k^*(y)] = 0.$$

The monodromy matrix $M(x,y,\lambda)$ satifies the forward and backward differential equations

$$\frac{\partial}{\partial x} M(x,y,\lambda) = :U(x,\lambda)M(x,y,\lambda): \qquad M(x,x,\lambda) = 1$$

$$\frac{\partial M(x,y,\lambda)}{\partial y} = -: M(x,y,\lambda)U(y,\lambda):$$

where $U(x,\lambda)$ is the linear operator on Fock space associated with the given Hamiltonian. For the class of systems considered here,

$$U(x,\lambda) = \lambda J + i\sqrt{c} \sum_{j=1}^{n} (u_j(x)e_{j\,n+1} - u_j^*(x)e_{n+1\,j})$$

where J is the $(n+1)\times(n+1)$ diagonal matrix $J = \text{diag}(i/2, i/2, \ldots -i/2)$, and $e_{j\,n+1}$ denotes the $(n+1)\times(n+1)$ matrix with a 1 in the j^{th} row and $(n+1)^{st}$ column, etc.

Written out explicitly, the forward and backward equations in normal order are

$$\frac{\partial M(x,y,\lambda)}{\partial x} = (\lambda J - i\sqrt{c} \sum_{j=1}^{n} e_{n+1\,j} u_j^*)M(x,y,\lambda) + i\sqrt{c} \sum_{j=1}^{n} e_{j\,n+1} M(x,y,\lambda) u_j$$

$$\frac{\partial M(x,y,\lambda)}{\partial y} = -M(x,y,\lambda)\,(\lambda J + i\sqrt{c} \sum_{j=1}^{n} e_{j\,n+1}\, u_j) + i\sqrt{c} \sum_{j=1}^{n} u_j^* M(x,y,\lambda) e_{n+1\,j}$$

We first derive the quantum analog of the classical relations (1.4). We prove :

Theorem 3.1 *The following commutators hold for the boson case ($y<x$):*

$$[u_j(z), M(x,y,\lambda)] = [u_j^*(z), M(x,y,\lambda)] = 0 \qquad y<x<z$$

$$[u_j(x), M(x,y,\lambda)] = -i\,(\sqrt{c}/2) e_{n+1\,j} M(x,y,\lambda)$$

$$[u_j(y)\,,M(x,y,\lambda)\,]=-i\,(\sqrt{c}/2)M(x,y,\lambda)\,e_{n+1\,j}$$

$$[u_j(z)\,,M(x,y,\lambda)\,]=-i\,\sqrt{c}\,M(x,z,\lambda)\,e_{n+1\,j}M(x,y,\lambda) \qquad y<z<x$$

$$[u_j^*(x)\,,M(x,y,\lambda)\,]=-i\;(\sqrt{c}/2)e_{j\,n+1}M(x,y,\lambda)$$

$$[u_j^*(y)\,,M(x,y,\lambda)\,]=-i\;(\sqrt{c}/2)M(x,y,\lambda)\,e_{j\,n+1}$$

$$[u_j^*(z)\,,M(x,y,\lambda)\,]=-i\;\sqrt{c}\,M(x,z,\lambda)\,e_{j\,n+1}M(z,y,\lambda) \qquad y<z<x$$

Proof: $M(z',y,\lambda)$ depends only on $u_j(s)$ and $u_j^*(s)$ for $y\leq s<z'$ and is therefore independent of $u_j(x)$ and $u_j^*(x)$ for $z'<x$. Since $u_j(x)$ and $u_j^*(x)$ commute with $u_j(s)$ and $u_j^*(s)$ for $s,s'<x$, they commute with $M(z',y,\lambda)$ for $y\leq z'<x$.This establishes the first commutation relation.

To prove the second, we transform the forward equation for $M(x,y,\lambda)$ to an integral equation:

$$M(x,y,\lambda) = 1+\int_y^x \lambda JM(z',y,\lambda) - i\sqrt{c}\sum_{j>0} e_{n+1\,j}u_j^*(z')M(z',y,\lambda) + e_{j\,n+1}\,M(z',y,\lambda)u_j(z')\;dz'$$

Multiply this equation on the left by $u_j(x)$. The first integral term on the right side is

$$\int_y^x u_j(x)\lambda JM(z',y,\lambda)\;dz'$$

We have already seen that $[u_j(x)\,,M(z',y,\lambda)\,] = 0$ for $y<z'<x$; the two operators do not commute at $z'=x$, but this commutation breaks down only at the endpoint of the integration and does not contribute to the integral. Implicit in this argument is the assumption that the commutator does not contain a delta function. We do not know of any argument that allows one to establish this assumption *a priori*; and so we must take it as an additional assumption. On this basis, $u_j(x)$ commutes with the first integral term in $M(x,y,\lambda)$.

Now go to the second term in the integral. Here we use the commutation relation $[u_j(x)\,,u_k^*(z')] = \delta_{jk}\delta(x-z')$. Therefore, when $u_j(x)$ is commuted to the right, the additional term results

$$-i\sqrt{c}\int_y^x \sum_{k=1}^{n} e_{n+1\,k}\delta_{kj}\delta(x-z')M(z',y,\lambda)\,dz' = -\frac{i\sqrt{c}}{2}\,e_{n+1\,j}M(x,y,\lambda)$$

The factor 1/2 arises, as explained in §2, because the delta function has its support at the endpoint of the interval of integration. Since $u_j(x)$ commutes past $u_k(z')$, and past $M(z'.y,\lambda)$ except at $z'=x$, it therefore commutes past all the other terms in the integral, and the second identity is established for $u_j(x)$.

The third relation in Theorem 3.1 is proved by converting the backward equation to an integral equation and arguing similarly. We remark that the integral equation from the forward equation depends on $M(z',y,\lambda)$, which in turn depends on $u_j(y)$ for all $y\leq z'\leq x$; hence it cannot be used to evaluate $[u_j(y), M(x,y,\lambda)]$.

The fourth relation in Theorem 3.1 follows immediately from the second and third. In fact,

$$[u_j(z), M(x,y,\lambda)] = [u_j(z), M(x,z,\lambda)M(z,y,\lambda)]$$

$$= M(x,z,\lambda)\,[u_j(z), M(z,y,\lambda)] + [u_j(z), M(x,z,\lambda)]\,M(z,y,\lambda)$$

$$= -i\ (\sqrt{c}/2)\,M(x,z,\lambda)e_{n+1\,j}M(z,y,\lambda) - i\ (\sqrt{c}/2)M(x,z,\lambda)e_{n+1\,j}M(z,y,\lambda)$$

$$= -i\ \sqrt{c}\,M(x,z,\lambda)e_{n+1\,j}M(z,y,\lambda).$$

The commutators with $u_j^*(x)$ are derived in a similar manner.

Using the second and third commutation relations in (3.1) we can rewrite the forward and backward equations in the forms

$$\frac{\partial M(x,y,\lambda)}{\partial x} = U(x,\lambda)M(x,y,\lambda) - \theta M(x,y,\lambda)$$

$$\frac{\partial M(x,y,\lambda)}{\partial y} = -M(x,y,\lambda)(U(y,\lambda) - \theta) \qquad (3.2)$$

where

$$\theta = \frac{c}{2}\sum_{j=1}^{n} e_{j\,n+1}\, e_{n+1\,j} = \frac{c}{2}\,\mathrm{diag}\,(1,1,\ldots 0)$$

The constant matrix θ is a quantum correction due to non-commutivity of the field variables. We shall say that equations (3.2) are in *standard order.*

The quantum analog of (1.1) is

Theorem 3.2. *The following identity holds for any quantum system that can be written in standard order, such as (3.2), with θ independent of x:*

$$[M(x,y,\lambda), W] = \int_y^x M(x,z,\lambda)\,[U(z,\lambda)\,,\,W\,]\,M(z,y,\lambda)\,dz \tag{3.3}$$

where W is an operator on Fock space.

Proof: The result follows immediately by integrating the following differential identity over the interval [y,x]:

$$\frac{d}{dz}M(x,z,\lambda)WM(z,y,\lambda) = -M(x,z,\lambda)\,[\,U(z,\lambda)\,,\,W]\,M(z,y,\lambda) \tag{3.4}$$

The differential identity (3.4) follows immediately from the standard forms (3.2) forms of the forward and backward equations.

We now turn to the derivation of the Yang–Baxter equation for the model discussed above. In general, a Yang-Baxter equation is an identity of the form

$$R(\lambda,\mu)\,M(x,y,\lambda) \otimes M(x,y,\mu) = M(x,y,\mu) \otimes M(x,y,\lambda)\,R(\lambda,\mu)$$

where R is a matrix on the tensor product space. We first prove the following result:

Theorem 3.3. *The Yang Baxter equation is equivalent to the differential identity*

$$\frac{d}{dz}M(x,z,\mu) \otimes M(x,z,\lambda)\ R(\lambda,\mu)\ M(z,y,\lambda) \otimes M(z,y,\mu) = 0\ .$$

Proof: Integrating this identity over the interval $y \le z \le x$, we get the Yang Baxter equation. Conversely, if the the Yang-Baxter equation holds for all x and y, then it certainly holds for y=z. Multiplying the Yang-Baxter equation for x,z on the right by $M(z,y,\lambda)\otimes M(z,y,\mu)$, we get

$$R(\lambda,\mu)\ [M(x,z,\lambda) \otimes M(x,z,\mu)]\ [\ M(z,y,\lambda)\otimes M(z,y,\mu)\ ]$$

$$= M(x,z,\mu) \otimes M(x,z,\lambda)\ R(\lambda,\mu)\ M(z,y,\lambda)\otimes M(z,y,\mu)$$

Since $M(x,z,\mu)$ and $M(z,y,\lambda)$ depend on field operators on non-overlapping intervals, and since the field operators commute on non-overlapping intervals, $[\ M(x,z,\mu)\ ,\ M(z,y,\lambda)]=0$, and $R(\lambda,\mu)$ $[M(x,z,\lambda) \otimes M(x,z,\mu)]\ [\ M(z,y,\lambda)\otimes M(z,y,\mu)\ ] = R(\lambda,\mu)\ M(x,y,\lambda) \otimes M(x,y,\mu)$. Therefore the left side above is independent of z, and differentiation of this identity proves the other half of Theorem 3.3.

Theorem 3.4 *For the generalized nonlinear Schrödinger model, the R matrix in the Yang-Baxter equation is*

$$R(\lambda,\mu) = \frac{-ic}{\lambda-\mu-ic}\,1 + \frac{\lambda-\mu}{\lambda-\mu-ic}\,P$$

where P is the permutation operator.

In order to prove this result we need to determine the forward and backward differential equations satisfied by the tensor product $M(x,y,\lambda) \otimes M(x,y,\mu)$. Let's consider the forward equation:

$$\frac{d}{dz}M(z,y,\lambda)\otimes M(z,y,\mu) = (U(z,\lambda) - \theta)M(z,y,\lambda)\otimes M(z,y,\mu)$$

$$+M(z,y,\lambda)\otimes(U(z,\mu)-\theta)M(z,y,\mu)$$

Using the commutation relations for $[u_j(z),M(z,y,\lambda)]$ and $u_j^*(z)\ ,\ M(z,y,\lambda)]$ from Theorem 3.1, the second term can be written

$$M(z,y,\lambda)\otimes(zJ + i\sqrt{c}\sum_{j=1}^{n} u_j(z)e_{j\,n+1} - u_j^*(z)e_{j\,n+1} - \theta)M(z,y,\mu)$$

$$=(\mu\, 1\otimes J - 1\otimes\theta)M(z,y,\lambda)\otimes M(z,y,\mu)$$

$$+ i\sqrt{c}\sum_{j=1}^{n} M(z,y,\lambda)u_j(z)\otimes e_{j\,n+1}\, M(z,y,\mu) - M(z,y,\lambda)u_j^*(z)\otimes e_{n+1\,j} M(z,y,\mu)$$

$$= (\mu\, 1\otimes J - 1\otimes\theta + i\sqrt{c}\sum_{j=1}^{n} u_j(z)(1\otimes e_{j\,n+1})(M(z,y,\lambda)\otimes M(z,y,\mu))$$

$$- u_j^*(z)(1\otimes e_{n+1\,j})(M(z,y,\lambda)\otimes M(z,y,\mu)$$

$$-\frac{c}{2}\sum_{j=1}^{n}(e_{n+1\,j}\otimes e_{jn+1} - e_{j\,n+1}\otimes e_{n+1\,j})M(z,y,\lambda)\otimes M(z,y,\mu)$$

$$= (\, 1\otimes U(\mu) - 1\otimes\theta - \frac{c}{2}\sum_{j=1}^{n} (e_{n+1\,j}\otimes e_{j\,n+1} - e_{j\,n+1}\otimes e_{n+1\,j})\,)M(z,y,\lambda)\otimes M(z,y,\mu)$$

Hence the forward equation in standard form is

$$\frac{d}{dz}M(z,y,\lambda)\otimes M(z,y,\mu) = \Gamma(z,\lambda,\mu)M(z,y,\lambda)\otimes M(z,y,\mu)$$

where

$$\Gamma(z,\lambda,\mu) = U(z,\lambda)\otimes 1 + 1\otimes U(z,\mu) - \theta\otimes 1 - 1\otimes\theta - \frac{c}{2}\sum_{j=1}^{n} e_{n+1\,j}\otimes e_{j\,n+1} - e_{j\,n+1}\otimes e_{n+1\,j}$$

By similar arguments the standard form of the backward equation is found to be

$$\frac{d}{dz}M(x,z,\lambda)\otimes M(x,z,\mu) = -\,M(x,z,\lambda)\otimes M(x,z,\mu)\,\Gamma(z,\lambda,\mu)$$

Theorem 3.5. *For any quantum model for which the forward and backward differential equations for the tensor products of the monodromy matrices can be written in standard form, the Yang-Baxter equation is equivalent to the commutator relation*

$$R(\lambda,\mu,)\,\Gamma(z,\lambda,\mu) - \Gamma(z,\mu,\lambda)R(\lambda,\mu) = 0$$

for the field operators.

Proof: The result follows immediately from the identity

$$\frac{d}{dz} M(x,z,\mu)\otimes M(x,z,\lambda)R(\lambda,\mu)M(z,y,\lambda)\otimes M(z,y,\mu)$$

$$= - M(x,z,\mu)\otimes M(x,z,\lambda)\,[\,-\Gamma(z,\mu,\lambda)R(\lambda,\mu) + R(\lambda,\mu)\,\Gamma(z,\lambda,\mu)]M(z,y,\lambda)\otimes M(z,y,\mu).$$

Now let us go to the proof of Theorem 3.4. We find

$$\Gamma(z,\lambda,\mu) - \Gamma(z,\mu,\lambda) = (\lambda - \mu)(J\otimes 1 - 1\otimes J),$$

$$P\Gamma(z,\lambda,\mu) - \Gamma(z,\mu,\lambda)P = -\frac{c}{2}\,[\,P\,, \sum_{j=1}^{n} e_{n+1\ j}\otimes e_{j\ n+1} - e_{j\ n+1}\otimes e_{n+1\ j}\,]$$

$$= c\,P\sum_{j=1}^{n} e_{j\ n+1}\otimes e_{n+1\ j} - e_{n+1\ j}\otimes e_{j\ n+1}$$

so we need to establish the identity

$$i(J\otimes 1 - 1\otimes J) = P\sum_{j=1}^{n} e_{j,n+1}\otimes e_{n+1,j} - e_{n+1,j}\otimes e_{j,n+1}$$

Using $P = \Sigma_{rs} e_{rs}\otimes e_{sr}$, we have

$$P\sum_{j=1}^{n} e_{j,n+1}\otimes e_{n+1,j} - e_{n+1,j}\otimes e_{j,n+1} = \sum_{r,s=1}^{n+1}\sum_{j=1}^{n} e_{rs}e_{j,n+1}\otimes e_{sr}e_{n+1,j} - e_{rs}e_{n+1,j}\otimes e_{sr}e_{j,n+1}$$

$$= \sum_{j,r,s} \delta_{sj} e_{r,n+1} \otimes e_{sj} \delta_{r,n+1} - e_{rj} \delta_{s,n+1} \otimes e_{s,n+1} \delta_{rj}$$

$$= \sum_{j=1}^{n} e_{n+1,n+1} \otimes e_{jj} - e_{jj} \otimes e_{n+1,n+1} = e_{n+1,n+1} \otimes 1 - 1 \otimes e_{n+1,n+1}$$

$$= (1/2 + iJ) \otimes (1/2 - iJ) - (1/2 - iJ) \otimes (1/2 + iJ)$$
$$= i\,(J \otimes 1 - 1 \otimes J).$$

4. Fermion Fields.

In this section we discuss the modifications of the preceeding section to take care of Fermion fields. We consider the Hamiltonian (3.1) for two Fermion fields $u_j(x)$, $j = 1,2$, which satisfy the anticommutation relations

$$\{u_j(x)\,,\, u_k{}^*(y)\} = \delta_{jk}\delta(x - y\,), \qquad \{u_j(x)\,,\, u_k(y)\} = \{u_j{}^*(x)\,,\, u_k{}^*(y)\} = 0,$$

where $\{A,B\} = AB+BA$. For the fermion version of the nonlinear Schrödinger equation, $U(x,\lambda)$ is the same as in the boson case. We define the supersymmetric tensor product $\otimes_s$ by

$$(A\otimes_s B)_{hj,kl} = A_{hk}B_{jl}(-1)^{p(j)(p(h)+p(k))}$$

where $p(j)$ is the parity of the j^{th} row or column of the matrix A. For fermion fields, the parity of an operator is even or odd according as the number of field operators is even or odd. Thus, $u_j(x)u_k{}^*(x)$ is even, while $u_j{}^*(x)u_k(x)u_l(x)$ is odd. When the monodromy operator is developed in a formal Neumann series expansion in the field operators, it is found [ ] to have the parities indicated in the diagram below, 0 denoting even and 1 denoting odd parity:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Thus $p(j) = 0$ for $j=1,2$ and $p(3)=1$.

The corresponding supersymmetric permutation matrix P_s is defined by

$$P_s = \sum_{j,k} e_{jk} \otimes e_{kj} \, (-1)^{p(j)p(k)} \tag{4.1}$$

As in the boson case, P_s has the property that $P_s(A \otimes_s B) = (B \otimes_s A)P_s$ for supermatrices A and B with entries of the appropriate parities.

Define

$$[M(x,y,\lambda) \,{}_{,}\!\otimes_s M(x,y,\mu)\,]_{ij,kl} = M(x,y,\lambda)_{ik} M(x,y,\mu)_{jl} (-1)^{p(j)(p(i)+p(k))}$$

$$- M(x,y,\mu)_{jl} M(x,y,\lambda)_{ik} (-1)^{p(l)(p(i)+p(k))}.$$

It is easy to see that

$$[M(x,y,\lambda) \,{}_{,}\!\otimes_s M(x,y,\mu)\,] = M(x,y,\lambda)M(x,y,\mu) - P_s\, M(x,y,\mu)M(x,y,\lambda)\, P_s.$$

The Yang-Baxter equation for the fermion case is

$$R(\lambda,\mu)\, M(x,y,\lambda) \otimes_s M(x,y,\mu) = M(x,y,\mu) \otimes_s M(x,y,\lambda)\, R(\lambda,\mu)$$

where now

$$R(\lambda,\mu) = \frac{ic}{\lambda - \mu} + P_s$$

As in the boson case, it is equivalent to the differential identity:

$$\frac{d}{dz} M(x,z,\mu) \otimes_s M(x,z,\lambda)\; R(\lambda,\mu) M(z,y,\lambda) \otimes_s M(z,y,\mu) = 0$$

To prove this differential identity we use

$$\frac{\partial}{\partial x} M(x,y,\lambda) \otimes_s M(x,y,\mu) = :U(x,\lambda) \otimes_s 1 + 1 \otimes_s U(x,\mu) + c\, e_{j3} \otimes_s e_{3j}) M(x,y,\lambda) \otimes_s M(x,y,\mu):$$

and

$$\frac{\partial}{\partial y} M(x,y,\lambda) \otimes_s M(x,y,\mu) = - : M(x,y,\lambda) \otimes_s M(x,y,\mu)\ (U(y,\lambda)\otimes_s 1 + 1\otimes_s U(y,\mu)$$

$$+c e_{j3}\otimes_s e_{3j}) :$$

Then

$$\frac{\partial}{\partial z} M(x,z,\mu)\otimes_s M(x,z,\lambda) P_s M(z,y,\lambda)\otimes_s M(z,y,\mu) =$$

$$c\ M(x,z,\mu)\otimes_s M(x,z,\lambda)\ P_s(-P_s e_{j3}\otimes_s e_{3j} P_s + e_{j3}\otimes_s e_{3l}) M(z,y,\lambda)\otimes_s M(z,y,\mu)$$

Now $-e_{j3}\otimes_s e_{3j} P_s + P_s e_{j3}\otimes_s e_{3j} = i\ (1\otimes J - J\otimes 1)$, so the expression above reduces to

$$ic\ M(x,z,\mu)\otimes_s M(x,z,\lambda)[1\otimes J - J\otimes 1] M(z,y,\lambda)\otimes_s M(z,y,\mu)$$

$$= ic\ [\ M(x,z,\mu)\ M(z,y,\lambda)\ \otimes_s M(x,z,\lambda)\ J\ M(z,y,\mu) -$$

$$M(x,z,\mu)\ J\ M(z,y,\lambda)\ \otimes_s M(x,z,\lambda)\ \otimes_s M(z,y,\mu)]$$

In the last step we used the fact that $M(x,z,\lambda)$ and $M(z,y,\lambda)$ commute since they are composed of field operators over non-overlapping intervals. On the other hand,

$$\frac{d}{dz} M(x,z,\mu)\ M(x,z,\lambda)\otimes_s M(z,y,\lambda)\ M(z,y,\mu)$$

$$= \frac{d}{dz} M(x,z,\mu) M(z,y,\lambda)\otimes_s M(x,z,\lambda) M(z,y,\mu)$$

$$= (\lambda-\mu)[M(x,z,\mu)\ J\ M(z,y,\lambda)\ \otimes_s M(x,z,\lambda) M(z,y,\mu) -$$

$$M(x,z,\mu)\ M(z,y,\lambda)\ \otimes_s M(x,z,\lambda)\ J\ M(z,y,\mu)]\ .$$

The Yang -Baxter equation now follows.

5. Discrete Case.

The Yang Baxter equation also arises in certain discrete (lattice) systems, for example in lattice models in statistical mechanics, such as the Heisenberg model.These situations can be treated by methods analogous to those used in the continuum cases, with difference operators and difference identities replacing the differential operators and identities.

Let us consider, for example, a monodromy matrix $M_{ij}(\lambda)$ defined on a one dimensional lattice, with the properties:

i) $M_{jj}(\lambda) = 1$
ii) $M_{jk}(\lambda)M_{km}(\lambda) = M_{jm}(\lambda)$
iii) $M_{j+1,j}(\lambda) = L_j(\lambda)$.

Then $M_{j,k}$ satisfies the difference equation $\Delta_j M_{j,k} = M_{j,k} - M_{j-1,k}$ and

$$M_{jk}(\lambda) = L_{j-1}(\lambda) \cdots L_k(\lambda).$$

Theorem 5.1 *Assume the field operators at different sites comute: $[L_j(\lambda), L_k(\mu)] = 0$. Then the tensor product $M_{jk}(\lambda) \otimes M_{jk}(\mu)$ satisfies the difference equations*

$$\Delta_j M_{jk}(\lambda) \otimes M_{jk}(\mu) = \Gamma_j(\lambda,\mu) M_{jk}(\lambda) \otimes M_{jk}(\mu)$$

$$\Delta_k M_{jk}(\lambda) \otimes M_{jk}(\mu) = -M_{jk}(\lambda) \otimes M_{jk}(\mu) \Gamma_k(\lambda,\mu)$$

where $\Gamma_k(\lambda,\mu) = L_k(\lambda) \otimes L_k(\mu) - 1 \otimes 1$.

Remark: The field operators $L_k(\lambda)$ are matrices with non-commuting operator entries; but as in the continuum case, operators at different sites are assumed to commute. The discrete case is somewhat simpler than the continuous case because the quantum corrections precisely cancel the errors in the Leibnitz rule for difference operators. Here is the proof:

$$\Delta_j M_{jk}(\lambda)\otimes M_{jk}(\mu) = M_{j+1,k}(\lambda)\otimes M_{j+1,k}(\mu) - M_{j,k}(\lambda)\otimes M_{j,k}(\mu)$$

$$=(L_j(\lambda) - 1)M_{jk}(\lambda) \otimes M_{j+1,k}(\mu) + M_{jk}(\lambda)\otimes(L_j(\mu) - 1)\, M_{jk}(\mu)$$

$$=L_j(\lambda)M_{jk}(\lambda)\otimes L_j(\mu)M_{jk}(\mu) - M_{jk}(\lambda) \otimes L_j(\mu)M_{jk}(\mu)$$

$$+M_{jk}(\lambda)\otimes L_j(\mu)M_{jk}(\mu) - M_{jk}(\lambda)\otimes M_{jk}(\mu)$$

$$=L_j(\lambda)M_{jk}(\lambda) \otimes L_j(\mu)N_{jk}(\mu) - M_{jk}(\lambda)\otimes M_{jk}(\mu).$$

Now note that $[M_{jk}(\lambda) , L_j(\mu)] = 0$ since M_{jk} depends only on the field operators $L_k, \ldots L_{j-1}$. Therefore we can rewrite the last term as

$$=(L_j(\lambda)\otimes L_j(\mu) - 1\otimes 1)M_{jk}(\lambda)\otimes M_{jk}(\mu),$$

which proves the proposition.

The Yang Baxter equation for the discrete case is similar to the continuous case; namely,

$$R(\lambda,\mu)M_{jl}(\lambda)\otimes M_{jl}(\mu) - M_{jl}(\mu)\otimes M_{jl}(\lambda)R(\lambda,\mu) = 0. \tag{5.1}$$

Theorem 5.2. *The Yang–Baxter equation is equivalent to the commutation relation*

$$\Gamma_k(\lambda,\mu)\, R(\lambda,\mu) - R(\lambda,\mu)\Gamma_k(\mu,\lambda) = 0, \tag{5.2}$$

where $\Gamma_k(\lambda,\mu) = L_k(\lambda)\otimes L_k(\mu) - 1\otimes 1$.

Proof. From Theorem 5.1 we have

$$\Delta_k M_{jk}(\lambda) \otimes M_{jk}(\mu)\ \ R(\lambda,\mu)\ \ M_{kl}(\mu) \otimes M_{kl}(\lambda)$$

$$= M_{jk}(\lambda) \otimes M_{jk}(\mu)\ [\, R(\lambda,\mu)\Gamma_k(\mu,\lambda) - \Gamma_k(\lambda,\mu)R(\lambda,\mu)]\ \ M_{kl}(\mu) \otimes M_{kl}(\lambda).$$

If (5.2) holds, then summing this identity over $l \le k \le j-1$ we get

$$0 = \sum_{k=l+1}^{j} \Delta_k M_{jk}(\lambda)\otimes M_{jk}(\mu)R(\lambda,\mu)M_{kl}(\mu)\otimes M_{kl}(\lambda)$$

$$= R(\lambda,\mu)M_{jl}(\mu)\otimes M_{jl}(\lambda) - M_{jl}(\lambda)\otimes M_{jl}(\mu)R(\lambda,\mu),$$

which is (5.1).

Conversely, applying Δ_j to (5.1), we get (5.2).

6. A general quantum integral identity.

In this section we derive a quantum integral relation which generalizes an identity obtained in the classical case by Faddeev and Takhtajan [2], p. 192 . It includes both the boson and fermion cases. The supersymmetric commutator is defined by

$$[A \otimes_s, B]_{ij,kl} = A_{ik} B_{jl}(-1)^{p(j)(p(i)+p(k))} - B_{jl}A_{ik}(-1)^{p(l)(p(i)+p(k))}$$

Theorem 6.1 *Let the monodromy operator satisfy the forward and backward differential equations in standard form:*

$$\frac{\partial}{\partial x} M(x,y,\lambda) = V(x,\lambda)\, M(x,y,\lambda), \qquad \frac{\partial}{\partial y} M(x,y,\lambda) = -\, M(x,y,\lambda)\, V(y,\lambda),$$

$$M(x,x,\lambda) = 1$$

Then the following supersymmetic integral identity holds:

$$[M(x,y,\lambda) \otimes_s, M(x,z',\lambda)] = \tag{6.1}$$

$$\int_y^x\int_y^x M(x,z,\lambda)\otimes_s M(x,z',\lambda)\ [V(z,\lambda)\ \otimes_s, V(z',\mu)]P_s M(z',\psi,\mu)\otimes_s M(z,y,\lambda)P_s\ dz\, dz'$$

In the classical case the commutators are replaced by a Poisson bracket, the Poisson brackets of matrix valued functionals being defined by [2] $\{F \otimes, G\}_{ij,kl} = \{F_{ik}, G_{jl}\}$. In the above identity we used the supersymmetric tensor product (4.1). The pure boson case is contained in Theorem 5.1 as a special case by taking all the parities p(j) = 0.

A similar result holds in the discrete case, namely

Theorem 6.2 *Let the monodromy matrix $M(i,j,\lambda)$ satisfy the forward and backward difference equations*

$$\Delta_i M(i,j,\lambda) = (L_i(\lambda) - 1)M(i,j,\lambda)\,, \qquad \Delta_j M(i,j,\lambda) = -M(i,j,\lambda)\,(L_j(\lambda) - 1).$$

Then

$$[M(i,j,\lambda) \otimes_{s'} M(i,j,\mu)]$$

$$= \sum_{k,k'=j}^{i-1} M(i,k+1,\lambda) \otimes_s M(i,k'+1,\mu)\ [L_k(\lambda) \otimes_{s'} L_{k'}(\mu)\ ]\ P_s M(k',j,\mu) \otimes_s M(k,j,\lambda)\ P_s$$

Again, this identity includes the pure boson case by taking all the parities equal to zero.

We shall give a proof of Theorem 6.1, the proof of Theorem 6.2 being entirely similar. The double integral in (6.1) can be written

$$\int_y^x\!\!\int_y^x dz\ dz'(M(x,z,\lambda)\otimes_s M(x,z',\mu))_{ij,mn}[V(z,\lambda)\otimes_{s'} V(z',\mu)]_{mn,rs} \times$$

$$(P_s M(z',y,\mu)\otimes_s M(z,y,\lambda)P_s)_{rs,kl}$$

where repeated indices denote summation over the pairs mn and rs. Writing out the indicated tensor products we get

$$\int_y^x\int_y^x dz\,dz'\,M_{im}(x,z,\lambda)M_{jn}(x,z',\mu)(-1)^{p(j)(p(i)+p(m))}[V_{mr}(z,\lambda)V_{ns}(z',\mu)(-1)^{p(n)(p(m)+p(r))}$$

$$-V_{ns}(z',\mu)V_{mr}(z,\lambda)(-1)^{p(s)(p(m)+p(r))}]\,M_{sl}(z',y,\mu)M_{rk}(z,y,\lambda)(-1)^{p(l)(p(r)+p(k))}$$

Using the forward and backward differential equations for $M(x,z',\mu)$ and $M(z',y,\mu)$ we can write the integral as

$$\int_y^x dz\,M_{im}(x,z,\lambda)\int_y^x dz' M_{jn}(x,z',\mu)V_{mr}(z,\lambda)\frac{\partial}{\partial z'}M_{nl}(z',y,\mu)\ (-1)^{p(n)(p(m)+p(r))}$$

$$+\frac{\partial}{\partial z}M_{js}(x,z',\mu)V_{mr}(z,\lambda)M_{sl}(z',y,\mu)(-1)^{p(s)(p(m)+p(r))}M_{rk}(z,y,\lambda)\times$$

$$(-1)^{p(j)(p(i)+p(m))+p(l)(p(r)+p(k))}$$

Now n and s are dummy indices of summation. Replacing s by n in the second term we get

$$\int_y^x dz\,M_{im}(x,z,\lambda)\int_y^x dz'\ (-1)^{p(n)(p(m)+p(r))}(M_{jn}(x,z',\mu)V_{mr}(z,\lambda)\frac{\partial}{\partial z'}M_{nl}(z',y,\mu)$$

$$+\frac{\partial}{\partial z}M_{jn}(x,z',\mu)\ V_{mr}(z,\lambda)\ M_{nl}(z',y,\mu))M_{rk}(z,y,\lambda)(-1)^{p(j)(p(i)+p(m))+p(l)(p(r)+p(k))}$$

$$\int_y^x dz\,M_{im}(x,z,\lambda)\int_y^x dz'\ \frac{\partial}{\partial z'}(M_{jn}(x,z',\mu)V_{mr}(z,\lambda)M_{nl}(z',y,\mu))\ \times$$

$$M_{rk}(z,y\ ,\lambda)(-1)^{p(j)(p(i)+p(m))+p(l)(p(r)+p(k))\ +\ p(n)(p(m)+p(r))}$$

$$=\int_y^x dz\,M_{im}(x,z,\lambda)\ [\delta_{jn}V_{mr}(z,\lambda)M_{nl}(x,y,\mu)\ (-1)^{p(n)(p(m)+p(r))}$$

$$-M_{jn}(x,y,\mu)V_{mr}(z,\lambda)\delta_{nl}(-1)^{p(n)(p(m)+p(r))}]\,M_{rk}(z,y,\lambda)(-1)^{p(j)(p(i)+p(m))\ +\ p(l)\ (p(r)+p(k))}$$

$$= \int_y^x dz\, M_{im}(x,z,\lambda)\ [\ V_{mr}(z,\lambda)\ M_{jl}(x,y,\mu)\ (-1)^{p(j)(p(i)+p(r))+p(l)(p(r)+p(k))}$$

$$-M_{jl}(x,y,\mu)V_{mr}(z,\lambda)\ (-1)^{p(l)(p(m)\ +\ p(k))+p(j)(p(i)+p(m))}]\ M_{rk}(z,y,\lambda)$$

Summing over m in the first term and r in the second, and using the forward and backward equations for $M(x,z,\lambda)$ and $M(z,y,\lambda)$ in standard form, *e.g.*

$$M_{im}(x,z,\lambda)V_{mr}(z,\lambda) = -\frac{\partial}{\partial z}M_{ir}(x,z,\lambda)$$

we obtain

$$= -\int_y^x dz\, \frac{\partial}{\partial z}M_{ir}(x,z,\lambda)\ M_{jl}(x,y,\mu)\ M_{rk}(z,y,\lambda)\ (-1)^{p(j)(p(r)+p(i)+p(l)(p(r)+p(k)}$$

$$+M_{im}(x,z,\lambda)M_{jl}(x,y,\mu)\frac{\partial}{\partial z}M_{mk}(z,y,\lambda)\ (-1)^{p(j)(p(i)+p(m))\ +\ p(l)(p(m)+p(k))}$$

Changing m to r in the summation in the second term, we get

$$= -\int_y^x dz\ [\ \frac{\partial}{\partial z}M_{ir}(x,z,\lambda)M_{jl}(x,y,\mu)M_{rk}(z,y,\lambda)\ +M_{ir}(x,z,\lambda)M_{jl}(x,y,\mu)\ \frac{\partial}{\partial z}M_{rk}(z,y,\lambda)\ ] \times$$

$$(-1)^{p(j)(p(r)+p(i))\ +\ p(l)(p(k)+p(r))}$$

$$= -\ M_{ir}(x,z,\lambda)M_{jl}(x,y,\mu)M_{rk}(z,y,\lambda)\ \ (-1)^{p(j)(p(r)+p(i))\ +\ p(l)(p(k)+p(r))}\Big|_y^x$$

$$= M_{ik}(x,y,\lambda)M_{jl}(x,y,\mu)\ (-1)^{p(j)(p(i)+p(k))} - M_{jl}(x,y,\mu)M_{ik}(x,y,\lambda)(-1)^{p(l)(p(k)+p(r))}$$

$$= [M(x,y,\lambda) \otimes_s M(x,y,\mu)\ ]_{ij,kl}\ .$$

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