REVELATION AND IMPLEMENTATION
UNDER DIFFERENTIAL INFORMATION

BY
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<table>
<thead>
<tr>
<th>Preprint #</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Workshop Summaries from</td>
<td>The September 1982 workshop on Statistical Mechanics,</td>
</tr>
<tr>
<td></td>
<td>the September 1982</td>
<td>Dynamical Systems and Turbulence</td>
</tr>
<tr>
<td>2</td>
<td>Raphael De la Llave</td>
<td>A Simple Proof of C. Siegel's Center Theorem</td>
</tr>
<tr>
<td>3</td>
<td>H. Simpson, S. Spector</td>
<td>On Copositive Matrices and Strong Ellipticity for Isotropic Elastic</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Materials</td>
</tr>
<tr>
<td>4</td>
<td>George R. Sell</td>
<td>Vector Fields in the Vicinity of a Compact Invariant Manifold</td>
</tr>
<tr>
<td>5</td>
<td>Milan Miklavcic</td>
<td>Non-linear Stability of Asymptotic Suction</td>
</tr>
<tr>
<td>6</td>
<td>Hans Weinberger</td>
<td>A Simple System with a Continuum of Stable Inhomogeneous Steady States</td>
</tr>
<tr>
<td>7</td>
<td>Bau-Sen Du</td>
<td>Period 3 Bifurcation for the Logistic Mapping</td>
</tr>
<tr>
<td>8</td>
<td>Hans Weinberger</td>
<td>Optimal Numerical Approximation of a Linear Operator</td>
</tr>
<tr>
<td>9</td>
<td>L.R. Angel, D.F. Evans,</td>
<td>Three Component Ionic Microemulsions</td>
</tr>
<tr>
<td></td>
<td>B. Ninham</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>D.F. Evans, D. Mitchell,</td>
<td>Surfactant Diffusion; New Results and Interpretations</td>
</tr>
<tr>
<td></td>
<td>S. Mukherjee, B. Ninham</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Leif Arkeryd</td>
<td>A Remark about the Final Aperiodic Regime for Maps on the Interval</td>
</tr>
<tr>
<td>12</td>
<td>Luis Magalhaes</td>
<td>Manifolds of Global Solutions of Functional Differential Equations</td>
</tr>
<tr>
<td>13</td>
<td>Kenneth Meyer</td>
<td>Tori in Resonance</td>
</tr>
<tr>
<td>14</td>
<td>C. Eugene Wayne</td>
<td>Surface Models with Nonlocal Potentials: Upper Bounds</td>
</tr>
<tr>
<td>16</td>
<td>George R. Sell</td>
<td>Smooth Linearization Near a Fixed Point</td>
</tr>
<tr>
<td>17</td>
<td>David Wolkind</td>
<td>A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification</td>
</tr>
<tr>
<td>18</td>
<td>Pierre Collet</td>
<td>Local ( C^m ) Conjugacy on the Julia Set for some Holomorphic Perturbations of ( z^2 + z )</td>
</tr>
<tr>
<td>19</td>
<td>Henry C. Simpson, Scott J.</td>
<td>On the Modified Bessel Functions of the First Kind</td>
</tr>
<tr>
<td></td>
<td>Spector</td>
<td>On Barrelling for a Material in Finite Elasticity</td>
</tr>
<tr>
<td>20</td>
<td>George R. Sell</td>
<td>Linearization and Global Dynamics</td>
</tr>
<tr>
<td>21</td>
<td>Constantin P, Folias</td>
<td>Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of</td>
</tr>
<tr>
<td></td>
<td></td>
<td>the Attractors for 2D Navier-Stokes Equations</td>
</tr>
<tr>
<td>22</td>
<td>Milan Miklavcic</td>
<td>Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator</td>
</tr>
<tr>
<td>23</td>
<td>P. Collet, H. Epstein, G.</td>
<td>Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties</td>
</tr>
<tr>
<td></td>
<td>Gallavotti</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>J.E. Dunn, J. Serrin</td>
<td>On the Thermomechanics of Interstitial working</td>
</tr>
<tr>
<td>25</td>
<td>Scott J. Spector</td>
<td>On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension</td>
</tr>
<tr>
<td>26</td>
<td>W.A. Coppel</td>
<td>Maps on an Interval</td>
</tr>
<tr>
<td>27</td>
<td>James Kirkwood</td>
<td>Phase Transitions in the Ising Model with Traverse Field</td>
</tr>
<tr>
<td>28</td>
<td>Luis Magalhaes</td>
<td>The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different</td>
</tr>
<tr>
<td>29</td>
<td>Charles Tresser</td>
<td>Homoclinic Orbits for Flow in ( \mathbb{R}^3 )</td>
</tr>
<tr>
<td>30</td>
<td>Charles Tresser</td>
<td>About some Theorems by L.P. Sil'nikov</td>
</tr>
<tr>
<td>31</td>
<td>Michael Aizenmann</td>
<td>On the Renormalized Coupling Constant and the Susceptibility in ( 4^4 ) Field Theory and the Ising Model in Four Dimensions</td>
</tr>
<tr>
<td>32</td>
<td>C. Eugene Wayne</td>
<td>The KAM Theory of Systems with Short Range Interactions I</td>
</tr>
</tbody>
</table>

(continued on back cover)
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REVELATION AND IMPLEMENTATION 
UNDER DIFFERENTIAL INFORMATION

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1. **INTRODUCTION**

Our goal in this paper is to merge several central ideas in economic theory: strategic behavior (incentive compatibility), differential (or incomplete) information, and the Arrow-Debreu model of general equilibrium. By strategic behavior we refer to the literature which models economic institutions as games in strategic form and uses Nash equilibrium as the solution concept. This literature, motivated by informational decentralization questions, deals not with a single economic environment and a single game, but rather considers a class of environments and a strategic outcome function (game form) which is applied uniformly to this class. The concept of differential information is that of Bayesian equilibrium as it has been applied in the literature on implicit contracts, principal-agent problems and bidding and auction models.

The differential information economy model used here was introduced in Postlewaite and Schmeidler (1983). That paper also contains motivation, interpretation, and examples of the model. Myerson (1983) contains an exposition of Harsanyi's (1967,8) model of differential information and Harsanyi-Nash (Bayesian) equilibria, and other results including the revelation principle. Several of the results proved in this paper extend to differential information economies results on implementation via Nash equilibria in a complete information framework presented in Maskin (1977) and Hurwicz, Maskin, and Postlewaite (1982). For exposition of these and other results for complete information see the surveys by Maskin (1983) and Postlewaite (1983).

Our model of a pure exchange economy differs from the Arrow-Debreu model in that an agent's characteristics include, in addition to his endowment and preferences, a specification of his information. This is done by making initial
endowments and preferences random variables on a set of (Savage) states of nature. An agent's information structure is then given by a partition of the states of nature. When a state of nature occurs, each agent is informed of the event in his partition containing this state. This description of an agent's information is incomplete. Since we are considering strategic behavior, an agent is concerned not only with which states of nature he can distinguish, but also which states can be distinguished by others, since others' behavior, which is a function of their information, affects him. Thus a complete specification of information must include information about others' information, others' information about others' information, and so on. We assume the partitions are common knowledge, thus all relevant information for an agent is captured by the event containing the true state of nature.

If any agent can distinguish two states, all agents' allocations may differ in these two states. An agent who must make a decision without being able to distinguish the states must trade off his welfare in these two states. Hence we are in a cardinal framework. Thus probability and von Neumann-Morgenstern utilities are included in the description of agents' characteristics as presented in detail in the next section.

The model of a differential information economy is quite flexible in its interpretation. One particular interpretation is of interest. Consider a subset of neo-classical economies with \( n \) agents, i.e., \( n \)-tuples of preferences and endowments. Each such economy is a trivial differential information economy with a single state of nature; each agent knows the entire vector of characteristics. We can associate with each economy a distinct state of nature and let each agent's information structure be such that all states of nature can be distinguished. The collection of trivial differential information economies becomes a single differential information economy itself.
in this way. Proposition 1 will show that an arbitrary combination of differential information economies will be a differential information economy.

The literature on implementation via Nash equilibria in complete information economies begins with the notion of a social welfare correspondence (or performance correspondence) which associates a non-empty set of allocations with each economy in a given set. Given our proposition stating that a combination of differential information economies will be a differential information economy itself, the analogue of a performance correspondence in our framework will be a social welfare correspondence which associates a non-empty set of allocations with every state of nature.

In the complete information framework the problem of implementation is to design a strategic outcome function whose Nash equilibria for any economy coincide with the allocations prescribed by the performance correspondence. Here we want to design a strategic outcome function for a differential information economy such that for every state of nature, the Bayes equilibria coincide with the social welfare correspondence.

One of the contributions of this paper is to provide a unified framework in which both differential information and the complete information Nash implementation approach can be accommodated. We described above how a set of complete information economies comprise a single differential information economy. A differential information equilibrium of this differential information economy assigns a Nash equilibrium to each complete information economy. Hence implementation for differential information economies subsumes implementation via Nash equilibria for complete information economies.

One of our main results states a necessary condition for implementability of a social welfare correspondence in a differential information economy. This reduces to Maskin's (1977) monotonicity condition for the complete information
case. We also provide a condition on agents' information structures, which together with monotonicity suffice for implementation. The informational condition is trivially satisfied in the complete information case. Hence, implementability for complete information economies is a special case of our results.

In the next section we present the model of a differential information economy and prove the proposition about combinations of differential information economies. In section three, strategic outcome functions (SOFs) are defined. Section four contains the definition of differential information equilibria (Harsanyi-Nash equilibria) and its existence is discussed. Section five contains a careful analysis of the concept of informational decentralization. The revelation principle is presented and discussed in Section six. Also, a condition is presented which guarantees that the revelation outcome function induced by a SOF has the same set of equilibria as the original SOF. Section seven contains our main results on implementation discussed earlier. Section eight introduces the concept of an extended revelation outcome function and relates its equilibria to the equilibria of the original outcome function. The concluding section, nine, contains two remarks. The first explains a possible simplification of the differential information economy. The second remark elaborates on the relationship between our implementation results and self selection models.

2. **DIFFERENTIAL INFORMATION ECONOMY**

A pure exchange economy with differential information is a list

\[ e = (\Omega_t, \hat{\nu}_t^T)_{t \in T}, (\hat{u}_t^T)_{t \in T}, (P_{t^{'}, t^{'}})_{t \in T} \].
The symbol \( T \) stands for an abstract set of \( n \) elements that represents the names of the economic agents. There are \( \ell \) Arrow-Debreu commodities in the economy and the consumption set is \( \mathbb{R}^\ell_+ \), the nonnegative orthant of euclidean space of dimension \( \ell \). Both \( T \) and \( \ell \) are fixed throughout this paper.

Initial endowments are elements of \( \mathbb{R}^\ell_+ \) and an agent's preferences are represented by a von Neumann-Morgenstern (NM) utility function on \( \mathbb{R}^\ell_+ \) which is assumed to be real-valued, continuous, quasiconcave, and increasing in the sense that an increase in all \( \ell \) coordinates increases the utility. The set of all such utility functions is denoted by \( U \).

Agents do not have complete information about basic parameters of the economy. Following Savage's (neobayesian) paradigm, the uncertainty is represented by a set of "states of the world" denoted by \( \Omega \). A state \( \omega \) in \( \Omega \) resolves, by definition, all uncertainties for all agents, with the possible exception of the uncertainties inherent in the definition of Arrow-Debreu contingent commodities. Thus in the differential information (D.I.) framework for all \( t \) in \( T \); \( \hat{w}_t : \Omega \times \mathbb{R}^\ell_+ \) and \( \hat{u}_t : \Omega \times \mathbb{R}^\ell_+ \times \mathbb{R} \) where \( \hat{u}_t(\omega, \cdot), \cdot) \in U \) for all \( \omega \) in \( \Omega \) (or equivalently \( \hat{u}_t : \Omega \times U \)). A certain measurability condition is imposed in the sequel on \( \hat{w}_t \). This condition implies that agent \( t \) knows the vector \( \hat{w}_t(\omega) \) before the act of exchange.

According to the neobayesian paradigm, every economic agent has a (prior) probability distribution over \( \Omega \). Since we are modeling differential information, one can think of a different prior for each \( t \) in \( T \). We postulate here, for all \( t \) in \( T \), the existence of a conditional (posterior) probability \( p_t : \Omega \times \Omega \times [0,1] \) such that for all \( \omega \) in \( \Omega \): \( \sum_{\omega'} \in \Omega \) \( p_t(\omega', \omega) = 1 \).

Before stating additional conditions on \( p_t(\cdot, \cdot) \) which imply that it is the usual conditional probability, an explanation is presented. The use of conditional probability instead of probability allows us to interpret the
difference in information among agents as resulting from different private
information, rather than from different basic models of the economy. Moreover,
the uncertainty embodied in the concept of state of the world relates not only
to an agent's information about the (neo-classical) parameters of the economy but
also about other agents' information about his information, etc. Conditional
probability allows representation of this uncertainty. Every agent knows e.
Suppose that \( P_t(\omega_1, \omega) > 0 \), \( P_t(\omega_2, \omega) > 0 \) and \( P_t(\cdot, \omega_1) \neq P_t(\cdot, \omega_2) \): Thus
agent \( t \) does not know the posterior probability of agent \( t' \), since either of
\( \omega_1, \omega_2 \) is possible. Our model of a differential information economy follows the
Harsanyi (1967-8) model of games with incomplete information. It is assumed that
every agent \( t \) in \( T \) knows e: Then some \( \omega \) in \( \Omega \) occurs and agent \( t \) is
informed of the probability \( P_t(\cdot, \omega) \), but not of \( \omega \).

**Informational Conditions.** For each \( t \) in \( T \) we assume the following on
\( P_t(\cdot, \cdot) : \Omega \times \Omega + [0,1] \).

**DI1:** For all \( \omega \) in \( \Omega \) the set \( \{ \omega' \in \Omega \mid P_t(\omega', \omega) > 0 \} \) is finite.

Hence the previously mentioned condition \( \sum_{\omega' \in \Omega} P_t(\omega', \omega) = 1 \) is
meaningful.

**DI2:** For all \( \bar{\omega} \) and \( \bar{\omega} \) in \( \Omega \): \( P_t(\bar{\omega}, \bar{\omega}) > 0 \) iff \( P_t(\cdot, \bar{\omega}) = P_t(\cdot, \bar{\omega}) \).

To explain and motivate DI2, suppose that agent \( t \) is informed of a
(posterior) probability \( \bar{P}_t \) on \( \Omega \). Denote

\[ B = \{ \omega' \in \Omega \mid \bar{P}_t(\cdot) = P_t(\cdot, \omega') \} . \]
Agent $t$ knows that some $\omega'$ in $B$ occurred, but any $\omega'$ in $B$ is possible. Hence, if $\bar{\omega}$ and $\tilde{\omega}$ belong to $B$ then $P_t(\bar{\omega}) > 0$ and $P_t(\tilde{\omega}) > 0$.

So if $P_t(\cdot, \bar{\omega}) = P_t(\cdot, \tilde{\omega})$ then $P_t(\bar{\omega}, \bar{\omega}) > 0$ and $P_t(\tilde{\omega}, \tilde{\omega}) > 0$. To explain the opposite implication note that if $\omega \notin B$ then $P_t(\omega) = 0$ because $P_t(\cdot, \omega) \neq P_t(\cdot)$. It goes without saying that Condition 2 implies the above inequalities and equalities. Hence a partition $\Pi_t$ is induced by this condition on $\Omega$. An element $B$ of this partition is finite by Condition 1.

We can now redefine $P_t(\cdot, \cdot)$ as the usual conditional probability $P_t: 2^\Omega \times \Pi_t + [0,1]$ by $P_t(A|B) = \sum_{\omega' \in \Pi} P_t(\omega', \omega)$ where $\omega \in B$.

The following condition subsumes DII.

**DII3:** Denote by $\Pi$ the finest partition of $\Omega$ which is coarser than each $\Pi_t$, $t$ in $T$. Then every $B$ in $\Pi$ is finite.

If $\omega$ in $\Omega$ occurs, every agent $t$ in $T$ is informed of the element $B_t$ in his partition $\Pi_t$ which contains $\omega$. The element $B$ of $\Pi$ which contains $\omega$ (and includes each $B_t$) is *common knowledge at $\omega$* according to the definition of Aumann(1976).

For future use we introduce some notation and redefine this concept. For all $t$ in $T$ define $I_t: \Omega + 2^\Omega$ by $I_t(\omega) = B_t \in \Pi_t$ such that $\omega \in B_t$.

Extend the definition of $I_t$ to $2^\Omega$ by $I_t(A) = \bigcup_{\omega \in A} I_t(\omega)$. Then $B = I(\omega)$ is *common knowledge at $\omega$* if $I(\omega) = I_t(\omega) \cup I_t(I_t(\omega)) \cup I_t(I_t(I_t(\omega))) \cup \ldots$

where the union is over all finite applications of the operators $I_t$, $t \in T$ according to some order $t_1, t_2, \ldots$ of agents in $T$. It is clear that $I(\omega)$ is independent of this order because $I(\omega) \in \Pi$ and is a finite set by DII3 (or denumerable if only DII is assumed). On a heuristic level it is assumed that $e$ is common knowledge, i.e., every agent knows $e$, every agent
knows that every agent knows e, every agent knows that every agent knows that every agent knows e, etc.

We now introduce the previously promised condition on the "random" variables $\hat{w}_t(\cdot)$, $t \in T$.

**DI4:** For all $t$ in $T$, $\hat{w}_t: \Omega \rightarrow \mathbb{R}^l_+$ is measurable with respect to $\Pi_t$, i.e., $P_t(\cdot, \omega) = P_t(\cdot, \bar{\omega})$ implies $\hat{w}_t(\omega) = \hat{w}_t(\bar{\omega})$.

The reason for the last condition becomes clearer after the formal definition of the market rules. We first start with a basic observation.

**Proposition 1:** Any combination of differential information economies is a differential information economy.

**Proof:** Let $(\Omega^a, (\hat{w}^a)_t, (u^a)_t, (P^a(\cdot, \cdot))_t)_{t \in T}$, $a \in A$ be a collection of D.I. economies. Defining their combination will clarify the proposition and its proof. Define $\Omega = \bigcup_{a \in A} \Omega^a$ to be the disjoint union of the $\Omega^a$'s. To define the other parameters of the combined economy note that for all $\omega$ in $\Omega$ there is a $\beta$ in $A$ such that $\omega$ belongs to $\Omega^\beta$.

We then define for all $t$ in $T$: $\hat{u}_t(\omega) = \hat{u}_t^\beta(\omega)$ in $U$, $\hat{w}_t(\omega) = \hat{w}_t^\beta(\omega)$, and $P_t(\omega', \omega) = P_t^\beta(\omega', \omega)$ for $\omega'$ in $\Omega^\beta$ and zero otherwise. O.E.D.

This section is concluded with an additional informational assumption.

**DI5:** The coarsest partition which is finer than each of the $\Pi_t$'s, $t$ in $T$, is the discrete partition. Equivalently, for all $\omega$ in $\Omega$: $\bigcap_{t \in T} I_t(\omega) = \{\omega\}$.
The only purpose of this assumption is to simplify presentation. If no one in the economy can distinguish $\omega$ and $\omega'$ in $\Omega$ they constitute one state of nature for all practical purposes. In a more involved model where the informational structure is not exogenously given and information can be acquired, such an assumption might be too restrictive.

**Example 1:** A special case of an economic environment is an environment such that for all $t$ in $T$ and for all $\omega$ in $\Omega$: $P_t(\omega, \omega') = 1$. Consider further the following specifications: $\Omega = R^T_{+} \times U^T$, and for all $t$ in $T$, $\hat{\omega}_t(\omega)$ and $\hat{u}_t(\omega)$ are the corresponding projections of $\omega$. Hence a single incomplete information environment $e$ represents the class of pure exchange economies with cardinal utilities. There exist strategic outcome functions (Schmeidler (1980), Postlewaite and Wettstein (1982)) that guarantee the existence of equilibrium for $e$ which is Walrasian for every $\omega$ in $\Omega$.

The most trivial special case is a complete information economy where $\Omega = 1$. The weaker restriction on $e$, that for all $t$ in $T$ and all $\omega, \omega'$ in $\Omega$, $P_t(\omega', \omega)$ is zero or one, is referred to as a common knowledge complete information economy.

3. **ECONOMIC INSTITUTIONS OF EXCHANGE**

Economic institutions of exchange are represented by the concept of a strategic outcome function (SOF). Given a vector $\omega$ in $R^T_{+}$, the set of acts or moves for an agent whose initial endowment is $\omega$ is denoted by the set $S(\omega)$. The domain of a SOF $f$ is $S = \bigcup_{\omega \in R^T_{+}} \bigcup_{t \in T} S(\omega)$. Its range is $R_{T}$. If we insist on total informational decentralization (see Schmeidler(1980)), then $S(\omega)$ is assumed to be independent of $\omega$. Thus, $f(\omega) = (f_t(\omega))_{t \in T}$.
\( (z_t)_{t \in T} \in R^{\mathcal{T}} \) is interpreted as a list of net trades where

\[ s = (s_t)_{t \in T} \] and \( s_t \) is agent \( t \)'s act. We will write \( (s|t,s'_t) \) to indicate the list of strategies with \( s'_t \) inserted in the \( t^{th} \) position of \( s \).

In this case the only restriction on an element of the range, \( f(s) \), is

\[ \sum_{t \in T} f_t(s) = 0. \]

If partial informational decentralization is assumed, and this is the case for the rest of the paper, \( f_t(s) \) is the final bundle of agent \( t \). The initial endowment of every agent is centrally known, i.e., \( \text{Dom}(f) \equiv D \subset R^{\mathcal{T}}_+ \times S \) where \( (w,s) \in D \) if and only if for all \( t \in T \), \( s_t \in S(w_t) \), \( f(w,s) \in R^{\mathcal{T}}_+ \), and

\[ \sum_{t \in T} f_t(w,s) = \sum_{t \in T} w_t. \]

We will also use the notation \( S(w) \) for \( X_{t \in T} S(w_t) \) where \( w = (w_t)_{t \in T} \).

**Remark:** The number \( n \) is the maximal number of agents in the economy. If there are less than \( n \) agents, the nonexistent agents are "dummy players". The endowment of such an agent is zero and \( S(0) \) is assumed to be a singleton.

The SOP \( f \) essentially disregards such agents and assigns each of them final outcome zero.

**Proposition 2:** Any finite combination of strategic outcome functions is a strategic outcome function.

**Proof:** The meaning and the proof of the proposition become clear when the combination of two SOFs is defined: Let \( f' \) and \( f^* \) be two SOFs with domains defined by agent's strategies sets \( S'(w) \) and \( S^*(w) \), respectively, for \( w \) in \( R^\mathcal{T}_+ \). Define \( S(w) = w \cup \{ S'(w') \times S^*(w^*) \} \) where the union is taken over all pairs \((w',w^*)\) in \( R^\mathcal{T}_+ \times R^\mathcal{T}_+ \) such that \( w' + w^* = w \). (If the institutions are exclusive then \( S(w) = S'(w) \times S^*(0) \cup S'(0) \times S^*(w) \).
After the set of acts in the combination has been defined for all \( w \) in \( R^+_T \), the set \( \tilde{S}(w) \) is well defined.

Given \( \tilde{s} \) in \( \tilde{S}(w) \), we now define \( f(\tilde{s}) \) in \( R^{LT} \). By our convention, for each \( t \) in \( T \), \( s_t = (\hat{w}_t, \tilde{s}_t^', \tilde{s}_t^*) \) where \( \tilde{s}_t^' = (\hat{w}_t, \tilde{s}_t^' \tilde{s}_t^* = (\hat{w}_t, \tilde{s}_t^*) \) with \( \hat{w}_t + \hat{w}_t^* = \hat{w}_t \). The decomposition of \( \hat{w}_t \) into \( \hat{w}_t^' \) and \( \hat{w}_t^* \) is part of a choice of act by agent \( t \). Furthermore \( \hat{s}_t \in \hat{S}(\hat{w}_t) \) and \( \hat{s}_t^* \in \hat{S}(\hat{w}_t^* \) .

Then for all \( t \) in \( T \):

\[
\hat{f}_t(\hat{s}_t) = \hat{f}_t^'(\hat{s}_t^') + \hat{f}_t^* (\hat{s}_t^*) .
\]

One can think of a different decomposition, say \( \hat{w}_t^' \rightarrow \hat{w}_t \), of \( \hat{w}_t \) for some \( t \) in \( T \) such that \( \tilde{s}_t^' \in \tilde{S}(\hat{w}_t^') \) and \( \tilde{s}_t^* \in \tilde{S}(\hat{w}_t^*) \). However, then \( \hat{s}_t = \hat{w}_t, (\hat{s}_t^', \hat{s}_t^*), (\hat{w}_t^*, \hat{s}_t^*) \) is then different from \( s_t \) and the corresponding value \( f(\hat{s}_t) \) may differ from \( f(s_t) \). So the SOF \( f \) is well defined.

O.E.D.

In a model of incomplete information the agents may be uncertain about the working of the economic institutions. This could be represented by defining a SOF as a mapping from \( S \times \hat{\Omega} \) to \( R^{LT} \). In this paper though, we deal with a somewhat restricted class of SOFs. Here \( \hat{u}_t \) is a function defined on \( \hat{\Omega} \times R^{LT} \), while \( f \) is defined on \( S \). Thus our SOF \( f \) is conceptually equivalent to a mapping from \( S \) to \( \hat{\Omega} \times R^{LT} \). However, the class of such mappings can be embedded as a strict subset of the class of mappings from \( S \times \hat{\Omega} \) to \( R^{LT} \). \( \hat{\Omega} \) also appears in a restricted way in the domain of our SOF. The set of acts available to agent \( t \), \( \hat{S}(\hat{w}(\cdot)) \) is a function of \( \hat{w} \) in \( \hat{\Omega} \).

Given a SOF \( f \) and a D.I. economy \( e \), we construct a strategic outcome function \( f_e \) defined on a subset, \( D \), of \( \hat{\Omega} \times S \). \( D \) contains those pairs \( (\hat{w}, \hat{g}) \) for which, for all \( t \in T \), \( \hat{s}_t \in \hat{S}(\hat{w}_t(\hat{w})) \). Then \( f_e (\hat{w}, \hat{g}) = f(\hat{w}, \hat{g}) \) where \( \hat{w} = \hat{w}(\hat{w}) \). In the sequel we will not always distinguish between \( f \) and \( f_e \) when it is clear from the context with which one we are dealing.

Typically \( e \) will be given and we deal with \( f_e \). We assume that \( f \) is
common knowledge as well as e. Finally, we point out that Proposition 2 applies to combinations of f, s for a fixed e.

**Notational Remark.** In the sequel when discussing a SOF f, we may sometimes omit from its domain R^k_+ and Ω and only consider S. It is always assumed that the combination of acts s are admissible and that the outcome is feasible.

4. **STRATEGIES AND EQUILIBRIA**

Given an economic environment e and a SOF f, a strategy for agent t in T is a mapping \( \sigma_t : \Omega \to S(R^k_+) = \bigcup_{w \in R^k_+} S(w) \), such that for all \( \omega \) in \( \Omega \); \( \sigma_t(\omega) \in S(\hat{\omega}_t(\omega)) \), and \( \sigma_t(\cdot) \) is measurable with respect to \( \Pi_t \).

The last condition means that for all \( t \) in \( T \) and all \( \omega, \omega' \) in \( \Omega \), if \( P_t(\omega, \omega') = P_t(\omega, \omega') \) then \( \sigma_t(\omega) = \sigma_t(\omega') \). The set of agent t's strategies for fixed e and f is denoted by \( \Theta_t \).

Given a list of strategies \( \sigma = (\sigma_t)_{t \in T} \), and agent v in T, v's **best response** to other agents' strategies \( (\sigma_t)_{t \neq v} \) is v's strategy \( \theta_v \) in \( \Theta_v \) such that for all \( \omega \) in \( \Omega \) and for all \( s_v \in S(\hat{\omega}_v(\omega)) \):

\[
\sum_{\omega', \Omega_v} \pi_v(\omega', \omega) u_v(\omega', f_v(\alpha(\omega)|v, \theta_v(\omega))) \geq \sum_{\omega', \Omega_v} \pi_v(\omega', \omega) u_v(\omega', f_v(\alpha(\omega)|v, s_v)).
\]

A list of strategies \( \Theta^* \) in \( \bigcap_{t \in T} \Theta_t \) is said to be a Differential Information equilibrium (D.I. equilibrium) (or Harsanyi-Nash equilibrium or Bayesian equilibrium) if for all \( t \) in \( T \), \( \Theta^* \) is a best response to other agents' strategies \( (\Theta^*)_{t' \neq t} \).

Note that for a fixed \( \omega \) in \( \Omega \), agent t's act is a best response
against a mixed act of others if $\# I_t(\omega) > 1$. However, in equilibrium as defined above, given $\omega$, agent $t$ plays some fixed act $\theta_t^*(\omega) = s_t^*$ in $S(\omega_t(\omega))$. Another agent, not knowing $\omega$, conceives it as a mixed act.

**Mixed Acts.** It may happen that for some $e$ and $f$ there does not exist a D.I. equilibrium as defined previously. In this case mixed acts have to be considered. A sufficient condition for the existence of a mixed strategy D.I. equilibrium (i.e. a strategy that assigns a mixed act to each $\omega$ in $\Omega$) is compactness of $S(\omega)$ in some Hausdorff topology for all $\omega$ in $R^\ell_+$ and continuity of the SOF $f$ in this topology and the euclidean topology on $R^{\ell T}$. A mixed act is a probability measure, say $\lambda_t$, on the Borel subsets of $S(\omega)$, $t$ in $T$. Given a list of mixed acts $\lambda(\omega) = (\lambda_t(\omega))_{t \in T}$ on the product space $\prod S(\omega(\omega))$ where $\hat{\omega}(\omega) = (\hat{\omega}_t(\omega))_{t \in T}$ for $\omega$ in $\Omega$, agent $v$'s expected utility of the outcome is defined as follows:

$$\sum_{\omega \in \Omega} p(t(\omega, \omega')) \int_{S(\omega(\omega))} u_t(\omega', f_t(\omega)) d \lambda(\omega).$$

The integration is done with respect to the product measure of the measures $\lambda(\omega)$. The definition of equilibrium in mixed strategies follows as usual. A special case of mixed strategies equilibria are such that for all $t$ in $T$ and all $\omega$ in $\Omega$: $\lambda_t(\omega)$ has a finite support in $S_t(\hat{\omega}_t(\omega))$.

In the sequel, existence of equilibrium will be assumed. If the equilibrium is not a pure strategy equilibrium it is assumed that the mixtures have finite support. We still denote such a mixed act by $s_t^*$, $t$ in $T$. But in this case $f(s)$ is a probability distribution with a finite support over $R^{\ell T}$. We refer to a random allocation with a finite range as a lottery and denote by $L = L(R^\ell_+)$ the set of lotteries. Lotteries over bundles $L(R^\ell_+)$
are denoted by $L$. If $\sigma^*$ is a D.I. equilibrium then for each $\omega$ in $\Omega$, $f(\sigma^*(\omega))$ belongs to $R^2_t$ or $L$.

**Purification.** Another way of dealing with the necessity of mixed acts is to allow every agent to privately observe a random variable over an atomless probability space. Formally let $\Omega_t$ denote the unit interval, $\bar{\Omega}$ its $T$-fold product. Given a D.I. economy $e$, its purification $\bar{e}$ is defined as follows: $\bar{\Omega} = \Omega \times \bar{\Omega}$, $\bar{w}_t(\bar{\omega}) = w_t(\omega)$ and $\bar{u}_t(\bar{\omega}) = u_t(\omega)$ where $\bar{\omega} = (\omega, \omega)$. Finally $\bar{F}_t(\cdot, \cdot)$ is interpreted as a density function, its projection on $\Omega_t$ uniformly distributed and all such projections independently distributed. $\bar{F}_t(\bar{\omega}', \bar{\omega}) = P_t(\omega', \omega)$ if the projections on $\Omega_t$ of $\bar{\omega}'$ and $\bar{\omega}$ coincide, zero otherwise. The informational conditions should be appropriately corrected and agent $t$'s strategy projection on $\Omega_t$ should be Borel measurable. If a SOF $f$ is continuous over a compact domain then $(\hat{e}, f)$ has a (pure strategy) D.I. equilibrium. Purification will not be used in this note.

5. **INFORMATIONAL DECENTRALIZATION**

We would now like to give formal content to the concept of informational decentralization within the framework we have constructed. We take as given a specific differential information economy $e$. We want a precise notion of what information is assumed to be centrally known, i.e., what information is used in the definition of $f = f_e$ defined on a subset of $\Omega \times S$. If $f(\omega, \cdot) \neq f(\omega', \cdot)$ it means that $\omega$ and $\omega'$ are centrally distinguishable. If $f(\omega, \cdot) = f(\omega', \cdot)$ it means that $\omega$ and $\omega'$ are centrally indistinguishable. **Total informational decentralization** means that $f(\cdot, \cdot)$ is independent of its first coordinate. A SOF $f$ is said to be **informationally more**
decentralized than the SOF \( g \) if the algebra of subsets of \( \Omega \) generated by \( g \) is a refinement of the algebra of subsets of \( \Omega \) generated by \( f \), i.e., for all \( \omega \) and \( \omega' \) in \( \Omega \): \( g(\omega, \cdot) = g(\omega', \cdot) \) implies \( f(\omega, \cdot) = f(\omega', \cdot) \).

We can now formalize the concept of **partial informational decentralization** in the sense that agents' initial endowments are centrally known. We had the set of available acts \( S(\omega) \) dependent on the endowment \( \omega \). Given an economy \( e \), agent \( t \)'s set of acts was defined as \( S(\omega_t(\omega)) \), \( \omega \in \Omega \). Thus agent \( t \)'s acts are defined by mappings \( S_t(\cdot) \) on \( \Omega \).

From now on the mapping \( S_t(\cdot) \) on \( \Omega \), \( t \) in \( T \), is one of our model's primitives when defining the SOF \( f \) given \( e \). The codomain of \( S_t(\cdot) \), i.e., the set of acts, is an abstract set. One can think of a subset of euclidean space or a set of functions on such subsets.

Now the Informational Decentralization (ID) conditions on \( (S_t(\cdot))_{t \in T} \) are introduced.

**ID1:** For all \( t \) in \( T \) and for all \( \omega \) and \( \omega' \) in \( \Omega \):

\[
S_t(\omega) = S_t(\omega') \iff \omega_t(\omega) = \omega_t(\omega').
\]

**ID2:** For all \( t \) in \( T \) and for all \( \omega \) and \( \omega' \) in \( \Omega \):

\[
I_t(\omega) = I_t(\omega') \implies S_t(\omega) = S_t(\omega').
\]

Note that ID2 and ID1 imply for all \( t \), \( \omega \) and \( \omega' \): If \( \omega_t(\omega) \neq \omega_t(\omega') \) then \( I_t(\omega) \neq I_t(\omega') \). Recall that for all \( \omega \), \( I_t(\omega) \in \Pi_t \), i.e., agent \( t \) is informed of \( I_t(\omega) \). The operational meaning that \( t \) knows \( I_t(\omega) \) if \( \omega \) in \( \Omega \) is that his set of strategies consists of all mappings from \( \Omega \) to
acts measurable with respect to $\mathcal{F}_t$. Note also that ID1 and ID2 imply informational condition 4 below.

The previously made assumption that an agent's set of acts depends only on his initial endowment is implied by assumption ID1.

The next conditions express (together with ID1) that the only centrally known information, given $\omega$ in $\Omega$, is the list of initial endowments of the agents in $T$, $\hat{w}(\omega)$. But first we introduce some notation and a definition.

Let $S(\omega) = \sum_{t \in T} S_t(\omega)$ for $\omega$ in $\Omega$ and $\underline{S} = \bigcup_{\omega \in \Omega} S(\omega)$. We can now define the domain of the SOF $f$,

$$\text{Dom}(f) = \{(\omega, s) \in \Omega \times \underline{S} \mid s \in S(\omega)\}$$

**ID3:** For all $\omega$ and $\omega'$ in $\Omega$; If for all $t$ in $T$,

$$\hat{w}_t(\omega) = \hat{w}_t(\omega') \text{ then } f(\omega, \cdot) = f(\omega', \cdot).$$

**ID4:** For all $(\omega, s)$ in $\text{Dom}(f)$,

$$\sum_{t \in T} f_t(\omega, s) = \sum_{t \in T} \hat{w}_t(\omega).$$

If the codomain of $f$ is $L = L(\mathbb{R}_+^{\infty})$, the lotteries over $\mathbb{R}_+^{\infty}$ instead of $\mathbb{R}_+^{\infty}$, then ID4 has to be strengthened to

**ID5:** For all $(\omega, s)$ in $\text{Dom}(f)$ and for all $x$ in the support of $f(\omega, s)$,

$$\sum_{t \in T} x_t = \sum_{t \in T} \hat{w}_t(\omega).$$
Remarks: Conditions ID4 and ID5 are feasibility conditions. A weaker feasibility notion is presented in Hurwicz, Maskin, and Postlewaite (1980).

Finally, we note that central knowledge should not be confused with common knowledge. If common knowledge is centrally known then condition ID3 is replaced by

**ID3*: For all \( \omega \) and \( \omega' \) in \( \Omega \): If for all \( t \) in \( T \)

\[
\hat{w}_t(\omega) = \hat{w}_t(\omega') \quad \text{and} \quad I(\omega) = I(\omega') \quad \text{then} \quad f(\omega, \cdot) = f(\omega', \cdot).
\]

Common knowledge is disregarded by the institutional rules if a person (or a party) is allowed to vote for \( A \) against \( B \) although it is commonly known that he (it) prefers \( B \) to \( A \). On the other hand we can think of societal rules which are parameterized by common knowledge, i.e., the rule may be different for different common knowledge events. In this case the incentive problem of embodying the specific common knowledge event into the rule is ignored. For example, agreement based on an index of prices assumes that the index will be common knowledge.

6. REVEALATION

Given a N.I. economy \( e \), a SOF \( g = g^e \) is said to be a revelation outcome function (ROF) if for all \( t \) in \( T \) and \( \omega \) in \( \Omega \),

\[
S^g_t(\omega) = \{ R \in \Pi_t \mid \hat{w}_t(\omega) = \hat{w}_t(\omega') \text{ for } \omega' \text{ in } B \}. 
\]
In other words, agent $t$ is called upon to announce the private information which could have been observed by him. Clearly "truth" is a well-defined strategy for each agent.

**Proposition 3:** (Revelation principle). Suppose that a D.I. economy $e$, a SOF $f = f_e$, and a D.I. equilibrium $\sigma^*$ for $(e,f)$ are given with conditions DI1-DI4 and ID2-ID5 satisfied. Then truth is a D.I. equilibrium for $e$ and the following ROF $g$ associated with $e$, $f$, and $\sigma^*$. For all $\sigma^*$ in $\mathcal{G}$ and all $\omega$ in $\Omega$: if for all $t$ in $T$ $\sigma^*_t(\omega) = s^*_t = I_t(\omega') \in S^g_t(\omega)$ then $g(\omega,\sigma^*) = f((\sigma^*_t(\omega'))_{t \in T})$. Furthermore, for every $\omega$ in $\Omega$ the truth yields the same outcome as $\sigma^*(\omega)$.

**Proof:** This form of the revelation principle appears in Postlewaite and Schmeidler (1983). It follows Myerson (1979) and the references there; the principle essentially originates with Gibbard (1973).

Truth means that for all $t$ and $\omega$, $s^*_t = I_t(\omega)$. If it is not an equilibrium act then there is another act $I_t(\omega') \in S^g_t(\omega)$, which is preferred by $t$ to truth at $\omega$. That is, $f_t(\sigma^*(\omega) \mid t, \sigma^*(\omega'))$ is preferred by $t$ to $f_t(\sigma^*(\omega))$ at $\omega$ and this act is feasible at $\omega$. This is a contradiction to the supposition that $\sigma^*$ was an equilibrium.

O.E.D.

There are several drawbacks to the revelation principle. First, for each D.I. equilibrium $\sigma^*$ of $(e,f)$ a separate ROF $g$ is constructed. Second, given such a ROF it may have D.I. equilibria in addition to the truth. Furthermore, these additional equilibria may yield outcomes which are not equilibria outcomes of the original SOF $f$. Such examples are presented by (among others)
Postlewaite and Schmeidler (1983). Another drawback not dealt with here is the possibility that the ROF is much more complicated than the original SOF.

**Proposition 4**: Let \( \sigma \) be a ROF associated with a SOF \( f \), a D.I. economy \( e \), and a D.I. equilibrium \( \sigma^* \) of \( (e,f) \). Suppose that for all \( t \in T \) and all \( \omega \) in \( \Omega \): \([\sigma_t^*(B)|B \in S_t^*(\omega)] = S_t(\omega)\). Then every D.I. equilibrium outcome of \( (e,g) \) is also an equilibrium outcome of \( (e,f) \).

**Proof**: Let \( \sigma^G \) be a D.I. equilibrium of \( (e,g) \) and let \( \sigma' \) be the corresponding strategies list for \( (e,f) \). That is, for all \( t \) and \( \omega \):

\[
\sigma'_t(\omega) = \sigma^*_t(\omega) \quad \text{where} \quad I_t(\omega') = \sigma_t^G.
\]

By definition \( \sigma' \) yields the same outcome as \( \sigma^G \).

If, by way of negation, \( \sigma' \) is not an equilibrium strategy for \( (e,f) \) then for some \( t \), \( \sigma'_t \) is not a best response to \( \sigma^G \). Let then \( \theta_t(\cdot) \) be a better response. But by the condition of the proposition, for each \( \omega \) in \( \Omega \) there exists \( B_t(\omega) \) in \( S_t^*(\omega) \) such that \( \sigma^*_t(B_t(\omega)) = \theta_t(\omega) \). Hence \( \sigma_t^G \) is not response to \( \sigma^G \), a contradiction. Q.E.D.

This result has been independently derived by Repullo [1983]. The question of constructing a separate ROF for every D.I. equilibrium in \( (e,f) \) is discussed in Section 8.

7. **IMPLEMENTATION**

Given a D.I. economy \( e \), a function \( \hat{x}: \Omega \times R^T_+ \) is a **D.I. allocation** (allocation for short) if for all \( \omega \) in \( \Omega \),

\[
\sum_{t \in T} \hat{x}_t(\omega) = \sum_{t \in T} \hat{w}_t(\omega).
\]

We
described above how a collection of complete information economies can be considered a single D.I. economy. In this case a D.I. allocation corresponds to a social welfare (choice) function on the collection of complete information economies. A set $F$ of D.I. allocations is termed a social welfare correspondence, SWC, (given $e$). A SOF $f$ is said to weakly implement the SWC $F$ if for every D.I. equilibrium $\sigma$ for $(e,f)$, $f(\sigma(\cdot)) \in F$, that is, $f(\sigma(\cdot))$ is a D.I. allocation in $F$. The SOF $f$ (faithfully) implements $F$ if $F = \{ f(\sigma(\cdot)) \mid \sigma$ is a D.I. equilibrium for $(e,f) \}$. We say a SWC $F$ is implementable if there exists a SOF $f$ which implements $F$. For completeness we introduce the following definition: A SWC $F$ is locally implementable if for each D.I. allocation $\hat{x} \in F$, there exists a SOF $f$ which weakly implements $F$ and a D.I. equilibrium $\sigma^*$ for $(e,f)$ such that $f(\sigma^*) = \hat{x}$.

We now state two simple observations.

Claim: Let $\sigma^*$ and $\sigma'$ be two D.I. equilibria for some $(e,f)$, and let $\{A, A^c\}$ be a partition of $\Omega$ which is measurable with respect to the common knowledge partition $\Pi$. Define a new list of strategies $\sigma$ by $\sigma(\omega) = \sigma^*(\omega)$ for $\omega \in A$ and $\sigma(\omega) = \sigma'(\omega)$ for $\omega \in A^c$. Then $\sigma$ is a D.I. equilibrium for $(e,f)$.

We say that $\sigma$ is a common knowledge concatenation of $\sigma^*$ and $\sigma'$. Similarly we use the notion of common knowledge concatenation for D.I. allocations.

Corollary: If a SWC $F$ is implementable for a given D.I. economy $e$, then for any $\hat{x}$, $\hat{y} \in F$ any common knowledge concatenation of $\hat{x}$ and $\hat{y}$ must be in $F$. In the sequel we use the notation $F|B$,
for \( B \in \Pi \), to mean the set of allocations in \( F \), each of them restricted to \( R \).

Next a condition on the informational structure of the D.I. economy \( e \) is introduced, which together with a monotonicity condition adjusted to D.I. economies, will yield an implementation result.

**NEI (Nonexclusivity in Information):** For all \( \omega \) in \( \Omega \) and \( t \) in \( T \):

\[
\bigcap_{t' \neq t} I_{t'}(\omega) = \{\omega\}.
\]

No agent has exclusively private information. This condition implies DI5. In the Example of Section 2 a much stronger assumption is satisfied; for all \( \omega \) and \( t \): \( I_t(\omega) = \{\omega\} \).

Given a D.I. economy \( e \) and a SWC \( F \) a monotonicity condition on \( F \) is defined.

**M (Monotonicity):** Given \( B \) and \( B' \) in \( \Pi \) with \( \alpha : B' \rightarrow B \) and given a selection \( \hat{x} \in F|B \) one also has \( \hat{x}(\alpha \omega)_{\omega \in B}, \in F|B' \) whenever the following three conditions hold:

(i) For every \( \omega \in B' \), \( \hat{w}(\omega) = \hat{w}(\alpha \omega) \).

(ii) \( \alpha \) preserves the information structure, i.e., for every \( t \in T \), and for every \( B_t \in \Pi_t, B_t \subseteq B; \alpha^{-1}(B_t) \in \Pi_t \).

(iii) For every \( \hat{y} : B' \rightarrow R^{\Pi} \) such that for all \( \omega' \in B' \), \( \sum_{t \in T} \hat{y}_t(\omega') = \sum_{t \in T} \hat{w}_t(\omega') \), for every \( t \in T \) and
for every \( \omega \in \Omega \), the following implication holds: If

\[
\sum_{\omega' \in \mathcal{I}_t(\omega)} p_t(\omega', \omega) u_t(\omega', \hat{y}_t(\omega')) >
\]

\[
\sum_{\omega' \in \mathcal{I}_t(\omega)} p_t(\omega', \omega) u_t(\omega', \hat{x}_t(\omega'))
\]

then

\[
\sum_{\omega' \in \mathcal{I}_t(\alpha \omega)} p_t(\omega', \alpha \omega) u_t(\omega', \hat{y}_t(\alpha \omega')) >
\]

\[
\sum_{\omega' \in \mathcal{I}_t(\alpha \omega)} p_t(\omega', \alpha \omega) u_t(\omega', \hat{x}_t(\omega'))
\]

A special but nontrivial case of the above condition occurs when \( B = B' \) but \( \alpha \) is not the identity.

**Proposition 5:** A social welfare correspondence \( F \) which is implemented on an economy \( e \) satisfying DI1 to DI4 is monotonic.

The proof follows immediately from the definitions of monotonicity and (faithful) implementation. To simplify matters an additional condition will be imposed on the D.I. economy \( e \).

**AF (Aggregate Feasibility):** For every \( B \) in \( \Pi \) and for every \( \omega \) and \( \omega' \) in \( B \):

\[
\sum_{t \in \mathcal{T}} \hat{w}_t(\omega) = \sum_{t \in \mathcal{T}} \hat{w}_t(\omega'), \quad \#\{t \in \mathcal{T} \mid \hat{w}_t(\omega) \neq 0\} \geq 3.
\]

**Proposition 6:** Given a D.I. economy \( e \) satisfying DI1-DI4, NEI, and AF, a SWC \( F \) satisfying M is implemented by a SOF, assuming free disposal.
Proof: The proof consists of constructing a SOF which implements $F$. Let $N = \{1, \ldots, n\}$ and $\rho : N \rightarrow T$ onto (hence $\rho$ is one to one).

$S_t(\omega) \subseteq S_t^1(\omega) \times S_t^2(\omega) \times S_t^3(\omega) \times S_t^4(\omega)$.

$S_t^1(\omega) = \{B_t \in \Pi_t \mid \omega' \in B_t \implies \hat{w}_t(\omega') = \hat{w}_t(\omega) \text{ and} \}$

$\sum_{t', \in T} (\hat{w}_{t',}(\omega') - \hat{w}_{t',}(\omega)) = 0\}$.

$S_t^2(\omega) = \{\hat{x} \in F \mid B_t \subseteq B \in \Pi \text{ for some } B_t \in S_t^1(\omega), \text{ and} \}$

$\text{for every } \omega' \text{ in } B : \sum_{t', \in T} (\hat{x}_{t',}(\omega') - \hat{w}_{t',}(\omega')) = 0\}.$

$S_t^3(\omega) = N$.

$S_t^4(\omega) = \{0, 1\}$, where $1$ may interpreted as an "objection" by agent $t$.

$S_t(\omega) = \{s_t = (B_t, \hat{x}_t, s_t^1, s_t^2, s_t^3, i_t) \in S_t^1(\omega) \times S_t^2(\omega) \times S_t^3(\omega) \times S_t^4(\omega) \mid B_t \text{ belongs to the domain of } \hat{x}_t\}.$

Given $\omega$ in $\Omega$ and $s = (s_t)_{t \in T}$ in $S(\omega)$ we define $f(\omega, s)$ separately for different subsets of $S(\omega)$. For convenience let us denote by $s^4 = \#\{t \in T \mid s_t^4(\omega) = 1\}$.

1) $s^4 = 0$, $\forall t \in T B_t = \{\omega\}$, and there is $\hat{x} \in F | B$ with $I(\hat{x}) = B$ such that for every $t$ in $T$, $\hat{x} = \hat{x}_t$. In this case $f(s) = \hat{x}(\omega)$. 

2) $s^4 = 0$ but not all of the remaining conditions of (1) hold, i.e., either 
\[ \bigcap_{t \in T} B_t \neq \emptyset \] 
or the agents do not agree on an allocation. In this case 
\[ f(g) = 0, \text{ i.e., all commodities are confiscated.} \]

3) $s^4 = 1 = s^4_t(\omega)$. Furthermore, \[ \bigcap_{t \neq \tau} B_t = \{ \omega \} \] 
and there is \[ \hat{x} \in P|B \] 
with \[ I(\omega) = B \] such that for every \[ t \neq \tau \], \[ \hat{x}^t = \hat{x}^\tau \]. Suppose further that 
\[ \sum_{\omega'} \epsilon B_{\tau} P_\tau(\omega' | \omega) u_\tau(\omega', \hat{x}^\tau(\omega')) \leq \sum_{\omega'} \epsilon I_{\tau}(\omega) P_\tau(\omega' | \omega) u_\tau(\omega', \hat{x}(\omega')). \] 

In this case \( f(g) \) belongs to \( L(R_{+}^T) \): It obtains the same values as \( \hat{x}^\tau \) with the probabilities induced by \( P_\tau(\cdot | \omega) \). That is, the probability of the allocation \( x \) is equal to the sum of \( P_\tau(\omega' | \omega) \) over all \( \omega' \) in \( B_\tau \) such that \( \hat{x}^\tau(\omega') = x \). Feasibility is guaranteed by the condition AF and the definition of \( S^2_T(\omega) \).

4) Same as (3) except that the inequality in (*) is reversed. We denote the reverse inequality \( > \) by (**)\( . \) In this case \( f(g) = \hat{x}(\omega), \) and finally

5) \( s \) does not satisfy either (1), (2), (3) or (4). In this case the remainder of \[ \sum_{t \in T} s^3_t \] 
divided by \( n \) is found, say \( k \), and let \( \rho(\tau) = k \) or if \( k = 0, \rho(\tau) = n \). Then \( f_{\tau}(g) = \sum_{\omega \in T} \hat{w}_\tau(\omega) \) and \( f_{\tau}(g) = 0 \) for \( t \neq \tau \).

Case (5) includes the situation where \( s^4 = 1 \) but the other conditions of cases (3) and (4) do not hold. It also covers the case where \( s^4 > 1 \).

Note that the sum \[ \sum_{t \in T} \hat{w}_\tau(\omega) \] is centrally known and constant over all \( \omega' \) in \( B_\tau \) in \( S^1_T(\omega) \) for all \( t \) by AF and the definition of \( S^1_T(\omega) \).
Furthermore, for every agent $\tilde{t}$ in $T$, given the strategies $s^3_t$ of others he can play $s^1_t$ such that in case (5) $\tilde{t} = t$, i.e., he gets everything.

The last observation implies that, except when $\sum_{t \in T} \hat{w}_t(\omega) = 0$, at equilibrium, given $\omega$, agents play $s^3_t$ of case (1). In cases (3) and (4) every agent except $\tau$ can get everything by changing $B_t$, $\hat{x}_t$ and choosing appropriate $s^3_t$. Clearly no $s^3_t$ of case (2) or (5) can be an equilibrium act.

So given an equilibrium strategy $\sigma^*$, $\sigma^*(\omega) = s^3_t$ is of case (1) (except when the aggregate endowment is 0). If $\sigma^*(\omega)$ is truthful, i.e., for every agent $t$ in $T$ the first component of his act $s_t$ at $\omega$ is $I_t(\omega)$ then $f(\sigma^*)|B \in F|B$. Otherwise on $B \in \Pi$ with $\omega \in B$, the agents play $\sigma^*(\cdot)$ which is not always truthful for everyone but for each $\omega$ in $B$, case (1) holds. Let us denote $\sigma^*_t(\omega) = (\sigma^1_t(\omega), \sigma^2_t(\omega), \sigma^3_t(\omega), \sigma^4_t(\omega))$ where $\sigma^i_t(\omega) \in S^i_t(\omega)$ for $i = 1, 2, 3, 4$. Thus $\hat{\gamma} \in T \sigma^1_t(\omega) = \{\omega\}$ but $\omega \neq \omega$. The outcome is then $\hat{x}(\omega) \in F(\omega)$. Every agent could have switched to a different strategy and received a different outcome according to case (3). Specifically, agent $t$ in $T$ could have played $I_t(\omega)$ instead of $\sigma^1_t(\omega) = I_t(\omega)$ and gotten an outcome which he prefers according to his $I_t(\omega)$ expected utility to $\hat{x}_t(\cdot)$ as long as it is not preferred according to his $I_t(\omega)$ expected utility to $\hat{x}_t(\cdot)$. Since $\sigma^*$ is the equilibrium strategy, i.e., no one benefits by defection, condition (iii) of $M$ is satisfied and $\hat{x}(\omega) \in F(\omega)$.

Hence it has been proven that if $\sigma^*$ is a D.I. equilibrium strategy then for all $B \in \Pi$, $f(\sigma^*)|B \in F|B$.

The other direction is obvious. Given any $\hat{x} \in F$ it can be obtained as an equilibrium outcome if $\sigma^1_t(\omega)$ is chosen to be the truth and $\sigma^2_t(\omega) = \hat{x}|I_t(\omega)$ for all $t$ and $\omega$.  

O.E.D.

Our definition of a social welfare correspondence is a collection of D.I.
allocations. Alternatively there is a concept of a **pointwise social welfare correspondence** \( F : \Omega \rightarrow R_+^T \) where \( F(\omega) \) is a collection of allocation for \( \omega \in \Omega \). Given a SWC \( F \), there is a pointwise SWC \( F^* \) naturally associated with \( F \). Different social welfare correspondences may be associated with a single pointwise SWC. Similarly, we can say that a SOP \( f^* \) pointwise implements a pointwise SWC \( F^* \) if for every \( \omega \in \Omega \): i) for every D.I. equilibrium \( \sigma^* \), \( f^*(\sigma^*(\omega)) \in F^*(\omega) \) and ii) for every \( x \in F^*(\omega) \) there is a D.I. equilibrium \( \sigma^* \) with \( x = f^*(\sigma^*(\omega)) \). It is obvious that Proposition 5 holds with pointwise SWC \( F^* \) substituted for SWC \( F \) and pointwise implements substituted for implements. On the other hand, if a pointwise SWC \( F^* \) is pointwise implementable, there exists at least one SWC \( F \) which is implementable and induces \( F^* \). However, there may exist a nonmonotonic SWC which also induces \( F^* \). Since it is not monotonic it cannot be implemented.

**Remark:** Propositions 5 and 6 extend Maskin's (1977) implementation results as applied to economic environments. (See Hurwicz, Maskin, and Postlewaite (1982)).

8. **EXTENDED REVEALATION**

Given a D.I. economy \( e \) and a SOP \( f \) denote by \( E^* \) the set of D.I. equilibria for \( (e,f) \). Given an equilibrium \( \sigma^* \) in \( E^* \) it was shown in Section 6 how to construct a revelation outcome function \( g = g(e,f,\sigma^*) \) where truth is a D.I. equilibrium of \( g \) resulting in the same outcome as \( \sigma^* \).

Here we construct a SOP \( h = h(e,f,E^*) \) associated with \( e,f \) and the set of all D.I. equilibria of \( (e,f) \). It will be referred to as the extended revelation outcome function, EROF for short. The participants in the EROF are
asked to reveal their private information and the equilibrium strategy 
$\hat{\sigma}^* \in \Sigma^*$ they agreed upon. In our interpretation the agents discuss and reach 
a nonbinding agreement about the equilibrium strategy to be used in the 
revelation game. Thus "extended truth" means the true private information and 
the equilibrium strategy agreed upon unanimously.

Formally we define for all $t$ in $T$ and all $\omega$ in $\Omega$:

$$S^h_t(\omega) = \{ B \in \Pi_t | \hat{\omega}_t(\omega) = \hat{\omega}_t(\omega') \text{ for } \omega' \text{ in } B \} \times \Sigma^*.$$ 

The first term in the definition of $S^h_t(\omega)$ is $S^g_t(\omega)$ of Section 6, 
Proposition 3. For a given $\omega$ in $\Omega$ and $t$ in $T$ let $\hat{\sigma}^h_t(\omega) = (I_t(\omega'), \hat{\sigma}^* t)$ 
in $S^g_t(\omega)$, then $h(\omega, \hat{\sigma}^h_t(\omega)) = f ((\hat{\sigma}^* t(\omega'))_{t \in T})$.

**Proposition 7:** (Extended revelation principle). Extended truth is a D.I. 
equilibrium of the EROF $h = h(e, f, \Sigma^*)$ defined above and the 
D.I. economy $e$. The extended truth equilibria outcomes 
coincide with the equilibria outcomes of $(e, f)$.

The proof is almost identical to that of Proposition 3. We can also 
state the condition analogous to that of Proposition 4 that prevents 
equilibria outcomes additional to those of the extended truthful revelation. 
Indeed the condition in Proposition 8, which follows, is weaker than that of 
Proposition 4 since the set of all equilibria strategies may use more acts 
than used by one specific equilibrium in this set.

**Proposition 8:** If for all $\omega$ in $\Omega$ and all $t$ in $T$ every act in $(e, f)$ 
of $t$ at $\omega$ is used by some equilibrium strategy of $t$ at
\[\omega,\] then the outcomes of all the D.I. equilibrium of the EROF h and e coincide with those of (e,f).

The proof of this Proposition is a repetition of the proof of Proposition 4. The only difference is in the statement of the condition. In the present proposition we have for all t and \(\omega\) \(S_t(\omega) = \{ B | B \in S_t^h(\omega) \} \) and \(\alpha \in \Delta^* \). In Proposition 4 there was a unique \(\alpha^* \) on the right hand side of the equality.

**Remark:** Proposition 3, 4, 7, and 8 hold in the original Harsanyi framework of Bayesian games where the game is given in utility payoffs rather than in our decomposed form. Proposition 3 reduces to Myerson's [1979] result in Harsanyi's framework. For related results under complete information see Maskin [1983].

9. **CONCLUDING REMARKS**

In our model of a differential information economy e presented in Section 2, every agent \(t\) has a von Neumann-Morgenstern utility function \(\hat{u}_t(\omega,x)\) and a conditional probability \(P_t(\omega,\omega')\) for all \(\omega \in \Omega\), or equivalently, for all \(\omega \in I_t(\omega')\). Every use of these characteristics involves only their product \(\hat{u}_t(\omega,x)P_t(\omega,\omega') \equiv \hat{v}_t(\omega,x)\). Since \(P_t(\omega,\omega') = 0\) for \(\omega \notin I_t(\omega')\) and \(P_t(\omega,\omega') = P_t(\omega'',\omega')\) for \(\omega'', \omega \in I_t(\omega')\), the substitution of the above product by \(\hat{v}_t(\omega,x)\) is possible on all of \(\Omega\). For all \(\omega' \in \Omega\), instead of \(\sum_{\omega \in I_t(\omega')}P_t(\omega,\omega')\hat{u}_t(\omega,x)\) we can write \(\sum_{\omega \in I_t(\omega')}\hat{v}_t(\omega,x)\). Thus we can formally substitute into our definition of a differential information economy

\[e = (\Omega, (\hat{\omega}_t)_{t \in T}, (\hat{u}_t)_{t \in T}, (P_t(\cdot, \cdot))_{t \in T})\]
the alternative

\[ e = (\Omega, (\hat{\omega}_t)_{t \in T}, (\hat{v}_t)_{t \in T}, (\Pi_t)_{t \in T}) \]

where \( \hat{v}_t \) is defined as above and \( \Pi_t \) is as defined in Section 2. This observation is due to Myerson (1983) (and the references there) for Harsanyi incomplete information games.

Note that the invariance properties of \( \hat{v}_t \) are the same as of \( \hat{u}_t \). The economy is invariant under positive linear transformations of the functions \( \hat{v}_t(\omega, \cdot) \), \( t \in T \), \( \omega \in \Omega \). The transformations may differ for different \( t \) in \( T \) and for any \( t \) they may differ for \( \omega \) and \( \omega' \) if \( \omega \notin \Pi_t(\omega') \). For \( t \in T \) and an element \( B_t \in \Pi_t \), if \( \omega_1 \in B_t \) \( i = 1, 2 \), then the transformations for \( \hat{v}_t(\omega_1, \cdot) \) and \( \hat{v}_t(\omega_2, \cdot) \) must be identical. The same invariance property holds for our original formulation with the functions \( \hat{u}_t(\omega, \cdot) \), \( t \in T \), \( \omega \in \Omega \).

We have presented our model in the more standard form separating the utilities and the probabilities for expositional purposes.

The second issue we wish to discuss is the relationship between our implementation results and the literature on partial equilibrium models using Bayes equilibrium as the solution concept, e.g., the literature on implicit contracts, the principal-agent problem, auctions, etc. In these latter models some agents typically have private information; at least one agent can distinguish two states which cannot be distinguished by any other agents. Thus these informational structures do not satisfy our pairwise nonexclusivity of information assumption. Hence our theorem giving sufficient conditions for implementation does not apply. These models rely on self-selection for implementation. If the social welfare correspondence gives an agent the proper incentive to "reveal" his information, it is not necessary that any other agent
possess this information. Our sufficiency results focus on the informational structure. Note, however, that monotonicity remains a necessary condition independent of the informational structure and the number of agents. Moreover, it is obviously implied by a careful statement of self-selection.

Consider a differential information economy $e$ and a social welfare correspondence $F$. We say agent $\tau$ has **exclusively complete information** (ECI) on a common knowledge event $B$ if for any $\omega \in B$, $I_{\tau}(\omega) = \{\omega\}$ and $I_{\tau}(\omega) = B$, $\tau \neq \tau$, i.e., agent $\tau$ can distinguish all elements of $B$ and no other agent can distinguish any elements of $B$.

We will now define a concept of self selection for an agent $\tau$ with exclusively complete information on an event $B \in \Pi$.

**SS (Self Selection):** A SWC $F$ satisfies self selection on $B \in \Pi$ for $\tau \in T$ if for any $\omega_1, \omega_2 \in B$, $x_1 \in F(\omega_1)$, $x_2 \in F(\omega_2)$, $u_\tau(\omega_1, x_1) \geq u_\tau(\omega_1, x_2)$.

**Claim:** Consider a D.I. economy $e$, a SWC $F$, and an agent $\tau$ with exclusively complete information on $B \in \Pi$. (i) If $F$ satisfies SS on $B$ for $\tau$, then (*) for all $\omega \in B$ and $x, y \in F(\omega)$, $u_\tau(\omega, x) = u_\tau(\omega, y)$ (ii) Further, the condition (*) together with monotonicity of $F$ implies SS holds for $F$ on $B$ for $\tau$.

The proof is a straightforward application of the definitions.

As mentioned above, monotonicity is a necessary condition for implementation independent of the informational structure and number of agents. We will now extend Proposition 6 to include the case of exclusively complete information. The extended result will include as a special case two person economies.
in which one has strictly better information (a strictly finer partition).
Thus our result applies to standard models of implicit contracts, principal-
agent models, etc.

**Proposition 9:** Consider a differential information economy $e$ satisfy DI1-
DI4, AF, and a SWC $F$ satisfying $M$. Further, suppose that
for each common knowledge event $B \in \Pi$ either (i) NEI is
satisfied when restricted to $B$ or (ii) there exists an
agent $\tau$ who has exclusively complete information on $B$ and
for whom condition (*) of the claim holds.

Then $F$ is implementable.

The concept NEI restricted to $R$ is self explanatory. The proof of
Proposition 9 follows that of Proposition 6 with minor changes.
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