BÄCKLUND TRANSFORMATIONS AND THE PAINLEVÉ PROPERTY

By

John Weiss

IMA Preprint Series # 509

April 1989

BÄCKLUND TRANSFORMATIONS AND THE PAINLEVÉ PROPERTY

JOHN WEISS*

Abstract. For systems with the Painlevé Property, Bäcklund transformations can be defined. These appear as specializations (truncations) of certain expansions of the solution about its' *singular manifold*. With reference to the Lax pair for a system, the Bäcklund transformations are equivalent to transformations of linear systems developed by Darboux (Bäcklund-Darboux transformations).

For specific systems the Bäcklund-Darboux transformations lead to a reformulation of these systems in terms of the Schwarzian derivative. We find the Bäcklund transformations of these system and study their periodic fixed points.

The periodic fixed points of the Bäcklund transformations determine a finite dimensional invariant manifold for the flow of the system. The resulting (ordinary) differential equations have a hamiltonian structure and the flow of the (partial) differential system is represented by commuting flows on the finite dimensional manifold.

Contents

1. Introduction: The Painlevé Property. We are interested in the behavior of solutions of nonlinear ordinary and partial differential equations. Amoung the simpler properties that a solution can have is whether the solution is single or multiple valued as a function of its' (complex) independent variables. We ask, at least locally, where the solution lives. Historically, this question was found to be interesting.

Kovalevsky [1] found that when the solutions of the spinning top equations are single valued the equations are completely integrable. Painlevé [2] considered a wide class of second order equations and classified these according to the nature of their singularities. Since the coefficients of the equations are allowed to depend on the independent variable, *fixed* singularities can arise at the fixed locations of singularities of the coefficients. Painlevé and his coworkers found essentially six different equations within the class considered whose solutions are single valued as functions of the complex independent variable, except possibly at the fixed singularities of the coefficients. These are known as the Painlevé transcendents

*6 Lockeland Avenue Arlington, Mass 02174

and have a great variety of interesting properties and applications. Singularities that are not fixed are said to be *movable.*

An ordinary differential equation is said to have the Painlevé Property when every solution is single valued, except at the fixed singularities of the coefficients. That is, the Painlevé Property requires that the movable singularities are no worse than poles.

Ablowitz et al [3] found that when certain integrable partial differential equatins have reductions to ordinary differential equations, the ordinary differential equations have the Painlevé Property. They conjecture that when a system is *integrable* its' reductions will have the Painlevé Property. This conjecture is supported by the results of McLeod and Olver [4]. Integrable here means there exists a nontrivial Lax Pair for the system.

The major difference between analytic functions of one complex variable and several complex variables is that, in general, the singularities of a function of several complex variables cannot be isolated [5]. If $f = f(z_1, \cdots, z_n)$ is a meromorphic function of n complex variables ($2n$ real variables), the singularities of f occur along analytic manifolds of (real) dimension $2n - 2$. These manifolds are determined by conditions of the form

$$\phi(z_1, \cdots, z_n) = 0, \tag{1.1}$$

where ϕ is an analytic function of $(z_1, \cdots, z_n)$ in a neighborhood of the manifold.

With reference to the above, we say that [6] *a partial differential equation has the Painlevé Property when the solutions of the PDE are single valued about the movable, singularity manifolds.* For partial differential equations we require that the solution be a *single-valued functional* of the data, i.e. *arbitrary functions*. This is a formal property and not a restriction on the data itself.

To verify if a PDE has the Painlevé Property we [6] introduce a method for expanding a solution of a nonlinear PDE about a movable, singular manifold (1.1). Let $u = u(z_1, \cdots, z_n)$ be a solution of the PDE and assume that

$$u = \phi^\alpha \sum_{j=0}^{\infty} u_j \phi^j \tag{1.2}$$

where ϕ and

$$u_j = u_j(z_1, \cdots, z_n)$$

are analytic functions of $(z_1, \cdots, z_n)$ in a neighborhood of the manifold (1.1). Substitution of (1.2) into the PDE determines the possible values of α and defines the recursion relations for $u_j, j = 0, 1, 2, \cdots$. When α is an integer and (1.2) is a valid and general expansion about the manifold (1.1), then the solution has a single valued representation about (1.1). If this representation is valid for all allowed movable singularity manifolds then the PDE has the Painlevé Property. For a specific PDE it is necessary to identify all possible values for α and then find what the form of the resulting *Psi* series [7] is.

A point that we will emphasize is that the Psi series for nonlinear PDE contain a lot of information about the solutions of the PDE. For equations which have the Painlevé property we have developed a method for finding the Lax pairs and Bäcklund transformations [8,9,10]. An outline of the ***singular manifold method*** is presented in section 2. For equations that do not have the Painlevé Property it is still possible to obtain single valued expansions by specializing the arbitrary functions that appear in the Psi series expansions. This specialization leads to a system of partial differential equations for the formally arbitrary data. For specific systems, and we conjecture in general, these equations are integrable. The form of the resulting reduction enables the identification of integrable reductions of the original system [11]. This is examined in section 3.

To illustrate the nature of the Painlevé Property it is worthwhile to examine a few examples of equations with and without the Painlevé Property.

A simple case of an equation with the Painlevé Property is **Burgers' Equation.**

$$u_t + uu_x = u_{xx} \tag{1.3}$$

It is not difficult to find the Psi series

$$u = \phi^{-1} \sum_{j=0}^{\infty} u_j \phi^j \tag{1.4}$$

is valid for (1.3). Examination of the recursion relations for the u_j obtains a system of the form

$$(j-2)(j+1)\phi_x^2 u_j = F_j(u_{j-1}, \cdots, u_0, \phi_t, \phi_x, \cdots).$$

For (1.4) to be valid F_2 must vanish identically. Evaluation of the recursions obtains

$$j = 0,\ u_0 = -2\phi_x$$

$$j = 1,\ \phi_t + u_1\phi_x = \phi_{xx}$$

$$j = 2,\ \partial_x(\phi_t + u_1\phi_x - \phi_{xx}) = 0.$$

The relation (compatibility condition) at $j = 2$ is satisfied identically and the expansion (1.4) is valid with *arbitrary* functions ϕ and u_2.

The Korteweg-de Vries equation

$$u_t + u_{xxx} + 3uu_x = 0 \tag{1.5}$$

has singularities of the form

$$u = \phi^{-2} \sum_{j=0}^{\infty} u_j \phi^j \tag{1.6}$$

with arbitrary functions ϕ, u_4 and u_6. The KdV equation has the Painlevé Property about singularities of the form (1.6).

The Schwarzian KdV equation [8]

$$\frac{\psi_t}{\psi_x} + \{\psi; x\} = \lambda, \tag{1.7}$$

where

$$\{\psi; x\} = \frac{\psi_{xxx}}{\psi_x} - \frac{3}{2}(\frac{\psi_{xx}}{\psi_x})^2$$

is the Schwarzian derivative , has singularities of the form

$$\psi = \phi^{-1} \sum_{j=0}^{\infty} \psi_j \phi^j \tag{1.8}$$

with arbitrary ϕ, u_0, u_1 if the non-characteristic condition $\psi_x \simeq \phi_x \neq 0$ is verified. If $\psi_x = 0$ the expansion about the *characteristic manifold* has the form

$$\psi = f(t) + \phi^3 \sum_{j=0}^{\infty} \psi_j \phi^j, \tag{1.9}$$

where $\phi = x - x_0(t)$ and $\psi_j = \psi_j(t)$.

The Schwarzian KdV equation has single valued expansions about both characteristic and noncharacteristic manifolds. *The Painlevé Property requires all movable singularity manifolds to be single valued, whether characteristic or not.* The above result runs counter to the observation of Ward [12,13] that direct consideration of expansions about characteristic manifolds cannot be allowed in the definition of the Painlevé Property since, for linear systems, *bad* singularities propagate along characteristics. For general systems, expansions about characteristics, when they exist, introduce certain arbitrary data [14]. If the data is *bad*, the expansion is still required to be a single valued **functional** of that data. In this sense, expansion (1.9) is a single valued functional of the data $f(t)$, $x_0(t)$, however multiple valued that data as a function of t may be. Of course, the same observation applies to the non-characteristic expansion (1.8). *The Painlevé Property is a statement of how the solutions behave as functionals of the data in a neighborhood of a singularity manifold and not a statement about the data itself.* This is an important point and should not be overlooked. The following example will illustrate this point.

A derivative Schwarzian equation

$$\frac{\psi_t}{\psi_x} + \frac{\partial}{\partial x}\{\psi; x\} = 0 \tag{1.10}$$

has non-characteristic singularities of the form

$$\psi = \phi^{-1} \sum_{j=0}^{\infty} \psi_j \phi^j \tag{1.11}$$

,where $\phi, \psi_0, \psi_1, \psi_2$ are arbitrary. Therefore, (1.10) has a single valued expansion depending on the maximum number of arbitrary functions allowed for by the order of the equation. However, about the characteristic manifold where $\psi_x = 0$

$$\psi = f(t) + \phi^4 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \psi_{jk} \phi^j \phi^{k\alpha} \tag{1.12}$$

,where $\alpha = \frac{7}{2} + \imath\sqrt{11}/2$ and $\phi = x - x_0(t)$. The expansions (1.12) are highly multilple valued *as functionals of* ϕ.

In general, to verify that an equation has the Painlevé Property it is necessary to show that **all** the allowed singularities are single valued (as functionals of the data). This requirement is also easily overlooked and has lead to some wrong conclusions.

Doktorov and Sakovich [15] claim that

$$\Delta\psi = P_n(\psi) \tag{1.13}$$

has the Painlevé property for any polynomial of degree n since if $\psi = u^{-1}$ then the equation for u has single valued poles depending on two arbitrary functions and therefore has the Painlevé property. In effect, they are only claiming that (1.13) has a single valued expansion about the simple zeros of ψ as is locally valid by the Cauchy-Kovalevsky Theorem. It is not difficult to see that (1.13) has singularities that are not single valued. For instance, the reduction to an ODE is within the class originally studied by Painlevé.

It is also thought that the **Clarkson equation**

(1.14) $$u_t^2 = 2uu_x^2 - (1+u^2)u_{xx}$$

has only meromorphic psi-series and has Painlevé Property [16,17]. However, this is not the case. Consider the points where

(1.15) $$u^2 + 1 = 0.$$

Rewriting the above we have

$$u_t^2 - 2iu_x^2 = (u-i)(2u_x^2 - (u+i)u_{xx})$$

and letting $G = u + i$

$$G_t^2 - 2iG_x^2 = (G-2i)(2G_x^2 - GG_{xx}).$$

To leading order

$$G = 2i + G_0\phi^\alpha + \cdots$$

where $\Re\alpha \geq 0$.
By substitution in the above

$$\alpha^2 G_0^2(\phi_t^2 - 2i\phi_x^2)\phi^{2\alpha-2} =$$
$$\alpha(\alpha+1)G_0^3\phi_x^2\phi^{3\alpha-2} - 2i\alpha(\alpha-1)G_0^2\phi_x^2\phi^{2\alpha-2}.$$

Since $\Re\alpha \geq 0$ we have

$$\alpha = 2i\phi_x^2/\phi_t^2.$$

For this leading order the balance equations are

$$G_t^2 + 2i(GG_{xx} - G_x^2) = 0$$

and the resonances

$$G = 2i + G_0\phi^\alpha + G_1\phi^{\alpha+r}$$

obtains

$$\alpha(\alpha + r)\phi_t^2 + i\{\alpha(\alpha - 1) + (\alpha + r)(r - \alpha - 1)\}\phi_x^2 = 0.$$

Using the leading order

$$r = -1, 0.$$

Thus G_0 is arbitrary. From the above, the Clarkson equation has a movable essential singularity and does not have the Painlevé Property.

A variant of the preceeding, a **modified Clarkson Equation**, [18] also has non-meromorphic singularities. The equation is

$$u_t = (1 + u^2)u_{xx} + (1 - 2u)u_x^2. \tag{1.16}$$

Again, expand about the points where $1 + u^2 = 0$. To see this, let $G = u + i$ and find

$$G_t = (G - 2i)(GG_{xx} - 2G_x^2) + (1 - 2i)G_x^2.$$

Let $G = 2i + G_0\phi^\alpha + \cdots$ and get to leading order

$$\alpha = 2i$$

and a resonance at $r = 0$.

The equation

$$(1 + u^2)u_{xx} = (2u - 1)u_x^2 \tag{1.17}$$

is the traveling wave of (1.14) and the steady state of (1.16). Thi equation was shown by Painlevé to have the general solution

$$u = \tan(\log(ax + b)).$$

The movable essential singularities are shown by the previous examples.

The steady state of (1.14) is,

$$(1 - u^2)u_{xx} + 2uu_x^2 = 0. \tag{1.18}$$

The paraphrase of the analysis for the first example detects no non-meromorphic singularity. The general solution is found to be

$$u = \tan(ax + b).$$

An example of an equation with a non-constant resonance is the **Rand equation** [18]

$$u^2 u_{xxx} = 3u_t^3 \tag{1.19}$$

It has the leading order $u = u_0\phi^\alpha + \cdots$ where it can be shown that

$$(\alpha - 1)(\alpha - 2) = 3(\phi_t^3/\phi_x^3)\alpha^2.$$

This quadratic equation for α determines the leading order. Of course α is a non-constant functional of ϕ_t and ϕ_x.

The resonance condition

$$u = u_0\phi^\alpha + u_1\phi^{\alpha+r} + \cdots$$

easily determines the resonances

$$r = -1, 0, 4 - 3\alpha.$$

It is the case here that one resonance, $4 - 3\alpha$, is a functional of the singular manifold, ϕ.

Finally, we consider the **inviscid Burgers equation**

$$u_t + uu_x = 0 \tag{1.20}$$

has a leading order of the form

$$u = u_0 + u_1\phi^\alpha + u_2\phi^{2\alpha} + \cdots.$$

By substitution into the above a solution is

$$\alpha = 1/2$$

and

$$\phi_t + u_0 \phi_x = 0.$$

The next term in the expansion is

$$u_{0,t} + u_0 u_{0,x} + 1/2 \phi_x u_1^2 = 0.$$

If $u_{0,t} + u_0 u_{0,x} = 0$ then $u_1 = u_2 = \cdots = 0$ and from the above

$$\phi_x^2 \phi_{tt} - 2\phi_x \phi_t \phi_{xt} + \phi_t^2 \phi_{xx} = 0. \tag{1.21}$$

This equation can be linearized by a Legendre transformation, as we shall show in section 3.

In the preceding paragraphs our intent is to illustrate both the definition of the Painlevé Property and the variety of singularities revealed by the functional Psi series. This approach is capable of substantial generalization. In this paper we will, for the most part, describe the applications to integrable systems.

2. The Singular Manifold Method. If an equation has the Painlevé Property we propose to calculate the Bäcklund transformations, Lax pair, Modified equations and Miura transformations through the expansions of the solutions about the singularity manifold. The *Singular Manifold Method* consists in truncating the Laurent Psi series after the *constant level* term. By construction, this forms a possible Bäcklund transformation. Depending on the distribution of *resonances*, a generally overdetermined system of equations are defined by the recursion relations for the coefficients of the Laurent expansion. Reduction of this system to consistent form defines the Bäcklund-Darboux transformation, the Schwarzian form of the modified equation and the related Miura transformation to the original system. The Lax pair can be found by linearizing the Miura transformation and modified equation, using the invariance of the Schwarzian derivative under the Moebius group to motivate the substitution for *linear* variables. The invariance under the Moebius group and the *discrete* symmetries of the modified equations are found to constitute a nontrivial Bäcklund transformation for these systems. We will examine the use of these transformations in finding explicit solutions, i.e. rational, finite-zone. We will also present the two cases where the method finds partial results. In one case, the Bullough-Dodd equation does not have a Bäcklund transformation, and in the other case, the Bäcklund transformation for the *modified Nonlinear Schrodinger* equations is a reduction of the system. Plausible extensions of the method, including applications to ODEs, are examined.

Since this is a method for the discovery of the structures associated with a given equation, we present the analysis for the KdV equation in detail.

The Korteweg-de Vries equation

(2.1) $$u_t + u_{xxx} + 3uu_x = 0$$

has meromorphic singularities of the form

(2.2) $$u = \phi^{-2} \sum_{j=0}^{\infty} u_j \phi^j$$

about the singularity manifold $\phi(x,t) = 0$. In the above we find that ϕ , u_4 and u_6 are arbitrary functions, and it is required that, for (2.2) to be well defined, ϕ be *non-characteristic.* That is, $\phi_x \neq 0$ when $\phi = 0$. For locally analytic data we will show that (2.2) converges in a neighborhood of $\phi = 0$.

It is also of interest to consider the slight generalization of (2.1)

(2.3) $$(\partial/\partial x)(u_t + u_{xxx} + 3uu_x) = 0$$

which has the expansion (2.2) with arbitrary functions ϕ, u_4, u_5 and u_6.

Now, we truncate (2.2) after the constant term to obtain

(2.4) $$u = u_0/\phi^2 + u_1/\phi + u_2.$$

Substitution into (2.1), or (2.3), obtains a system of four equations in the four functions ϕ, u_0, u_1, u_2. This is most readily seen for (2.3) since setting the *arbitrary* functions $u_4 = u_5 = u_6 = 0$ and requiring $u_3 = 0$ obtains (2.4) and four equations. From these we find

(2.5) $$u = -4\phi_x^2/\phi^2 + 4\phi_{xx}/\phi + u_2$$

(2.6) $$u_2 + \lambda = -(\phi_{xxx}/\phi_x) + \frac{1}{2}(\phi_{xx}/\phi_x)^2$$

(2.7) $$\phi_t/\phi_x + \{\phi; x\} = \lambda$$

where u and u_2 satisfy the KdV equation. Again,

$$(2.8) \qquad \{\phi; x\} = \phi_{xxx}/\phi_x - \frac{3}{2}(\phi_{xx}/\phi_x)^2,$$

the Schwarzian derivative [7,8], is the unique differential invariant [19] of the Moebius group

$$(2.9) \qquad \phi = (a\psi + b)/(c\psi + d).$$

The Moebius group is the unique group of conformal (monodromny preserving) automorphisms of the complex ϕ (Riemann) sphere.

The relations (2.6) and (2.7) imply that u_2 satisfy (2.1) since

$$u_{2,t} = -(\frac{\partial}{\partial x} + V)\frac{\partial}{\partial x}(\frac{\partial}{\partial x} - V)\frac{\phi_t}{\phi_x}$$

where

$$(2.10) \qquad V = \frac{\phi_{xx}}{\phi_x}.$$

This definition of V obtains the modified KdV equation

$$(2.11) \qquad V_t + V_{xxx} - \frac{3}{2}V^2 V_x = \lambda V$$

from (2.6).

The Bäcklund-Darboux equation (2.5) may be written in the form

$$(2.12) \qquad u = 4\frac{\partial^2}{\partial x^2}\ln\phi + u_2$$

where

$$(2.13) \qquad u_2 + \lambda = -\frac{\partial}{\partial x}(\phi_{xx}/\phi_x) - \frac{1}{2}(\phi_{xx}/\phi_x)^2$$

and ϕ satisfies the Schwarzian-KdV equation (2.6).

We regard (2.13) as a *Miura transformation* from (2.6) to (2.1). It also has the form of a *Ricati equation* in the variable $W = \phi_{xx}/\phi_x$ and can be linearized by the substitution $W = -2v_x/v$. This obtains the linear equation for v

$$(2.14) \qquad 2v_{xx} = (u_2 + \lambda)v$$

and the identification $\phi_x = v^{-2}$. From (2.6) the additional linear equation

$$v_t = (2\lambda - u_2)v_x + \frac{1}{2}u_{2,x}v \tag{2.15}$$

is found. By construction (2.14) and (2.15) are the Lax pair for the KdV equation and imply u_2 is a KdV solution.

The linearizing substitution for ϕ has the form

$$\phi = v_1/v_2$$

where v_1 and v_2 are solutions of (2.14).

In terms of the linear equation (2.14) the Bäcklund transformation is the classical Darboux transformation for adding elements to the spectra [20,21].

Now, consider the Bäcklund transformations for the Schwarzian KdV equation (2.6). From the invariance of the Schwarzian derivative we have the invariance of (2.6) under the Moebius group (2.9). This invariance is also found by examining the singularities of (2.6), (1.8), and finding that the truncated expansion

$$\phi = \psi^{-1}\phi_0 + \phi_1 \tag{2.16}$$

requires constant ϕ_0, ϕ_1 and ψ satisfies (2.6). In other words, we find the Moebius invariance. An additional Bäcklund transformation is found from the *discrete symmetries* of the modified equation (2.11). That is, $V \Rightarrow -V$ implies the invariant transformation

$$\phi_x = 1/\psi_x \tag{2.17}$$

for (2.6). The time dependent form of this transformation is found from (2.6) and (2.17). That is,

$$\phi_t/\phi_x + \psi_t/\psi_x + (\phi_{xx}/\phi_x)(\psi_{xx}/\psi_x) = 2\lambda \tag{2.18}$$

and (2.17) imply by $\phi_{xt} = \phi_{tx}$ and $\psi_{xt} = \psi_{tx}$ that ϕ and ψ both satisfy (2.6). Therefore, we have a strong Bäcklund transformation for (2.6). In section 4 we will show how the periodic fixed points of the Bäcklund transformations for (2.6) define finite dimensional invariant manifolds and as commuting hamiltonian flows *factor* the KdV flow on this invariant manifold.

Using the Bäcklund transformations, it is simple to show that the Laurent expansion

$$\phi = \xi^{-1}\phi_0 + \phi_1 + \cdots$$

converges in a neighborhood of $\xi(x,t) = 0$. The symmetry $\psi = \phi^{-1}$ maps the *pole* into a simple *zero* and the data satisfies the conditions of the *Cauchy-Kovalevsky Theorem*, being non-characteristic and locally analytic. Therefore, both expansions converge in a (punctured) neighborhood. Furthermore, the symmetry (2.17) maps the characteristic, non-conformal singularity

$$\phi = f(t) + \xi^3 \phi_3 + \cdots$$

into a simple pole

$$\psi = \xi^{-1}\psi_0 + \cdots$$

and by the previous argument , this also converges. Using this result and the relation between the modified and KdV systems, the pole singularities for the KdV equation also converge. The characteristic singularities for the KdV are trivial.

The KdV (2.1) and modified KdV (2.11) have a hamiltonian structure [9,10]

$$u_t + \frac{\partial}{\partial x}(u_{xx} + \frac{3}{2}u^2) = 0 \tag{2.19}$$

$$V_t + \frac{\partial}{\partial x}(V_{xx} - \frac{1}{2}V^3) = 0 \tag{2.20}$$

and are connected by the Miura transformation

$$u = \pm V_x - \frac{1}{2}V^2. \tag{2.21}$$

Note

$$u_{xx} + \frac{3}{2}u^2 = \delta_u \int (-u_x^2 + \frac{1}{2}u^3)dx$$

$$V_{xx} - \frac{1}{2}V^3 = \delta_V \int (-V_x^2 - \frac{1}{8}V^4)dx.$$

Using the Miura transformation finds the *second hamiltonian* structure for the KdV equation from the first hamiltonian structure of (2.20). By the change of variable formula $\Omega = (\delta_V u)\partial_x(\delta_V u)^t$. It is

$$\Omega = \partial_x^3 + 2u\partial_x + u_x = (\partial_x - V)(\partial_x)(\partial_x + V) \tag{2.22}$$

where $u = V_x - \frac{1}{2}V^2$ and the KdV equation is

$$u_t + \Omega\delta_u H_2 = 0$$

where $H_1 = \int \frac{1}{2}u^2$.

The gradient of the integrals for the KdV equation satisfy the Lenard recursion formula

$$\partial_x \delta_u H_{j+1} = \Omega \delta_u H_j \tag{2.23}$$

and the higher order equations are

$$u_t + \partial_x \delta_u H_j = 0 \tag{2.24}$$

for $j = 1, 2, \cdots$. Now, putting together the above we have the following result [9].

The sequence of *higher-order KdV equations*

$$u_t + \partial_x \delta_u H_{j+1} = 0 \tag{2.25}$$

for $j = 1, 2, 3, \cdots$ has the Bäcklund-Darboux transformation

$$u = 4\frac{\partial^2}{\partial x^2} \ln \phi + u_2 \tag{2.26}$$

where

$$u_2 = -\frac{\partial}{\partial x}(\phi_{xx}/\phi_x) - \frac{1}{2}(\phi_{xx}/\phi_x)^2 \tag{2.27}$$

and

$$\frac{\phi_t}{\phi_x} + \delta_u H_j(\{\phi; x\}) = 0. \tag{2.28}$$

Furthermore,

$$u_3 = \{\phi; x\}$$

and u_2 satisfy (2.25) and the sequence (2.28) is invariant under the Moebius group and the symmetry (2.17).

Note that the first few gradients are:

$$\delta_u H_1 = u$$

$$\delta_u H_2 = u_{xx} + \frac{3}{2}u^2$$

$$\delta_u H_3 = u_{xxxx} + 5uu_{xx} + \frac{5}{2}u_x^2 + \frac{5}{2}u^3.$$

Now, using a simple leading order argument and the Bäcklund transformations to raise and lower the *weight* of the Laurent expansions it is not difficult to see that the higher-order systems must have the Painlevé Property and the formal Laurent expansions converge in a punctured neighborhood of the singularity manifold when the data is locally analytic in this neighborhood [9].

To summarize, the Painlevé expansion truncated after the constant level term defines a form of modified equation that is expressed in terms of the Schwarzian derivative.

By linearizing the Miura transformation the Lax pair is found. The discrete symmetries of the modified equations and the Moebius group are Bäcklund transformations for the Schwarzian equations. The Miura transformation allows the calculation of second hamiltonian structures and tha associated recursion operators for the gradients of conserved densities. The action of Báckund transformations on the singularity structure of equation sequences allows the conclusion that the sequence is Painlevé and the formal Laurent expansions converge.

The analysis for the Boussinesq equation [10]

$$u_{tt} + \frac{\partial^2}{\partial x^2}(\frac{1}{3}u_{xx} + u^2) = 0 \tag{2.29}$$

finds the Bäcklund - Darboux transformation

$$u = 2\frac{\partial^2}{\partial x^2}\ln\phi + u_2 \tag{2.30}$$

where $\phi = v_1/v_2$ and $v_1,\ v_2$ satisfy

$$4v_{xxx} + 6uv_x + 3(u_x + h)v = 0$$

$$v_t = v_{xx} + (u+\lambda)v$$

$$h_x = u_t.$$

The Schwarzian modified equation is

$$\frac{\partial}{\partial t}(\phi_t/\phi_x) + \frac{1}{3}\frac{\partial}{\partial x}(\{\phi;x\} + \frac{3}{2}(\phi_t/\phi_x)^2) = 0. \tag{2.31}$$

The discrete symmetry for this system has the form

$$\frac{\phi_{xx}}{\phi_x} = -\frac{1}{2}\frac{\psi_{xx}}{\psi_x} \mp \frac{3}{2}\frac{\psi_t}{\psi_x}$$

$$\frac{\phi_t}{\phi_x} = \pm\frac{1}{2}\frac{\psi_{xx}}{\psi_x} - \frac{1}{2}\frac{\psi_t}{\psi_x}. \tag{2.32}$$

The singular manifold method can also be applied to ordinary differential equations [22,23]. For instance, the Bäcklund transformation for the Henon-Heiles system

$$x_{tt} = -Ax - 2dxy$$

$$y_{tt} = -By + cy^2 - dx^2$$

with $d/c = -\frac{1}{6}$ has the form

$$x = \phi^{-1}x_0 + x_1$$

$$y = \phi^{-2}y_0 + \phi^{-1}y_1 + y_2$$

where $y_0 = -\phi_t^2$, $y_1 = \phi_{tt}$, $y_2 = \frac{1}{12}(4\lambda - B - 3V - 3(\phi_{tt}/\phi_t)^2)$ and $x_0^2 = \phi_t^2 V$, $x_1 = -\frac{1}{2}(V_t/V + \phi_{tt}/\phi_t)V^{1/2}$. The variable V is

$$V = \{\phi; t\} + \lambda$$

where

$$\frac{1}{2}V_t^2 + \frac{1}{2}V^3 + (\frac{1}{3}B - 2A + \frac{2}{3}\lambda)V^2 + (\frac{1}{6}B^2 - \frac{2}{3}\lambda^2)V = 0.$$

This defines V as a Weierstrass elliptic function and $\phi = u_1/u_2$, where u_1, u_2 are solutions of the linear equation

$$u_{tt} = -\frac{1}{2}(V + \lambda)u.$$

The procedure as described has been applied to many systems. Amoung these are the Boussinesq sequence, the Caudrey-Dodd-Gibbon sequence and Hirota-Satsuma sequences. As described above the method does not directly apply to systems that do not have Bäcklund transformations. A system of this type is the Bullough-Dodd equation.

$$U_{xt} = e^U - e^{-2U}$$

Surprisingly, the Bullough-Dodd equation is a specialization of an equation in the Caudrey-Dodd-Gibbon sequence. The Bäcklund transformation for this sequence does not preserve the form of the specialization. Hence, there is no Bäcklund transformation in the usual form for this equation [24,25].

A different sort of problem occurs for the nonlinear Schrodinger equation [10]. Here, the method works in so far as finding the Lax pair and Schwarzian modified equations are concerned. However, the modified equations themselves have a Bäcklund transformation that acts as a reduction on the system to a pair of Burgers equations.

The singular manifold method also applies to ordinary differential equations [23]. We have found for the Henon-Heiles system a Bäcklund transformation and a three parameter class of associated solutions. The general solution depending on four parameters is not directly found by the singular manifold method. However, by direct formulation in terms of the Schwarzian derivative we have also found an identification of the integrable instances of the Henon-Heiles system with the KdV and Caudrey-Dodd-Gibbon equations. This allows an immediate identification of the Lax pairs for the Henon-Heiles system.

When the singular manifold method does not find a Lax pair with a parameter, it is possible to effect a resummation of terms in the truncation so as to allow an explicit dependence on the singular manifold itself. This procedure was introduced by Weiss in reference [9] and applied to the Caudrey-Dodd-Gibbon equation. Later, B. Gaffet in references [26,27] found a Bäcklund transformation for the Bullough-Dodd equation that is in the form of an expansion about the singular manifold. Whether this transformation is an involution or not, the form of the transformation involves a resummation of terms involving an explicit dependence on the singular manifold. Finally, Newell et al [28] and Gibbon et al [29] apply this same method to introduce a spectral parameter in the Lax pair for the Henon-Heiles system. Therefore, a resummation of terms involved in the singular manifold truncation seems to be required for certain problems. In this sense the method requires further development to be truly algorithmic.

A recent paper by R. Conte [30] examines the question of why the *Painlevé-Bäcklund equations* that arise on application of the singular manifold method are invariant under the Moebius group. This would explain why the Schwarzian derivative naturally occurs in the formulation of integrable systems, since, as explained earlier, the Schwarzian derivative is the unique differential invariant of the Moebius group. This question is also examined in reference [9] where it is conjectured that the appearence of the Schwarzian derivative is related to the conformal geometry of the *complex Riemann sphere* and not as might be supposed, from an associated second order Lax pair operator. This question is also considered by Wilson [31].

3. Conditional or Partial Integrability. For systems without single-valued expansions (Painlevé Property), it is possible to constrain the *arbitrary* functions in the expansion so as to restore the single-valued behavior. Depending on the number of constraints the resulting expansion will represent a solution of reduced dimensions.

The systems of constraints are expressed as a system of partial differential equations for the previously arbitrary functions (data) in the expansion. For this system of partial differential equations we make the following conjecture.

Conjecture: *The constraint equations are completely integrable.*

To illustrate this point we will present several examples from reference [11].

The first example is the **Double Sine-Gordon** equation

$$u_{xt} = 4a\sin(u/2) + 4\sin u. \tag{3.1}$$

To apply the Painlevé analysis we set

$$V = e^{iu/2}$$

and find

$$VV_{xt} - V_xV_t = a(V^3 - V) + V^4 - 1. \tag{3.2}$$

The expansion about the singular manifold takes the form

$$V = \phi^{-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} V_{jk} \phi^j (\phi^2 \ln \phi)^k. \tag{3.3}$$

From the recursion relations

$$V_{00}^2 = \phi_x \phi_t$$

$$V_{10} = -\frac{1}{2} \frac{\phi_{xt}}{\phi_x \phi_t} V_0 - \frac{a}{2}.$$

The arbitrary functions are ϕ and V_{20}. At the *resonance* $j = 2$ and $k = 0$ we can cancel the $\phi^2 \ln \phi$ terms by requiring that ϕ satisfy the constraint equation

$$\phi_x^2 \phi_{tt} - 2\phi_x \phi_t \phi_{xt} + \phi_t^2 \phi_{xx} = 0. \tag{3.4}$$

This equation is identical to (1.21) and is integrable by a Legendre transformation [11]

$$\xi = \phi_x, \ \eta = \phi_t$$

$$x = W_\xi, \ t = W_\eta$$

$$\phi(x, y) + W(\xi, \eta) = x\xi + y\eta$$

and has the result that

$$\xi^2 W_{\xi\xi} + 2\xi\eta W_{\xi\eta} + \eta^2 W_{\eta\eta} = 0. \tag{3.5}$$

Let

$$\frac{d}{ds} = \xi \frac{\partial}{\partial \xi} + \eta \frac{\partial}{\partial \eta} \tag{3.6}$$

and (3.4) becomes

$$\frac{d^2}{ds^2} W = \frac{d}{ds} W. \tag{3.7}$$

The complete solution of (3.7) is

$$W = W_0 + W_1 \tag{3.8}$$

where W_0 is homogenous of degree zero in (ξ, η) and W_1 is homogenous of degree one. This implies, using the definition of the Legendre transformation, that

$$\phi(x, t) = -W_0(\xi, \eta)$$

and

$$\xi x + \eta t = W_1(\xi, \eta).$$

The Legendre transformation is invertible when $\omega = \phi_{xx}\phi_{tt} - \phi_{xt}^2 \neq 0$. When $\omega = 0$, then $\phi = f(x+t)$ and we have a traveling wave form that is integrable for (3.1). Some simple closed form solutions when $\omega \neq 0$ are

$$\phi = f(x/t)$$

for arbitrary f.

In the cases where ϕ satisfies the constraint equation the expansion (3.3) becomes single-valued

$$V = \phi^{-2} \sum_{j=0}^{\infty} V_j \phi^j \tag{3.8}$$

where V_2 is arbitrary.

Next, we consider the **N dimensional elliptic Sine-Gordon** equation [11]

$$- \Delta u = \sin u \tag{3.9}$$

where

$$\Delta = \sum \partial_{x_j}^2 = \nabla^t \nabla.$$

Using $V = e^{iu}$

$$-V \Delta V + \nabla V \cdot \nabla V = \frac{1}{2}(V^3 - V) \tag{3.10}$$

The Painlevé expansion

$$V = \phi^{-2} \sum_{j=0}^{\infty} V_j \phi^j \tag{3.11}$$

is valid with arbitrary V_2 iff

$$\nabla \phi \cdot D \nabla \phi = 0 \tag{3.12}$$

where

$$D_{ii} = \frac{1}{2} \sum_{l=1, l\neq i}^{N} \sum_{m=1, m\neq i}^{N} (\phi_{lm}^2 - \phi_{ll}\phi_{mm}) \tag{3.13}$$

$$D_{ij} = \sum_{m=1}^{N} (\phi_{ij}\phi_{mm} - \phi_{im}\phi_{jm}). \tag{3.13}$$

The matrix D is symmetric and equation (3.12) is invariant under arbitrary scalings and translations in the independent variables, and orthogonal changes of independent variables

$$\nabla = B\nabla'$$

where

$$B^t = B^{-1}.$$

Using these properties it can be shown that the hypersurface M defined by the level sets

$$M = \{\hat{x}; \phi(\hat{x}) = \phi_0\}$$

has the property that principal curvatures of M as a manifold in R^N , $K_j; j = 1, \cdots, N-1$ verify the condition

$$K_1K_2 + K_1K_3 + \cdots + K_{N-2}K_{N-1} = 0. \tag{3.14}$$

When $N = 2$ the condition is trivial and (3.9) is integrable. When $N = 3$ equation (3.12) is

$$\begin{aligned} &\phi_t^2(\phi_{xx}\phi_{yy} - \phi_{xy}^2) + \phi_x^2(\phi_{tt}\phi_{yy} - \phi_{yt}^2) + \phi_y^2(\phi_{tt}\phi_{xx} - \phi_{xt}^2) \\ &+2\phi_x\phi_t(\phi_{ty}\phi_{yx} - \phi_{xt}\phi_{yy}) \\ &+2\phi_y\phi_t(\phi_{tx}\phi_{xy} - \phi_{yt}\phi_{xx}) \\ &+2\phi_x\phi_y(\phi_{xt}\phi_{yt} - \phi_{xy}\phi_{tt}) = 0. \end{aligned} \tag{3.15}$$

Equation (3.15) may be integrated by a Legendre transformation, $\xi_1 = \phi_t$, $\xi_2 = \phi_x$, $\xi_3 = \phi_y$ and $t = W_{\xi_1}$, $x = W_{\xi_2}$, $y = W_{\xi_3}$ where

$$\phi(t, x, y) + W(\xi_1, \xi_2, \xi_3) = t\xi_1 + x\xi_2 + y\xi_3$$

and

$$W = W_0 + W_1$$

where W_0 and W_1 are homogenous of degree zero and one, respectively. Again, the form of ϕ might be used to find integrable reductions.

When $N \geq 4$ it is not known if (3.12) is integrable. *Our conjecture states that it is integrable for all N.*

4. Periodic fixed points of Bäcklund transformations. A Bäcklund transformation (BT) maps solutions of a nonlinear system into solutions of the same nonlinear system. The BT is applied iteratively to define a sequence of solutions beginning from a known *seed* solution. Rather than study solutions defined in this manner we consider the periodic fixed points of the Bäcklund transformations. These are integrable systems of ordinary differential equations that define finite dimensional invariant manifolds of the flow associated with the infinite dimensional partial differential system. We find this flow is expressed as commuting flows on the invariant manifold [32].

In section 2 we have seen that the Schwarzian KdV equation

$$\phi_t/\phi_x + \{\phi; x\} = \lambda \tag{4.1}$$

has the Bäcklund transformations

$$\phi = (a\psi + b)/(c\psi + d) \tag{4.2}$$

and

$$\phi_x = 1/\psi_x. \tag{4.3}$$

$$\phi_t/\phi_x + \psi_t/\psi_x + \left(\frac{\phi_{xx}}{\phi_x}\right)\left(\frac{\psi_{xx}}{\psi_x}\right) = 2\lambda. \tag{4.4}$$

We compose $\phi = -1/\psi$ and (4.3), (4.4) and get

$$\phi_{j+1,x} = \phi_j^2/\phi_{j,x} \tag{4.5}$$

$$\frac{\phi_{j+1,t}}{\phi_{j+1,x}} + \frac{\phi_{j,t}}{\phi_{j,x}} = \left(\frac{\phi_{j,xx}}{\phi_{j,x}}\right)^2 - 4\frac{\partial^2}{\partial x^2}\ln\phi_j + 2\lambda \tag{4.6}$$

where $j = 1, 2, 3, \cdots,\ \bmod N$. The periodic fixed points continue to define a strong Bäcklund transformation since

$$\phi_{j+1,xt} = \phi_{j+1,tx}$$

continues to imply that the set $\{\phi_j, j = 1, 2, 3, \cdots,\ \bmod N\}$ are solutions of (3.1).

Now, define the variables $\xi_j = \phi_{j,x}/\phi_j$ and find that (4.5) is $\xi_{j+1}\xi_j = \phi_j/\phi_{j+1}$. By logarithmic differentiation

$$\xi_{j+1,x}/\xi_{j+1} + \xi_{j,x}/\xi_j = \xi_j - \xi_{j+1} \tag{4.7}$$

where $j = 1, 2, 3, \cdots \ \bmod N$.

We claim that (4.7) is a completely integrable hamiltonian system.

First, we introduce the *circulant matricies* [33] A and B where

$$A = [1, 1, 0, \cdots, 0]$$

$$B = [1, -1, 0, \cdots, 0]. \tag{4.8}$$

We note that $\det B = 0$ for any N and the null vector of B is $(1, 1, \cdots, 1)^t$. We also define the variable $\hat{\beta} = (\beta_1, \beta_2, \cdots, \beta_N)^t$ where $\beta_j = \ln \xi_j$. Then, system (4.7) is

$$A\hat{\beta}_x = B\hat{\xi}. \tag{4.9}$$

Applying the null vector of B to (4.9) we find the *Casimir* integral

$$H_N = \prod_{j=1}^{N} \xi_j. \tag{4.10}$$

The complete set of independent integrals in involution for the system (4.9) are

$$H_{N-2m} = L^m \circ H_N, \tag{4.11}$$

where

$$L = \sum_{j=1}^{N} \frac{\partial^2}{\partial \xi_j \partial \xi_{j+1}}. \tag{4.12}$$

To see this we need the following identities, for each j

$$\xi_j \partial_j L^m \circ H_N = L^m \circ H_N - m\partial_j(\partial_{j-1} + \partial_{j+1}) L^{m-1} \circ H_N \tag{4.13}$$

$$\xi_j \partial_j (\partial_{j-1} - \partial_{j+1}) L^m \circ H_N = (\partial_{j-1} - \partial_{j+1}) L^m \circ H_N. \tag{4.14}$$

Then

$$\partial/\partial_x (L^m \circ H_N) = \sum \xi_{j,x} (\partial_j \circ H_N)$$

$$= \sum \frac{\xi_{j,x}}{\xi_j} \circ \xi_j \partial_j L^m \circ H_N$$

$$= -m \sum \frac{\xi_{j,x}}{\xi_j} \circ \partial_j (\partial_{j-1} + \partial_{j+1}) L^{m-1} \circ H_N$$

$$= -m \sum (\frac{\xi_{j,x}}{\xi_j} + \frac{\xi_{j+1,x}}{\xi_{j+1}})\partial_j \partial_{j+1} L^{m-1} \circ H_N$$

$$= -m \sum (\xi_j - \xi_{j+1})\partial_j \partial_{j+1} L^{m-1} \circ H_N$$

$$= -m \sum \xi_j \partial_j (\partial_{j+1} - \partial_{j-1}) L^{m-1} \circ H_N$$

$$= -m \sum (\partial_{j+1} - \partial_{j-1}) L^{m-1} \circ H_N$$

$$= 0.$$

The higher order flows

$$A\hat{\beta}_x = B\,\mathrm{diag}[\xi_1, \cdots, \xi_N]\nabla H_k, \tag{4.15}$$

where $k = N, N-2, N-4, \cdots$, $\beta_j = \ln \xi_j$, can be shown to commute by an argument similar to that above.

If N is odd then A is invertible and $\Omega = A^{-1}B = [0, -1, 1, -1, 1, \cdots, -1, 1]$ is anti-symmetric. The anti-symmetric matrix

$$M_{\hat{\xi}} = \mathrm{diag}[\xi_1, \cdots, \xi_N]\Omega\,\mathrm{diag}[\xi_1, \cdots, \xi_N] \tag{4.16}$$

defines the bracket

$$\{G, H\} = (\nabla G)^t M_{\hat{\xi}} \nabla H \tag{4.17}$$

and the systems (4.15) are

$$\hat{\xi}_x = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_k. \tag{4.18}$$

If N is even the flow is *constrained* by the contraction with the null vector of A with (4.7). The constraint is

$$\sum_{j=1}^{N} (-1)^j \xi_j = 0. \tag{4.19}$$

It can be shown that the flow for even N is equivalent to two copies of the periodic Toda lattice of period $N/2$ [34].

In general, the KdV flow *factors* on the finite dimensional invariant manifold

$$A\hat{\beta}_x = B\,\mathrm{diag}[\xi_1, \cdots, \xi_N]\nabla H_1 \tag{4.19}$$

(4.20) $$A\hat{\beta}_t = B\,\mathrm{diag}[\xi_1, \cdots, \xi_N]\nabla H_3.$$

The flows of the higher order flows factor according to (4.19) and

(4.21) $$A\hat{\beta}_t = B\,\mathrm{diag}[\xi_1, \cdots, \xi_N]\nabla H_k.$$

Again, for odd N the KdV flow is

(4.21) $$\hat{\xi}_x = M_{\hat{\xi}}\nabla H_1$$

(4.22) $$\hat{\xi}_t = M_{\hat{\xi}}\nabla H_3$$

where $\{H_1, H_3\} = 0$.

The *Generic system* of equations [34] generalizes the KdV system and, as we shall see in the next section, describes the periodic fixed points of the Bäcklund transformations for several hierarchies of equations.

Define the circulant shift matrix

(4.32) $$C = \circ[0, 1, 0, 0, \cdots, 0]$$

and the associated coefficient matricies

(4.33) $$A = I + C + \cdots + C^p$$

(4.34) $$B = I - C^p.$$

Then, with $\beta_j = \ln \xi_j$, the generic system is

(4.35) $$A\hat{\beta}_x = B\hat{\xi}.$$

With operator

(4.36) $$L = \sum \partial_j \partial_{j+1} \cdots \partial_{j+p}$$

there are the integrals

(4.37) $$H_{N-p(m+1)} = L^m \circ H_N.$$

for the system (4.35).

The KdV sequence corresponds to $p = 1$, the Boussinesq sequence corresponds to $p = 2$ and every sequence corresponds to the periodic fixed points of the Bäcklund transformations for the two dimensional Toda lattice.

5. Caustic surfaces, and Factoring the Laplace-Darboux transformation. The KdV and Boussinesq systems are instances of the general system in component form [33]

$$\frac{\xi_{j,x}}{\xi_j} + \frac{\xi_{j+1,x}}{\xi_{j+1}} + \cdots + \frac{\xi_{j+p,x}}{\xi_{j+p}} = \xi_j - \xi_{j+p} \tag{5.1}$$

where $j = 1, 2, \ldots \mod N$. The KdV systems correspond to $p = 1$ and the Boussinesq to $p = 2$. Let the circulant forward shift matrix [33] be

$$C = \mathbf{circ}[0, 1, 0, 0, \ldots, 0].$$

In the N-vector form equations (5.1) are

$$A \begin{pmatrix} \frac{\xi_{1,x}}{\xi_1} \\ \vdots \\ \frac{\xi_{N,x}}{\xi_N} \end{pmatrix} = B\hat{\xi} \tag{5.2}$$

with

$$\begin{aligned} A &= I + C + \cdots + C^p \\ B &= I - C^p. \end{aligned}$$

The casimir integrals of (5.2) correspond to the null vectors of B. The null vectors of A produce the constraints.

Associated with the principal Casimir, for any N

$$H_N = \prod_{j=1}^{N} \xi_j$$

we find the principal integrals of (5.2)

$$H_{N-pm-m} = L^m \circ H_N, \tag{5.3}$$

where $m = 0, 1, 2, \ldots$ and

$$L = \sum_{j=1}^{N} \partial_{\xi_j} \partial_{\xi_{j+1}} \ldots \partial_{\xi_{j+p}}.$$

The systems (5.2) have a Hamiltonian structure

$$A \begin{pmatrix} \frac{\xi_{1,x}}{\xi_1} \\ \vdots \\ \frac{\xi_{N,x}}{\xi_N} \end{pmatrix} = B \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_N \end{pmatrix} \nabla_{\hat{\xi}} H_1, \tag{5.4}$$

where $H_1 = \sum_{j=1}^{N} \xi_j$.

The higher-order equations associated with the integrals (5.3) are

$$A \begin{pmatrix} \frac{\xi_{1,x}}{\xi_1} \\ \vdots \\ \frac{\xi_{N,x}}{\xi_N} \end{pmatrix} = B \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_N \end{pmatrix} \nabla_{\hat{\xi}} H_{N-pm-m}. \tag{5.5}$$

When A is invertible, then

$$\Omega = A^{-1} B$$

is an antisymmetric circulant matrix.

We have the systems

$$\hat{\xi}_{,x} = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_1 \tag{5.6}$$

and

$$\hat{\xi}_{,x} = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_{N-pm-m} \tag{5.7}$$

where

$$M_{\hat{\xi}} = \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_N \end{pmatrix} \Omega \begin{pmatrix} \xi_1 & & \\ & \ddots & \\ & & \xi_N \end{pmatrix}$$

is the co-symplectic form.

Darboux [20] has shown that the parameters (x, y) for surfaces in three dimensions can be defined so the coordinates (z_j) of the surface satisfy a partial differential equation of the form:

$$\partial^2 z/\partial x \partial y + a \partial z/\partial x + b \partial z/\partial y + cz = 0, \tag{5.8}$$

where (a, b, c) are functionals of the first fundamental form in the (x, y) parameters.

Under the gauge transformation $z \to \lambda z$, the form of (5.8) is preserved and:

$$h = \partial a/\partial x + ab - c$$

$$k = \partial b/\partial y + ab - c$$

are invariant.

The *Laplace transformation* of a surface is a partial factorization of (5.8) in terms of the *invariants*[42].

$$(5.9) \qquad z_1 = \partial z/\partial y + az$$

$$\partial z_1/\partial x + bz_1 = hz$$

Equations (5.9) imply that z satisfies (5.8) and z_1 satisfies the system

$$\partial^2 z_1/\partial x\partial y + a_1\partial z_1/\partial x + b_1\partial z_1/\partial y + c_1 z_1 = 0$$

where

$$(5.10) \qquad a_1 = a - \partial \ln h/\partial y$$

$$b_1 = b$$

$$c_1 = c - \partial a/\partial x + \partial b/\partial y - b\partial \ln h/\partial y.$$

From (5.10) the Laplace transformation of the invariants is

$$(5.11) \qquad h_1 = 2h - k - \partial^2 \ln h/\partial x\partial y$$

$$k_1 = h.$$

Darboux [20] studied the periodic fixed points of the Laplace transformation and found that these surfaces are related as a sequence of *focal surfaces*. From (5.11), the periodic fixed points are

$$(5.12) \qquad \partial^2 \ln h_j/\partial x\partial y = -h_{j+1} + 2h_j - h_{j-1},$$

where $j = 1, 2, 3, \cdots \pmod{n}$ and n is the order of the fixed point. The substitution

$$h_j = e^{\theta_{j+1}-\theta_j}$$

obtains the *two dimensional periodic Toda lattice*

$$(5.13) \qquad \theta_{j,xy} = -e^{\theta_{j+1}-\theta_j} + e^{\theta_j-\theta_{j-1}}$$

We now find Bäcklund transformations for the *Darboux equations* (5.12) and the Toda lattice equations (5.13). With reference to systems (5.6) and (5.7), without loss of generality normalize the casimir, $H_N = 1$, and set

(5.14) $$\hat{\xi}_{,x} = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_1$$

(5.15) $$\hat{\xi}_{,y} = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_{N-p-1},$$

where $H_{N-p-1} = L \circ H_N$. Then, let $\xi_j = e^{\psi_j - \psi_{j+1}}$ and find that (5.14), (5.15) imply

(5.16) $$\psi_{j,xy} = e^{\psi_{j+p} - \psi_j} - e^{\psi_j - \psi_{j-p}},$$

where $j = 1, 2, 3, \cdots \pmod N$.

To see this let $\xi_j = e^{\theta_j}$ and find

$$\hat{\theta}_{,x} = \Omega \nabla_\theta H_1$$
$$\hat{\theta}_{,y} = \Omega \nabla_\theta G$$

where

$$G = \sum e^{-\theta_j - \theta_{j+1} - \cdots - \theta_{j+p}}$$
$$= \sum 1/\xi_j \xi_{j+1} \cdots \xi_{j+p} = H_{N-p-1}/H_N.$$

It can be shown that

$$\hat{\theta}_{,xy} = C^{-p}(I - C^p)(I - C) \begin{pmatrix} e^{-\theta_1 - \theta_2 - \cdots - \theta_p} \\ \vdots \\ e^{-\theta_N - \theta_1 - \cdots - \theta_{p-1}} \end{pmatrix}.$$

Let $\hat{\theta} = (I - C)\hat{\psi}$ and find (5.16).

When $p = 1$ (5.16) are the Toda lattice of period N. If N and p are relatively prime (5.16) is again a Toda lattice of length N. If $N = mp$ (5.16) is p distinct lattices of length m. When N and p have common factors (5.16) there is one lattice for each distinct orbit of translation by $p \pmod N$. In all cases the set of fields ξ_j are directly related to the set of invariants H_j. When A is not invertible we find for equations (5.4) and (5.5) a similar connection with the Toda lattice. In this case one must take into account the *constraints* that apply to these systems to obtain a valid correspondence. A more comprehensive analysis of (5.4) and (5.5) as completely integrable hamiltonian systems and the properties of their related surfaces is currently in progress [35,36].

Consideration of the form of (5.6), (5.7) and the possible relations between p and N determine that for a lattice of fixed length m there will exist an infinite sequence of distinct

Bäcklund transformations. For instance, we have a Bäcklund transformation for a lattice of length m when $N = pm$ for $p = 1, 2, 3, \cdots$.

Finally, the Bäcklund transformation for the Toda lattice that was reported in ref.[37] corresponds in our formulation to the system (5.7) with $p = 1$

$$\hat{\xi}_{,x} = M_{\hat{\xi}} \nabla_{\hat{\xi}} H_{N-2}$$

$$\hat{\xi}_{,y} = -M_{\hat{\xi}} \nabla_{\hat{\xi}} H_{N-2}.$$

6. Conclusions and Comments. The *Painlevé test*, as described in section 1, is proposed as a sufficient condition for integrability. The Painlevé Property is a statement about how the solutions behave as functionals of the data in the neighborhood of a singularity manifold and not a statement about the data itself. Examples of this phenomenon are examined. Expansions about *characteristic manifolds* are required to be single-valued. Essential singularities are found to be determined by certain *Psi series* involving nonconstant leading orders and resonances.

The *singular manifold method* finds Bäcklund transformations by truncating the functional Laurent series after the constant level term. This results in the formulation of *modified* equations in terms of the *Schwarzian derivative*. The Miura transformation between the modified and given system can be used to determine the *Lax pair* and recursion operators for the gradients of conserved densities. The symmetries of the modified equation and the invariance under the Moebius group are a form of Bäcklund transformation for the modified equation. The periodic fixed points of these Bäcklund transformations are finite dimensional invariant manifolds for the flow of the system. The dynamics occurs as commuting hamiltonian flows on this finite dimensional manifold.

Constrained *Psi* series expansions are applied to non-Painlevé systems. The constraints are expressed as nonlinear partial differential equations. We conjecture that these are integrable and provide the integrable reductions of the original system.

Acknowledgments.

This work was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation, and by the National Science Foundation Research Grant No. DMS-8607576.

REFERENCES

[1] S. Kowalevskaya, Acta Mathematica, 14 (1890), pp. 81.
[2] E.L. Ince, *Ordinary Differential Equations*, Dover, New York, 1956.
[3] M.J Ablowitz, A. Ramani and H. Segur,, J. Math. Phys., 21 (1981), pp. 715.
[4] J.B. McLeod and P.J. Olver, SIAM J. Math. Anal., 14 (1983), pp. 1566.
[5] W.F. Osgood, *Topics in the Theory of Functions of Several Complex Variables*, Dover, New York, 1966.
[6] J. Weiss, M. Tabor and G. Carnevale, J. Math. Phys., 24 (1983), pp. 522.

[7] E. HILLE, *Ordinary Differential Equations in the Complex Plane*, John Wiley, New York, 1976.
[8] J. WEISS, J. Math. Phys., 24 (1983), pp. 1405.
[9] J. WEISS, J. Math. Phys., 25 (1984), pp. 13.
[10] J. WEISS, J. Math. Phys., 26 (1985), pp. 258.
[11] J. WEISS, J. Math. Phys., 25 (1984), pp. 2226.
[12] R.S. WARD, Phys Letts, 102A (1984), pp. 279.
[13] R.S. WARD, Phil. Trans. R. Soc. Lond. A, 315 (1985), pp. 451.
[14] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, Interscience, New York, 1962.
[15] E.V. DOKTOROV AND S. YU. SAKOVICH, J. Phys. A, 18 (1985), pp. 3327.
[16] P.A. CLARKSON, Phys. Letts, 109A (1985), pp. 205.
[17] P.A. CLARKSON, Physica, 18D (1986), pp. 209.
[18] WILLY HEREMANN, PRIVATE COMMUNICATION.
[19] M. LAVIE, Canadian J. Math., 21 (1969), pp. 235.
[20] G. DARBOUX, *Théorie Générale des Surfaces, II*, Chelsea, New York, 1972.
[21] P. DEIFT AND E. TRUBOWITZ, Comm. Pure Appl. Math., 32 (1979), pp. 121.
[22] J. WEISS, Phys. Letts., 102A (1984), pp. 329.
[23] J. WEISS, Phys. Letts., 105A (1984), pp. 387.
[24] J. WEISS, J. Math. Phys., 27 (1986), pp. 1293.
[25] M.J. ABLOWITZ AND H. SEGUR, *Solitons and the Inverse Scattering Transformation*, SIAM, Philadelphia, 1981.
[26] B. GAFFET, Physica, 26D (1987), pp. 123.
[27] B. GAFFET, J. Phys. A, 21 (1988), pp. 2491.
[28] A.C. NEWELL, M. TABOR AND Y.B. ZENG, Physica, 29D (1987), pp. 1.
[29] J.D. GIBBON, A.C. NEWELL, M. TABOR AND Y.B. ZENG, Nonlinearity, 1 (1988), pp. 481.
[30] R. CONTE, Phys. Letts, 184A (1988), p. 100.
[31] GEORGE WILSON, Phys. Letts, 132A (1988), pp. 445.
[32] J. WEISS, J. Math. Phys., 27 (1986), pp. 2647.
[33] J. WEISS, *Circulant Matricies*, Wiley, New York, 1983.
[34] J. WEISS, J. Math. Phys., 28 (1987), pp. 2025.
[35] J. WEISS, *Bäcklund transformations, Focal surfaces and the Two dimensional Toda lattice , submitted to*, Phys Letts. A (1988).
[36] J. WEISS, WORK IN PROGRESS.
[37] A.P. FORDY AND J. GIBBONS, Commun. Math. Phys., 77 (1980), pp. 21.