

**EXISTENCE OF KAM TORI IN THE PHASE SPACE
OF VORTEX SYSTEMS**

By

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EXISTENCE OF KAM TORI IN THE PHASE SPACE OF VORTEX SYSTEMS

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Abstract. Under very mild conditions on the circulations, the existence of a positive measure set of quasi-periodic solutions for a lattice vortex model, is shown via a new approach in the application of KAM theory. In this approach, the size of the perturbation is controlled by choosing special regions in phase space and scaling the hamiltonian, instead of restricting the values of the circulations. An essential tool in this procedure is a class of canonical transformations which are generated by binary trees. These transformations give a completely integrable unperturbed term, which in fact, is the hamiltonian for decoupled oscillators. The KAM nondegeneracy condition takes a simple form, which has a combinatorial interpretation in terms of the topological structure of the tree associated with the canonical transformation used. The existence of KAM tori in special regions of phase-space implies the physical phenomenon of clustering of vortices.

I. Introduction. In previous applications of the Kolmogorov-Arnold-Moser theory [1, 2], largely to the field of celestial mechanics, the hamiltonian formulation of the problem contains an explicit small parameter ϵ , i.e. (cf. also [3], [4, 5], [6])

$$(1) \quad H = H_0 + \epsilon H_1$$

This parameter ϵ may represent the small mass ratio of the celestial bodies. It gives one explicit control of the size of the perturbation term H_1 in (1) and makes the application of the KAM theory to those problems a little easier. Clearly the smallness of ϵ , will restrict the values that coefficients such as mass ratios can have.

We will apply the KAM theorem [1] to a different class of hamiltonians which (after some work) can be put in the form,

$$(2) \quad H = H_0 + H_1.$$

where H_0 is completely integrable. Note that the explicit small parameter ϵ is no longer present in (2). In this case, severe restrictions on the coefficients of the particles will not be necessary because control of the size of H_1 in (2) is based on the phase-space dynamics rather than on externally assigned properties of the particles. To elaborate a little, we will control the size of H_1 by restricting the size of special regions in phase-space and by scaling time appropriately. (cf. important remarks at the end of section III).

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The class of hamiltonians is given by the general form (cf. [7]).

$$(3) \quad \begin{aligned} H(q_1, \dots, q_n; p_1, \dots, p_n) \\ = \sum_{j \neq k=1}^n \Gamma_j \Gamma_k \log |F(z_j - z_k)| \end{aligned}$$

with the following properties:

(A) $\sqrt{\Gamma_j} z_j = q_j + ip_j$ and Γ_j are real coefficients

(B) F is an entire function of one complex variable, and $F(0) = 0$.

These hamiltonians arise in several models for two-dimensional vortex dynamics, as well as plasma physics. For the following functions F , (3) represents well-known vortex models:

(4a) $F = \text{identity} \quad \longleftrightarrow \quad \text{point vortex model}$

(4b) $F = \sin \pi \quad \longleftrightarrow \quad \text{lattice vortex models}$

(4c) $F = \sin \pi(\cdot) + \delta \quad \longleftrightarrow \quad \text{desingularized numerical vortex models}$

The point vortex model [8], first discussed by Kirchhoff is basic in the sense it will appear (as the unperturbed term) in the other models (4b, 4c).

In this paper, we will prove the existence of KAM tori in the phase-space of the lattice vortex models, (4b). The third case (4c) is sketched in Lim [9], using the approach in this paper. Also mentioned in the above note is the Desingularized Elliptic Moment Model proposed by Melander, Styczek and Zabusky [10]. Details of the proofs in [9] can be filled in by following the steps in this paper.

Section II will be devoted to a brief survey of the literature on the lattice vortex model and some physical consequences of the results in this paper. Section III will focus on the steps required to put the hamiltonian (3) (with F given by (4b)) in the form of (2) and section IV will *discuss* the steps that give us control over the size of the perturbation H_1 , and thus permit the application of the KAM theory.

An essential tool in our approach is a special group of canonical transformations which are associated with *binary trees* (cf. Lim [11, 12]). Surprisingly, it turns out that the topological structure of the tree that is associated with the definition of a transformation, actually gives a combinatorial interpretation of the KAM-nondegeneracy condition [1] for H_0 . We will not pursue this here because our aim here is to construct the most direct proof of the existence of KAM tori in a class of lattice vortex models (cf. Lim [13] for details of this fact).

II. Lattice Vortex Model. The n degrees of freedom hamiltonian for this model is

$$(5) \quad H(z_1, \dots, z_n) = \frac{1}{2\pi} \sum_{j \neq k=1}^n \Gamma_j \Gamma_k \log |\sin \pi(z_j - z_k)|$$

where the z_j 's are complex numbers representing the positions of the n vortices in a unit cell of the lattice, and $\Gamma_j \in \mathbb{R}$, represents the circulation of the j -th vortex in all the cells. In the hydrodynamics literature [14], the equations of motion are given by

$$(6) \quad \Gamma_k \dot{\bar{z}}_k = i \frac{\partial H}{\partial z_k}$$

where the over bar denotes complex conjugation. This immediately implies the standard hamilton's equations for the canonical variables, (where $z_j = x_j + iy_j$)

$$(7) \quad \begin{aligned} q_j &= \sqrt{|\Gamma_j|} x_j, \\ p_j &= \text{sgn}(\Gamma_j) \sqrt{|\Gamma_j|} y_j \end{aligned}$$

It is well-known that the quantities [14]

$$(8) \quad C_R = \sum_{j=1}^n \Gamma_j x_j, \quad C_I = \sum_{j=1}^n \Gamma_j y_j$$

are conserved by the flow, (6). In fact, this follows directly from the translational symmetry of the hamiltonian function, (5). The complex quantity

$$C = C_R + iC_I = \sum_{j=1}^n \Gamma_j z_j$$

is nothing more than the "center of vorticity" of the n vortices. Later we will give derive canonical transformations that will reduce the hamiltonian (5) by one degree of freedom precisely by making C_R and C_I the new conjugate parts of the n -th degree of freedom.

Another symmetry of the hamilton's equations, (6) is the invariance under the transformation,

$$t \longrightarrow -t, \quad \Gamma_j \longrightarrow -\Gamma_j \quad \forall j = 1, \dots, n.$$

This can be immediately ascertained by inspection of the form of (5) and (6), noting that the hamiltonian function (5) is quadratically homogeneous in the coefficients Γ_j . The homogeneity of (5) in the circulations implies an important fact, namely:

$$(9) \quad \begin{aligned} &\text{Under the transformation } (t \rightarrow \frac{1}{\eta} t, \Gamma_j \rightarrow \sqrt{\eta} \Gamma_j, j = 1, \dots, n) \\ &\text{where } \eta \text{ is some positive real numbers, the hamilton's equation (6)} \\ &\text{are invariant} \end{aligned}$$

which, in fact, generalizes the previous symmetry. This property (9) will be used in the proofs of our results.

We now give a brief summary of the hydrodynamical background for this model. There are two main origins of (5), namely,

- (a) the von Karman lattice model for the vortex street shed by a cylinder [15], and
- (b) the Rosenhead numerical method for computing the evolution of a periodic vortex sheet [16].

The von Karman 1-dimensional lattice model consists of two doubly infinite rows of vortices, of opposite circulations. The dynamical behavior of this lattice when subjected to large amplitude periodic excitations is modelled by (5) which was first derived by Kochin [14]. For a discussion of some numerical results concerning the asymptotic large amplitude behavior of this lattice, see [17]; discussions of the wave properties of this lattice are given in [18, 19]. A similar model for the 2-dimensional jet, is analyzed rigorously in [20]. This paper complements the results obtained in [20].

The Rosenhead method [21] discretizes the vortex sheet by point vortices and tracks the evolution of the sheet by interpolating the future positions of the vortices. In general, the Rosenhead numerical method refers to any scheme using point vortices for the study of two-dimensional hydrodynamical structures. A modern proponent of this method is Krasny [21] who applied the Rosenhead lattice with some modifications to the problem of the periodic vortex sheet before the critical time, t_c (after which the sheet rolls up). Recently, Cafilisch and Lowengrub [22] proved that this numerical method converges.

We now turn to the inevitable questions: apart from the mathematical value (whatever it is) of the new techniques invented here in order to apply the KAM theory in the novel situation given by (2), does the main result in this paper have any physical value? If so, what are the physical statements that we can make regarding the dynamics of vortices from the theorem on the existence of quasi-periodic solutions?

The following physical conclusions can be drawn from the mathematical results in this paper:

- (A) The quasi-periodic solutions on KAM tori correspond to the clustering of vortices, because (as will be explained in details later) the special regions in phase-space in which the existence of tori is proved, can be interpreted in terms of small relative distances between particles, i.e. clusters. Furthermore, the main theorem states that the set of tori has positive measure in phase-space which means that there are "plenty" of these clustered states. And these clusters of vortices never disintegrate, despite their inherent motion.

We follow with an application of the statements in (A) to a numerical model:

- (B) We recall that the hamiltonian (5) is for a lattice vortex model. Thus consider the Rosenhead discretization of an infinite periodic row of small vortex patches of

the same total circulation. Each similar vortex patch is now represented by many ($N = 100$ say) point vortices. Rosenhead discretization implies that the initial state of the N point vortices is a tightly clustered one. We assume that there are no wave excitations of the vortex patches in the $1 - d$ lattice because it is well known that the single infinite row is highly unstable [8]. On physical grounds, the vortex patches should remain quiescent, and the periodic row of patches should therefore be a permanent structure if viscosity is neglected. On the other hand, there is no reason why the clustered state of point vortices should not disintegrate as it evolves, on a priori dynamical grounds alone. Yet, the positive measure set of tori means that “large” set of such initial clustered states (obtained by Rosenhead discretization), is preserved by the dynamics. On this note, we conclude that the Rosenhead method may produce spurious results.

We conclude this section with the observation that the freedom from severe restrictions on the values of the circulations Γ_j , inherent in the approach based on (2), is fundamental for the successful applications of the model (5). Meanwhile the restriction to small special regions in phase-space, in the main results, fortunately coincide with the physical phenomenon of clusters. Similar questions can be raised in the model (4c), which is a lattice version of desingularized vortex methods (cf. Chorin [23] for the origin of this family of vortex models).

III. Main Result and Proofs. It is necessary in our approach to first perform a canonical transformation which reduces (5) to $n - 1$ degrees of freedom. Consider the following equations which define a linear transformation from the old canonical variables, $\{q_j, p_j\}$ given by (7), to the new variables $\{Q_j, P_j\}$:

(10)

$$Q_1 = \left| \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \right|^{1/2} \left(\frac{q_2}{\sqrt{|\Gamma_2|}} - \frac{q_1}{\sqrt{|\Gamma_1|}} \right)$$

$$P_1 = \operatorname{sgn} \left(\frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \right) \left| \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2} \right|^{1/2} \left(\frac{p_2}{\operatorname{sgn}(\Gamma_2) |\Gamma_2|^{1/2}} - \frac{p_1}{\operatorname{sgn}(\Gamma_1) |\Gamma_1|^{1/2}} \right)$$

$$Q_2 = \left| \frac{(\Gamma_1 + \Gamma_2) \Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3} \right|^{1/2} \left(\frac{q_3}{\sqrt{|\Gamma_3|}} - \frac{\sqrt{|\Gamma_1|} q_1 + \sqrt{|\Gamma_2|} q_2}{|\Gamma_1 + \Gamma_2|} \right)$$

$$P_2 = \operatorname{sgn} \left(\frac{(\Gamma_1 + \Gamma_2) \Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3} \right) \left| \frac{(\Gamma_1 + \Gamma_2) \Gamma_3}{\Gamma_1 + \Gamma_2 + \Gamma_3} \right|^{1/2} \left(\frac{p_3}{\operatorname{sgn}(\Gamma_3) |\Gamma_3|^{1/2}} - \frac{\operatorname{sgn}(\Gamma_1) |\Gamma_1|^{1/2} p_1 + \operatorname{sgn}(\Gamma_2) |\Gamma_2|^{1/2} p_2}{|\Gamma_1 + \Gamma_2|} \right)$$

⋮

$$Q_m = \left| \frac{\left(\sum_{i=1}^m \Gamma_i \right) \Gamma_{m+1}}{\sum_{i=1}^{m+1} \Gamma_i} \right|^{1/2} \left(\frac{q_{m+1}}{|\Gamma_{m+1}|} - \frac{\sum_{j=1}^m \sqrt{|\Gamma_j|} q_j}{\left| \sum_{j=1}^m \Gamma_j \right|} \right)$$

$$P_m = \operatorname{sgn} \left| \frac{\left(\sum_{i=1}^m \Gamma_i \right) \Gamma_{m+1}}{\sum_{i=1}^{m+1} \Gamma_i} \right|^{1/2} \left(\frac{p_{m+1}}{\operatorname{sgn}(\Gamma_{m+1}) |\Gamma_{m+1}|^{1/2}} - \frac{\sum_{j=1}^m \operatorname{sgn}(\Gamma_j) |\Gamma_j|^{1/2} p_j}{\left| \sum_{j=1}^m \Gamma_j \right|} \right)$$

⋮

$$Q_n = \frac{\sum_{j=1}^n \sqrt{|\Gamma_j|} q_j}{\left| \sum_{j=1}^n \Gamma_j \right|}$$

$$P_n = \frac{\sum_{j=1}^n \operatorname{sgn}(\Gamma_j) |\Gamma_j|^{1/2} p_j}{\left| \sum_{j=1}^n \Gamma_j \right|}$$

Obviously, this transformation is not well-defined on the following set of nonzero values

for the circulations:

$$(11) \quad \begin{aligned} \Omega &= \{\Gamma = (\Gamma_1, \dots, \Gamma_n) \mid \Gamma_1 + \Gamma_2 = 0\} \\ &U\{\Gamma \mid \Gamma_1 + \Gamma_2 + \Gamma_3 = 0\} \\ &U \dots U\{\Gamma \mid \sum_{j=1}^n \Gamma_j = 0\} \end{aligned}$$

However the complement of Ω in the nonzero reals has full measure

The characteristic equations in Γ that define Ω are equivalent to the Kolmogorov-Arnold Moser non-degeneracy condition, (36) for this problem. Moreover, the new variables $\{Q_j, P_j\}_{j=1}^{n-1}$ give relative distances between the particles, z_j while (Q_n, P_n) represents the "center of vorticity". Thus the physical interpretation of certain invariant regions in phase-space in terms of clusters.

By a direct verification that the jacobian of (10), i.e. the transformation matrix M , is a real $(2n \times 2n)$ symplectic matrix, we can prove the following

LEMMA 1. *In the complement of Ω , the linear transformation (10) is canonical, where (Q_n, P_n) is the new conjugate pair that corresponds to the conserved quantities, C_R and C_I , (8). The new hamiltonian is independent of the pair (Q_n, P_n) , and is given by*

$$(12) \quad \begin{aligned} &H(Q_1, \dots, Q_{n-1}; P_1, \dots, P_{n-1}) \\ &= \frac{1}{2\pi} \sum_{j \neq k=1}^n \Gamma_j \Gamma_k \log |\sin \pi(F_{jk}(\mathbf{Q}; \mathbf{P}))| \end{aligned}$$

where

$$(13) \quad F_{jk}(\mathbf{Q}; \mathbf{P}) = z_j - z_k$$

is a linear combination of the new variables, which can be obtained explicitly by inverting (10).

The proof of lemma 1 is quite long and we refer the interested reader to a full account of the relationship between canonical transformations like (10) and graph theory, given in Lim [11, 12]. We are content here to remark that (10) represents just one example (the simplest) of the so-called Jacobi transformations, and more importantly, each one of these transformations could have been used in the proofs that follow.

Now we are ready to state the main theorem:

THEOREM 1. *For each value of Γ in the complement of Ω , there exists an invariant set, $\mathcal{K}(\Gamma)$, of positive measure in the phase space of the hamiltonian function, (5). \mathcal{K} consists of the product $T^{(n-1)} \times \mathbf{R}^2$ where $T^{(n-1)}$ is composed entirely of $(n-1)$ -dimensional*

invariant tori for the reduced flow of the new hamiltonian (12), while the \mathbf{R}^2 part is given by the conserved quantities C_R, C_I . The reduced flow of (12) on a torus in $T^{(n-1)}$ is quasi-periodic with $(n-1)$ strongly irrational frequencies.

The proof of this theorem will be executed in several steps (lemmas). First, it will be shown in the remainder of this section that the hamiltonian (12) can be reformulated in the form (2). Secondly, in section IV we will prove that in special regions of phase-space called cone sets (to be defined later), a rescaling argument based on the homogeneity (9) of the hamiltonian, gives us control over the perturbation term H_1 in (2).

Lemma 1 implies that we can work with the reduced flow of (12) in the $2(n-1)$ dimensional phase space coordinatized by $\{Q_j, P_j\}$. If the invariant tori of (12) constitutes a set, $T^{(n-1)}$ of positive measure in $\mathbf{R}^{(2n-2)}$ then the product $T^{(n-1)} \times \mathbf{R}^2$ clearly constitutes an invariant set, that have positive measure in the $2n$ -dimensional phase space of the original hamiltonian (5).

We collect together some definitions which will be needed later. By inverting the transformation (10) to solve for the z_j 's in terms of $\{Q_j, P_j\}_{j=1}^n$, we find that the solutions must have the form

$$(14) \quad z_j = \sum_{l=j-1}^n C_l(j) (Q_l + iP_l) \quad \text{except for } j=1, \text{ in which case } l \text{ starts at } 1$$

where all the $C_l(j)$ are nonzero. In fact, it is easy to see that the following relation between the coefficients must hold.

$$(15) \quad C_l(j) = C_l(k) \quad \text{iff } n \geq l \geq k,$$

It is assumed that $1 \leq j < k \leq n$ in the above relation. If we write the difference $z_j - z_k = F_{jk}$ in a linear combination of $\{Q_l, P_l\}$, i.e.

$$(16) \quad F_{jk} = \sum_{l=1}^n C_l(\{j, k\}) (Q_l + iP_l)$$

then (14) and (15) imply

$$(17) \quad C_l(\{j, k\}) \stackrel{d}{=} C_l(j) - C_l(k) = \begin{cases} 0 & \text{for } k \leq l \leq n \\ 0 & \text{for } 1 \leq l \leq j-2 \\ \text{nonzero} & \text{for } \max\{1, j-1\} \leq l \leq k-1 \end{cases}$$

Therefore the largest integer l for which the coefficient $C_l(\{j, k\})$ is nonzero, is $k-1$. Moreover, (10) and (14) imply that the $C_l(\{j, k\})$ are invariant under a uniform scaling of the circulations, i.e. $\Gamma_j \rightarrow \Delta \Gamma_j$. This leads to the definitions:

DEFINITION 1. The function $S(\{j, k\}) = k - 1$ gives the largest integer l such that $C_l(\{j, k\}) \neq 0$.

DEFINITION 2. $\chi(s)$ consists of pairs $\{j, k\}$ such that $S(\{j, k\}) = s$.

Now, a simple calculation gives the size of $\chi(s)$.

$$(18) \quad \#\chi(s) = s \quad \text{for all } s = 1, \dots, n-1.$$

This formula can be immediately derived if one observes that $S(\{j, k\}) = s = k - 1$ for all values of j from 1 to $k - 1$.

Now, we can begin to formulate the hamiltonian (12) in the form of (2). First we give a formal result:

LEMMA 2. In terms of $\rho_j = Q_j + iP_j$ the hamiltonian (12) can be written in the form:

$$(19) \quad H(\rho_1, \dots, \rho_{n-1}) = H_0(|\rho_j|) + H_1(\rho_j) + H_2(\rho_j)$$

where

$$(20a) \quad H_0(|\rho_j|) = \frac{1}{2\pi} \sum_{s=1}^{n-1} \left[\left(\sum_{\{j,k\} \in \chi(s)} \Gamma_j \Gamma_k \right) \log |\rho_s| \right]$$

$$(20b) \quad H_1 = \frac{1}{2\pi} \sum_{s=1}^{n-1} \sum_{\{j,k\} \in \chi(s)} \Gamma_j \Gamma_k \left(\log \left| 1 + \sum_r \frac{C_r(\{j, k\}) \rho_r}{C_s(\{j, k\}) \rho_s} \right| \right)$$

$$(20c) \quad H_2 = \frac{1}{2\pi} \sum_{j \neq k=1}^n \Gamma_j \Gamma_k \log \left| \frac{\sin \pi F_{jk}(\rho)}{\pi F_{jk}(\rho)} \right|$$

Proof. First, write (12) in the form

$$\begin{aligned} H(\rho) &= H'_0(\rho) + H_2(\rho) \\ &= \frac{1}{2\pi} \sum_{j \neq k=1}^n \Gamma_j \Gamma_k \log |\pi F_{jk}(\rho)| \\ &\quad + \frac{1}{2\pi} \sum_{j \neq k} \Gamma_j \Gamma_k \log \left| \frac{\sin \pi F_{jk}}{\pi F_{jk}} \right| \end{aligned}$$

which can always be done. Next, factor F_{jk}

$$\begin{aligned} F_{jk}(\rho) &= C_s(\{j, k\}) \rho_s + \sum_{r < s} C_r(\{j, k\}) \rho_r \\ &= C_s \rho_s \left(1 + \sum_{r < s} \frac{C_r \rho_r}{C_s \rho_s} \right) \end{aligned}$$

where $s = S(\{j, k\})$ and C_r , $r < s$ obey (17). Without loss of generality we can drop the factor π in the following expansions of $H'_0(\rho)$ since it will only change the hamiltonian by an additive constant,

$$(21) \quad \Gamma_j \Gamma_k \log |F_{jk}(\rho)| = \Gamma_j \Gamma_k \log |\rho_s| + \Gamma_j \Gamma_k \log \left| 1 + \sum_{r < s} \frac{C_r \rho_r}{C_s \rho_s} \right|$$

We have also dropped the constant $C_s(\{j, k\})$ for the same reason. For each ρ_s , $s \in \{1, \dots, n-1\}$, there are exactly s pairs $\{j, k\}$ such that the associated term in $H'_0(\rho)$ can be written in the form of (21). Thus there are s terms of the form $\Gamma_j \Gamma_k \log |\rho_s|$ for each ρ_s and summing over them, we obtain

$$\left(\sum_{\{j, k\} \in \chi(s)} \Gamma_j \Gamma_k \right) \log |\rho_s|$$

where the bracketed expression is now the coefficient for $\log |\rho_s|$. Next sum over the set $s \in \{1, \dots, n-1\}$ and we obtain $H_0(|\rho_j|)$. The same double summation for the second term in (21) gives $H_1(\rho)$. \square

Next write the expanded hamiltonian (19) in action-angle form, i.e. in terms of (q_j, p_j) where

$$(22) \quad \rho_j = p_j e^{iq_j}$$

These variables should not be confused with the (q_j, p_j) introduced earlier.

$$(23a) \quad H_0(p_j) = \frac{1}{2\pi} \sum_{s=1}^{n-1} \left\{ \left(\sum_{\{j, k\} \in \chi(s)} \Gamma_j \Gamma_k \right) \log(p_s) \right\}$$

$$(23b) \quad H_1(q_j, p_j) = \frac{1}{2\pi} \sum_{s=1}^{n-1} \sum_{\{j, k\} \in \chi(s)} \Gamma_j \Gamma_k \left(\log \left| 1 + \sum_{r < s} \frac{C_r p_r e^{iq_r}}{C_s p_s e^{iq_s}} \right| \right)$$

$$(23c) \quad H_2(q_j, p_j) = \frac{1}{2\pi} \sum_{j \neq k} \Gamma_j \Gamma_k \log \left| \frac{\sin \pi F_{jk}(p_l, q_l)}{\pi F_{jk}} \right|$$

We will need the following definitions of special regions in phase space. The aim here is to exclude all logarithmic singularities of (12) or (23) from these regions, which we shall call *cone sets*.

DEFINITION 3. A cone set in action-space $\{p_j\} \in \mathbb{R}^{n-1}$ is

$$(24) \quad M'_\delta(\epsilon) = \left\{ \mathbf{p} = (p_1, \dots, p_{n-1}) \mid p_j \neq 0, \right. \\ \left. p_1 < \delta p_2 < \delta^2 p_3 < \dots < \delta^{n-2} p_{n-1}, \right. \\ \left. \text{and } \left(\frac{\epsilon}{n}\right)^2 < \sum_{j=1}^{n-1} p_j^2 < \epsilon^2. \right\}$$

The parameter ϵ , give the bounds for the euclidean norm of $\mathbf{p} \in (p_1, \dots, p_{n-1})$, and δ defines a geometrical ratio between the actions p_j , thus suggesting the name, *cone set*.

So far, the hamiltonian (23) is defined for real values of (q_j, p_j) . In order to apply Arnold's real-analytic version of the KAM theorem (cf. [1], pg. 15), this hamiltonian has to be extended analytically into the complex region. Consider

$$(25) \quad F = \left\{ \mathbf{p} = (p_1, \dots, p_{n-1}) \in G \subset \mathbb{C}^{n-1}, \right. \\ \left. \mathbf{q} = (q_1, \dots, q_{n-1}) \text{ s.t. } |Im(q_j)| < \rho \right\}.$$

where G to begin with, is the set,

$$(26) \quad \{\mathbf{p} \in \mathbb{C}^{n-1} \mid Re(p_j) > 0\}.$$

and ρ will be fixed later. In extending the formula (23a) for H_0 to complex valued actions, we choose the logarithm function to be the principal branch logarithm, which is well-defined and has no singularities in G . Keeping this choice of the logarithm, we will have to further restrict the domain of the complex actions in order for the perturbation terms H_1, H_2 (cf. (23b, c)) to be well-defined.

We fix the "height" of the strip for the complex angles q_j in F to be

$$(27) \quad \rho = \frac{1}{10}$$

for the rest of the proofs (other small values for ρ will also work in the proof of Lemma 3).

For the restricted subsets of G , we pick the following extensions of the real cone set, (24).

DEFINITION 4. The complex cone set is

$$(28) \quad M_\delta(\epsilon) = \left\{ \mathbf{p} \in G \mid \left| \begin{array}{l} |p_1| < \delta |p_2| < \delta^2 |p_3| < \dots < \delta^{n-2} |p_{n-1}| \\ \text{and } \left(\frac{\epsilon}{n}\right)^2 < \sum_{j=1}^{n-1} |p_j|^2 < \epsilon^2 \end{array} \right. \right\}$$

Now, we prove that this choice of the complex domain F is adequate for the required extension of all the terms (23 a, b, c) in the hamiltonian to complex actions and angles. To fix the notation, F now denotes the set (25) with $\rho = \frac{1}{10}$ and G replaced by the subset $M_\delta(\epsilon)$ where the parameters δ and ϵ will be determined in the proof.

LEMMA 3. In F , for $\epsilon < \epsilon^*(\Gamma)$ and $\delta < \delta^*(\Gamma)$, the hamiltonian, (12) (in action-angle form, (23)) is analytic, 2π -periodic in each q_j , and is a small perturbation of order 10^{-2} of the completely integrable term $H_0(p_j)$, which in fact is decoupled in all its $(n-1)$ degrees of freedom.

Proof. We will show that there exists $\epsilon^*(\Gamma)$ and $\delta^*(\Gamma)$, depending on the circulation $\Gamma = (\Gamma_1, \dots, \Gamma_n)$ of the given hamiltonian function, (23), such that the desired results hold in the associated F . It should be clear that Γ is in the complement of Ω (cf. (11)).

- (a) First, $H_0(p_j)$, (23a) is obviously *completely integrable* because all the angles, q_j are cyclic variables, i.e. do not appear in H_0 . In fact, the form of (23 a), being a sum of $(n-1)$ terms, each dependent on its own p_j , implies that H_0 is the hamiltonian for $(n-1)$ decoupled singular oscillators. $H_0(p_j)$ is analytic in the set G , where the actions, p_j are restricted to the open right half complex plane, because the principal branch logarithm has no singularities there. Therefore, H_0 is analytic in F for arbitrary ϵ and δ .
- (b) Next, we expand $H_1(q_j, p_j)$, (23b); treating the variables (q_j, p_j) as real quantities, and after using the standard formula for the modulus of a complex number, we obtain,

$$(29) \quad H_1(q_j, p_j) = \frac{1}{4\pi} \sum_{s=1}^{n-1} \sum_{\{j,k\} \in \chi(s)} \Gamma_j \Gamma_k \left\{ \log \left[1 + \sum_{r < s} \frac{C_r^2 p_r^2}{C_s^2 p_s^2} \right. \right. \\ \left. \left. + 2 \sum_{r < s} \frac{C_r p_r}{C_s p_s} \cos(q_r - q_s) \right. \right. \\ \left. \left. + 2 \sum_{l, m < s} \frac{C_l C_m p_l p_m}{C_s^2 p_s^2} \cos(q_l - q_m) \right] \right\}$$

Observe that once the circulation Γ is given, the coefficients, $\left\{ \begin{array}{l} C_r(\{j, k\}), r \leq s, \{j, k\} \in \chi(s), \\ s \in (1, \dots, n-1) \end{array} \right\}$ are completely determined. Since there are only a finite number of quotients,

$$\frac{C_r p_r}{C_s p_s} \quad \text{and} \quad \frac{C_l C_m p_l p_m}{C_s^2 p_s^2}$$

in (29), we can choose $\delta = \delta(\Gamma)$ so that these quotients are as small as we wish in the complex cone set $M_\delta(\epsilon)$. For $\rho = \frac{1}{10}$, the trigonometric terms in (29), i.e.

$$\cos(q_r - q_s), \quad \cos(q_l - q_m)$$

are $O(1)$ in modulus, in the complex strip, $|I_m(q_j)| < \rho$. Combining the above statements, we conclude that the complex arguments of logarithm in (29) can be made as close to 1

as we wish by proper choice of $\delta(\Gamma)$ in F . Since the principal branch logarithm has no singularities near 1, H_1 is therefore, analytic in F for $\delta < \delta^*(\Gamma)$ and the choice $\delta^*(\Gamma)$ is now determined to make the modulus of H_1 small, i.e. $|H_1| < \frac{1}{100}$. Obviously, H_1 is 2π -periodic in each q_j

(c) Finally, the expression

$$(30) \quad \frac{\sin \pi F_{jk}(q_l, p_l)}{\pi F_{jk}(q_l, p_l)}$$

is nearly 1 if

$$F_{jk}(q_l, p_l) = C_s(\{j, k\})p_s e^{iq_s} + \sum_{r < s} C_r(\{j, k\})p_r e^{iq_r}$$

is small in F . For $\rho = \frac{1}{10}$, all the exponentials $|e^{iq_r}| \sim 0(1)$ in the strip $|Im(q_j)| < \rho$. Since the coefficients are determined by Γ , one can choose $\epsilon = \epsilon(\Gamma)$ so that $|p_j|$ are small, and therefore, F_{jk} can be made as small as desired in F , with the cone set $M_\delta(\epsilon(\Gamma))$ where δ is arbitrary. Substituting the expression F_{jk} in (30), and using the usual formula for sine, we find that (30) is analytic in F for $\epsilon < \epsilon^*(\Gamma)$ and $\delta < \delta^*(\Gamma)$. The principal branch logarithm is therefore analytic in F . $\epsilon^*(\Gamma)$ is now determined by the condition,

$$|H_2| < \frac{1}{100}.$$

Since the angles q_j appear only in the trigonometric terms in the expansion of (30), H_2 is obviously 2π -periodic in each q_j . Putting (a), (b), and (c) together, we have the proof. \square

With Lemma 3, we have succeeded in putting the hamiltonian (12) in the form of (2), as intended. The following remarks are very important for understanding the approach in section IV. Although the proof of Lemma 3 indicates that one can make the perturbation $H_1 + H_2$ arbitrarily small by simultaneously choosing ϵ and δ sufficiently small, we *do not* use this way to control the size of the perturbation because of one possible difficulty, i.e.

as the cone set shrinks in "size", ϵ and "angle", δ , the set of invariant tori of the unperturbed term, H_0 changes and the KAM-bound [1] for the size of the perturbation may change with ϵ and δ in such a way that the KAM bound is never satisfied no matter how small both ϵ and δ are.

Instead, we will fix δ at $\delta^*(\Gamma)$ and use rigorous scaling arguments to show that the KAM bound is independent of ϵ , as it tends to zero. We have shown that for given circulation Γ that is not in Ω (cf. (11)), there is a choice $\epsilon^*(\Gamma)$ and $\delta^*(\Gamma)$ and $\rho = \frac{1}{10}$ of the parameters for the complex domain F , in which the total perturbation is bounded, i.e. $|H_1 + H_2| < \frac{1}{50}$. The choice of $\frac{1}{50}$ is quite arbitrary and any small number will serve as well to fix the complex domain F for now.

IV. KAM nondegeneracy condition and the KAM bound. In Arnold's real analytic version of the KAM-theorem (cf. Theorem 1 on page 15 in [1]), the KAM-bound, denoted by M depends on several factors, i.e. $M = M(\rho, G, H_0, N)$ where ρ is the "height" of the strip in F , G is the complex domain of the actions, which in our case, is the complex cone set $M_{\delta^*}(\Gamma)(\epsilon^*(\Gamma))$, H_0 is the completely integrable term and N is the degrees of freedom, which in our case, is equal to $n - 1$, where n is the degrees of freedom of the original hamiltonian, (5). For our problem, $\rho = \frac{1}{10}$ is fixed and so is the degrees of freedom, $n - 1$. Thus we need only study the dependence of the KAM-bound on $G = M_{\delta^*}(\epsilon^*)$ and H_0 , via the parameters θ and Θ , defined below [1].

Following Arnold (cf. [1], pg. 16), define a diffeomorphism from action-space to frequency-space, i.e.

$$(32) \quad A : \{p_j\} \longrightarrow \left\{ \omega_j = \frac{\partial H_0}{\partial p_j} \right\}$$

where ω_j is the frequency of the j -th independent oscillator (cf. Lemma 3, part (a)). Since H_0 is given by (23a), we get the frequencies,

$$(33) \quad \omega_s \stackrel{\text{def}}{=} \frac{\partial H_0}{\partial p_s} = \left[\sum_{\{j,k\} \in X(s)} \Gamma_j \Gamma_k \right] \frac{1}{p_s}.$$

Bounds on the range of frequencies, $\omega = (\omega_1, \dots, \omega_{n-1})$ in the image of the complex cone set, G under the map A , are given by the following (cf. [1])

$$(34) \quad \theta |dp| \leq |dA| \leq \Theta |dp|, \quad 0 < \theta < 1 < \Theta < \infty.$$

Equation (33) implies that θ and Θ are functions of ϵ and δ in the cone set G and the given circulations $\Gamma = (\Gamma_1, \dots, \Gamma_n)$ of H_0 . Now $\delta = \delta^*$ is fixed from the start in our complex domain F . We will show that the θ and Θ do not change as we reduce the "size", ϵ of the cone set, $M_{\delta^*}(\epsilon)$ and simultaneously scale the circulations Γ uniformly.

LEMMA 4. *The KAM-bound $M = M(\theta, \Theta)$ for (23) is invariant under the following scaling transformations of the complex domain F and circulations Γ ,*

$$(35a) \quad p_j \longrightarrow \Delta p_j \quad \text{for} \quad j = 1, \dots, n - 1$$

$$(35b) \quad \Gamma_k \longrightarrow \Delta \Gamma_k \quad \text{for} \quad k = 1, \dots, n.$$

Proof. Note that the scaling transformations, (35) do not change the "angle", δ^* of the cone set, $M_{\delta^*}(\epsilon)$ nor the "height" $\rho = \frac{1}{10}$ of the complex strip for $\{q_j\}$ in F . The first transformation reduces ϵ by a factor of Δ . Now, equation (33) and (34) imply that θ and Θ are invariant under (35a, b). Since ρ, δ^* and n are fixed, $M = (\theta, \Theta)$ is invariant under (35a, b). \square

LEMMA 5. The perturbation $H_1 + H_2$, (23b, c) decreases quadratically in Δ under (35).

Proof. (a) The perturbation term H_1 , (23b) depends quadratically in the circulations $\{\Gamma_j\}$, i.e. $\Gamma_j \Gamma_k$ are multiplicative factors of the logarithm functions. Recall the discussion following (17), that the coefficients, $C_r(\{j, k\})$ are invariant under (35b). Moreover, the actions $\{p_j\}$ enter the arguments of the logarithm in H_1 , only in the form of homogeneous ratios. Therefore the logarithm functions in H_1 are invariant under (35a, b). Thus $|H_1|$ is quadratic in Δ .

(b) The perturbation term H_2 , (23 c) also depends quadratically in $\{\Gamma_j\}$ in a similar way. Now the logarithm functions

$$\log \left[\frac{\sin \pi F_{jk}}{\pi F_{jk}} \right]$$

actually decreases as the actions, $\{p_j\}$ and thus, $F_{jk}(q_l, p_l)$ decreases under (35a, b). Therefore $|H_2| \sim \Delta^2 o(\Delta)$.

Combining (a) and (b) we obtain the result. \square

Lemma 4 and 5 show that after repeated applications of transformations (35a, b), the size of the perturbation terms should satisfy the KAM-bound, i.e. $|H_1 + H_2| < M$ eventually. Now, the reader may raise the objection:

repeated use of transformation (35b) appears to imply that the circulations $\{\Gamma_j\}$ are severely restricted, and moreover, are different from the circulations in the original hamiltonian (5) or (23).

There is a trick around this obstacle, which is based on the invariance of the hamilton's equations under transformation (9). This says that a uniform scaling of the circulations $\{\Gamma_j\}$ such as (35b) will only change the frequencies of the motion uniformly, or equivalently, change the time-scale. Therefore, if the KAM-theorem implies that T is a KAM torus for the hamiltonian (23) with circulations Γ , then it is also an invariant torus for the hamiltonian (23) with circulations $\Gamma' = (\Delta)^m \Gamma$ where m is some large negative integer.

Only the KAM nondegeneracy condition (cf. [1], pg. 15 eqn. (1)) remains to be verified in the following lemma.

LEMMA 6. The unperturbed term $H_0(p_j)$, (23a) satisfies the KAM-nondegeneracy condition if the following sufficient condition holds,

$$(36) \quad \prod_{s=1}^{(n-1)} \left[\sum_{\{j,k\} \in \chi(s)} \Gamma_j \Gamma_k \right] \neq 0.$$

Proof. The KAM nondegeneracy condition, expressed in terms of

$$\det \left[\frac{\partial^2 H_0}{\partial p_j \partial p_k} \right] \neq 0$$

is equivalent to condition (36) because the Hessian is given by the diagonal matrix,

$$(37) \quad - \begin{bmatrix} \left(\sum_{x(1)} \Gamma_j \Gamma_k \right) / p_1^2 & & & \\ & \ddots & & \\ & & \left(\sum_{x(s)} \Gamma_j \Gamma_k \right) / p_s^2 & \\ & & & \ddots \\ & & & & \left(\sum_{x(n-1)} \Gamma_j \Gamma_k \right) / p_{n-1}^2 \end{bmatrix}$$

Finally, we give the proof of the main result:

Proof of Theorem 1.

- (a) Lemma 1 and the fact that if the invariant tori $T^{(n-1)}$ has positive measure in the reduced system, then the product $T^{(n-1)} \times \mathbb{R}^2$ is invariant and has positive measure in \mathbb{R}^{2n} , imply that Theorem 1 can be established by proving that the KAM-theorem [1] is valid for (12).
- (b) Lemma 2 and 3 state that the hamiltonian (12) takes the form of a nearly-integrable hamiltonian (23), which is moreover, analytic in the complex domain

$$F = \left\{ \begin{array}{l} \mathbf{p} \in M_{\delta^*}(\epsilon^*) \subset \mathbb{C}^{n-1}, \\ \mathbf{q} \text{ such that } |Im(q_j)| < \rho = \frac{1}{10} \end{array} \right\}$$

for complex action-angle variables, (22), and 2π -periodic in each q_j .

- (c) Lemma 6 implies that the *completely integrable* term $H_0(p_j)$ satisfies the KAM nondegeneracy condition if (36) holds which is equivalent! to the condition that Γ is in the complement of Ω (cf. (11)).
- (d) Lemma 4 and 5 imply that after m (some large integer) applications of (35 a,b) on the complex domain F and circulations Γ , the perturbation is less than the KAM-bound which is invariant under (35) i.e. $|(H_1 + H_2)| < M(\rho, n, \delta^*)$. Moreover, $\Gamma' = (\Delta)^m \Gamma$ still satisfies the KAM nondegeneracy condition. With circulation, Γ' the hamiltonian

(23) satisfies all the conditions of the KAM-theorem (cf. [1] theorem 1) in the reduced domain F' . Therefore, there is an invariant set of tori, $T^{(n-1)}$ with positive measure in $\mathbb{R}^{(2n-2)}$, and moreover, each such torus T supports, quasi-periodic motion with $(n-1)$ *strongly nonresonant* frequencies, $\omega_j = \frac{\partial H_0}{\partial p_j}$, given by (33) (with Γ').

- (e) By the invariance of the hamilton's equations of (12) under (9), each torus T for circulations Γ' is again an invariant torus for the original circulations Γ . Thus we have established the results of Theorem 1. \square

Since the equivalence between (36) and (11) is such a surprising and delightful by-product of our calculations, we restate the result in a corollary:

COROLLARY 1. *The KAM nondegeneracy condition (36) for hamiltonians of the form (3) is equivalent to a combinatorial condition that the canonical transformation (10) is well-defined.*

It is important to note that any relabeling of the vortices by a permutation on n integers, will not affect the canonical transformation (10) provided, of course, that the corresponding condition for well-definition, (11) is again satisfied. Thus new cone sets, one for each permutation on n integers, can be defined in phase-space, and all the steps of the proof of Theorem 1 can be executed. We therefore have the

COROLLARY 2. *For each permutation, P of the conditions (11), there belongs a permuted set of circulations $P(\Omega)$. Suppose Γ is in the complement of $P(\Omega)$, then there is an invariant set $P(\mathcal{K})$ of positive measure in the permuted cone set $P(M_\delta^{(\epsilon)})$.*

From this, we draw the conclusion that there can be several distinct invariant regions in phase-space if the circulations Γ lies in the complement of the union of several $P(\Omega)$. Moreover, the condition in theorem 1 that Γ is in the complement of Ω is *not* a necessary condition for obvious reasons.

VI Conclusions. More important than the existence of KAM tori for the lattice vortex models, or the physical and numerical consequences it implies, we have invented an alternative method for applying KAM theory, which does not impose *a priori* restrictions on the given or "external" coefficients of the hamiltonian, i.e. the circulations Γ or "weights" of the particles. This method can be applied to classes of hamiltonians, other than those discussed here.

Furthermore, the need for the quintessential tools of canonical transformation theory in this approach, have prompted our study of the relation between graph theory and symplectic matrices, which has produced some results that are interesting in their own rights [11, 12]. Finally, the discovery of a surprising link between the structure of the canonical transformation used, and the resulting KAM-nondegeneracy condition, is a special bonus, which gives a combinatorial interpretation of this KAM condition.

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