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Abstract. The abelian Higgs models with sources (e.g. an arbitrarily distributed external magnetic field) on a torus T^d ($d = 2, 3, 4$) and on an Euclidean space-time $\mathbb{R}^4$ are studied. In the former case, there exist regular solutions which minimize the total (Gibbs) energy and are in the Coulomb gauge. The method of investigation is then used to prove the existence of Abrikosov's mixed state solutions in type II superconductivity with prescribed flux strength. In the latter case, the existence is established for energy minimizing solutions among all field configurations in the Coulomb gauge.

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1. Introduction. Recently, the abelian Higgs models with sources have attracted considerable attention. Although the solutions of the models cannot be found analytically except for some extreme cases, the comparison of the models with the Ginzburg-Landau theory [7] in superconductivity and the use of the lattice gauge theories proposed by Wilson [16] already allow a fairly good understanding of a wide range of problems. In particular, the ground state behavior of the abelian Higgs models has been discussed in [9] at the static tree level approximation which shows that a sufficiently strong external magnetic field can alter the ground state of the theory by either restoring a spontaneously broken symmetry or creating a qualitatively different mixed phase ground state corresponding to a sector of the theory where the Higgs field has a non-constant expectation value; in [11] the critical field strengths are calculated and a picture of phase transitions is depicted in terms of the ratio of the Higgs boson mass and the gauge boson mass determined in the absence of an external field; and these heuristic arguments have found support from various numerical simulations [4,5,8] based upon computations (usually via a Monte Carlo approach) of the corresponding lattice gauge theories with sources.

Although all the above and other studies have respected the least action principle (namely, the system will exist in the state of the lowest (Gibbs) energy), unfortunately, mathematically rigorous existence results for energy minimizing solutions have not been well-established due to the lack of a suitable function space setting of the problem. The main purpose of the present paper is to address these existence problems.

The paper may be viewed as a continuation of our earlier work [14] where the energy minimizing solutions among all field configurations in the Coulomb gauge of the static abelian Higgs model on $\mathbb{R}^3$ (or the Ginzburg-Landau theory) with sources have been shown to exist and the expected vacuum decay properties proved. The method succeeded in [14] is a modified functional trick: the modified "energy" functional fails to be gauge-invariant but its critical points will automatically be in the Coulomb gauge, and as a

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consequence, the original (gauge-invariant) Gibbs energy is dramatically recovered. An outline of the contents of the paper is as follows. In the next section we consider a compact version of the abelian Higgs model defined on a torus of dimensions $= 2, 3, 4$. Although the model has interest of its own right, the motivation of such a study can also be thought to be originated from the periodic structure of the lattice gauge theories and the compact version of gauge theories [3,13] on a box where periodic boundary conditions are imposed. We shall show that under the influence of an arbitrary external source (mostly our discussions are carried out under the specific assumption that the source is an external magnetic field for simplicity), the minimal energy value can be attained by a field configuration in the Coulomb gauge. The additional difficulty is that the space-average of a vector potential cannot be controlled by its Dirichlet integral. Fortunately, the structure of a torus enables us to make suitable gauge transformations to overcome this difficulty. In section 3, we consider a simulation of Abrikosov's mixed states. In his original paper [1] Abrikosov predicted the square lattice structure of the magnetic field in a two-dimensional superconductor of type II. We shall study a prescribed magnetic flux problem where the field strength is periodic on a square. To see that our model is physically relevant, we will give a partial proof of the Meissner effect on magnetic screening. Along this direction, there was only one mathematical study [10] where no external field was present and solutions were shown to exist under some restrictive assumptions. In Section 4, we prove the existence of the least energy solutions of the abelian Higgs models with sources on $\mathbf{R}^4$. As in Section 2, the method of [14] is not directly applicable due to certain restrictions in a Sobolev embedding relation. However, if we make a suitable choice of the energy minimizing sequences, this difficulty can be tackled. In Section 5 we summarize the results for the case where the external field is a general source current. Section 6 is a brief remark about the asymptotic decay properties of finite energy solutions on $\mathbf{R}^4$. Note that, in order to achieve desired decay results in higher dimensions, we have to assume that the external field enjoys some stronger integrability conditions.

2. The abelian Higgs model with sources on a torus. As usual, let us first specify an equivalence relation $\sim$ on $\mathbf{R}^d$. We say $x \sim y$ for $x = (x^j)$, $y = (y^j) \in \mathbf{R}^d$ if $x^j = y^j \mod (T_j)$ for some fixed $T_j > 0$, $j = 1, 2, \ldots, d$. Then the torus T^d with periods $\{T_j\}_{j=1}^d$ is defined as the quotient space $\mathbf{R}^d / \sim$ endowed with the standard topology. With the notation $\Omega^d = \Omega(T^d) = [0, T_1] \times \cdots \times [0, T_d]$, the Gibbs energy of the (static) abelian Higgs model with an external magnetic field F_{jk}^{ex} on T^d can be written in the form

$$E(A,\phi) = \int_{\Omega^d} dx \left\{ \frac{1}{4} F_{jk}F_{jk} + \frac{1}{2}(D_j^A\phi)^*(D_j^A\phi) + \frac{\lambda}{8}(|\phi|^2 - 1)^2 - \frac{1}{2}F_{jk}F_{jk}^{\mathrm{ex}} \right\}$$

where (A, ϕ) and F_{jk}^{ex} satisfy the periodic boundary conditions on $\partial\Omega^d$, $F_{jk} = \partial_j A_k - \partial_k A_j (j, k = 1, \cdots, d)$ is the excited magnetic field induced from the vector potential $A = (A_j)_{j=1}^d$, ϕ is the complex scalar Higgs field, $D_j^A\phi = \partial_j\phi - iA_j\phi$ is the gauge-covariant derivative of ϕ, and the summation convention over repeated indices is observed. Therefore the Euler-Lagrange equations are:

$$
\begin{cases} D_j^A D_j^A \phi + \frac{\lambda}{2}(1-|\phi|^2)\phi = 0, \\ \partial_k F_{kj} + \frac{i}{2}(\phi(D_j^A\phi)^* - \phi^*(D_j^A\phi)) = \frac{1}{2}\partial_k(F_{kj}^{\mathrm{ex}} - F_{jk}^{\mathrm{ex}}). \end{cases} \tag{2.1}
$$

Let us introduce the function spaces:

$H(\Omega^d) =$ the completion in $W^{1,2}(\Omega^d)$ of the set of all (vector-valued) C^∞ functions on $\mathbf{R}^d$ with multiple periods $\{T_j\}_{j=1}^d$,

$$
\overset{\circ}{H}(\Omega^d) = \left\{ \alpha \in H(\Omega^d) \mid \int_{\Omega^d} \alpha dx = 0 \right\}.
$$

To simplify the statements slightly, we shall assume that F_{jk}^{ex} is smooth.

After the above preparation, we can state the main result of this section as follows:

THEOREM 2.1. *(Existence of an energy minimizing solution for the model on T^d) For $d = 2,3,4$ and fixed $T_j > 0$, $j = 1,\cdots,d$, the equations (2.1) have a smooth solution $(\tilde{A},\tilde{\phi})$ on $\mathbf{R}^d$ with multiple periods $\{T_j\}_{j=1}^d$. This solution solves the following minimization problem*

$$
m = \min\{E(A,\phi) \mid (A,\phi) \in H(\Omega^d)\} \tag{2.2}
$$

and enjoys the properties

$$
\partial_j \tilde{A}_j = 0 \quad \text{in } \mathbf{R}^d \text{ and } \left| \int_{\Omega^d} \tilde{A} dx \right| < (T_1 \cdots T_d) \sqrt{\sum_{j=1}^d \left(\frac{2\pi}{T_j} \right)^2}.
$$

Proof. First we observe that the gauge transformation

$$
\phi \mapsto e^{i\eta}\phi, \quad A \mapsto A + \nabla\eta
$$

will keep the periodicity of (A,ϕ) provided we require:

$$
\eta(x) = \eta(y) \mod (2\pi) \text{ if } x^j = y^j \mod (T_j), \quad j = 1,\cdots,d.
$$

Hence the following phase changes are allowed:

$$\eta(x) = 2\pi\frac{z_1 x^1}{T_1} + \cdots + 2\pi\frac{z_d x^d}{T_d}, \quad z_1, \cdots, z_d \in \mathbf{Z}. \tag{2.3}$$

For $A \in H(\Omega^d)$ we have the unique decomposition:

$$A = A^0 + \alpha \text{ with } A^0 \in \mathbf{R}^d \text{ and } \alpha \in \mathring{H}(\Omega^d).$$

In fact, A^0 can always be decided by the simple formula

$$A^0 = (T_1 \cdots T_d)^{-1} \int_{\Omega^d} A dx. \tag{2.4}$$

Fixing $A \in H(\Omega^d)$ and calculating A^0 through (2.4), then we can choose $z_1, \cdots, z_d \in \mathbf{Z}$ such that

$$A^0 = \left(B_1^0 + 2\pi\frac{z_1}{T_1}, \cdots, B_d^0 + 2\pi\frac{z_d}{T_d} \right)$$

and $|B_j^0| < 2\pi/T_j$, $j = 1, \cdots, d$.

As a consequence, for any field configuration $(A, \phi) \in H(\Omega^d)$, we may make a suitable gauge transformation induced by a phase shift of the type (2.3) to pull down the space-average of A so that

$$\left| \int_{\Omega^d} A dx \right| < (T_1 \cdots T_d) \sqrt{\sum_{j=1}^{d} \left(\frac{2\pi}{T_j} \right)^2} \equiv M.$$

Therefore (2.2) is equivalent to the minimization problem:

$$m = \min \left\{ E(A, \phi) \mid (A, \phi) \in H(\Omega^d), \left| \int_{\Omega^d} A dx \right| < M \right\}. \tag{2.5}$$

Now we can use the modified functional method introduced in [14] to tackle the problem (2.5).

Define a new "energy" for any $(A, \phi) \in H(\Omega^d)$ as:

$$\tilde{E}(A, \phi) = \int_{\Omega^d} dx \left\{ \frac{1}{2}(\partial_j A_k)(\partial_j A_k) + \frac{1}{2}(D_j^A \phi)^*(D_j^A \phi) + \frac{\lambda}{8}(|\phi|^2 - 1)^2 - \frac{1}{2} F_{jk} F_{jk}^{\text{ex}} \right\}.$$

Consider the minimization problem associated with $\tilde{E}$:

$$\tilde{m} = \min \left\{ \tilde{E}(A, \phi) \mid (A, \phi) \in H(\Omega^d), \left| \int_{\Omega^d} A dx \right| < M \right\}. \tag{2.6}$$

Let $\{(A^n, \phi^n)\}$ be a minimizing sequence of the problem (2.6) and set the decomposition:

$$A^n = A^{n0} + \alpha^n, \quad A^{n0} \in \mathbf{R}^d, \quad \alpha^n \in \overset{\circ}{H}(\Omega^d), \quad n = 1, 2, \cdots .$$

In virtue of the definition of $\tilde{E}$ and the fact that Poincaré's inequality holds on the space $\overset{\circ}{H}(\Omega^d)$, we can conclude that $\{\alpha^n\}$ is a bounded sequence in $\overset{\circ}{H}(\Omega^d)$. From the standard Sobolev embeddings

$$(2.7) \qquad W^{1,2}(\Omega^2) \to L^p(\Omega^2)\ (p \geq 1) \quad \text{and} \quad W^{1,2}(\Omega^d) \to L^{\frac{2d}{d-2}}(\Omega^d)\ (d > 2)$$

it follows that $\{\alpha^n\}$ is also bounded in $L^p(\Omega^d)$ for $1 \leq p < \infty$ if $d = 2$; $1 \leq p \leq 2d/(d-2)$ if $d \geq 3$.

Let us choose first the easier cases $d = 2, 3$ to discuss. Hence we may assume $\{\alpha_n\}$ is bounded in $L^6(\Omega^d)$ (see (2.7)).

From the readily inferred inequality

$$(2.8) \qquad 2\tilde{E}(A^n, \phi^n) \geq \int_{\Omega^d} (D_j^{A^n} \phi^n)^* (D_j^{A^n} \phi^n) dx$$

$$\geq \frac{1}{2} \|\nabla \phi^n\|^2_{L^2(\Omega^d)} - 2\|(|\phi^n|^2 - 1)\|^2_{L^2(\Omega^d)} - C_1$$

where $C_1 > 0$ is a constant depending on $\sup_n \|\alpha^n\|_{L^4(\Omega^d)}$, and $|\phi^n|^4 \leq 2(|\phi^n|^2 - 1)^2 + 1$, one sees that $\{\phi^n\}$ is a bounded sequence in $H(\Omega^d)$.

Therefore, using the compact inclusion $H(\Omega^d) \to L^p(\Omega^d)\ (1 \leq p < 6, d = 2, 3)$, we may assume for simplicity that there exists $(\tilde{A}, \tilde{\phi}) \in H(\Omega^d)$ such that

$$(2.9) \qquad \begin{aligned} (\tilde{A}, \tilde{\phi}) &= (w) - \lim_{n \to \infty} (A^n, \phi^n) \quad \text{in } H(\Omega^d), \\ (\tilde{A}, \tilde{\phi}) &= (s) - \lim_{n \to \infty} (A^n, \phi^n) \quad \text{in } L^p(\Omega^d) \quad (1 \leq p < 6), \end{aligned}$$

where $\tilde{A}^0 = \lim_{n \to \infty} A^{n0}$ and hence $|\tilde{A}^0| \leq \mu$.

Now, on applying (2.9), one easily finds

$$(2.10) \qquad \tilde{E}(\tilde{A}, \tilde{\phi}) \leq \lim_{n \to \infty} \tilde{E}(A^n, \phi^n) = \tilde{m}.$$

For $d = 4$, we have to take a somewhat indirect path to reach (2.10).

Set $\tilde{E}_n = \tilde{E}(A^n, \phi^n)$. Then there is a smooth pair $(\mathcal{A}^n, \psi^n) \in H(\Omega^4)$ such that $|\int_{\Omega^4} \mathcal{A}^n dx| < M$ and

$$(2.11) \qquad \tilde{E}(\mathcal{A}^n, \psi^n) < \tilde{E}_n + \frac{1}{n}.$$

Choose $\chi^n \in H(\Omega^4)$ such that

$$\tilde{E}(\mathcal{A}^n, \chi^n) = \min\{\tilde{E}(\mathcal{A}^n, \psi) \mid \psi \in H(\Omega^4)\}. \tag{2.12}$$

(The proof of existence of such a minimizer to the problem (2.12) is elementary.) Then, as a critical point of the functional $F(\psi) = \tilde{E}(\mathcal{A}^n, \psi)$ defined on $H(\Omega^4)$, χ^n is smooth and satisfies the elliptic equation:

$$D_j^{\mathcal{A}^n} D_j^{\mathcal{A}^n} \chi^n = \frac{\lambda}{2}(|\chi^n|^2 - 1)\chi^n. \tag{2.13}$$

From (2.13) we have the inequality:

$$\Delta(1 - |\chi^n|^2) \le \lambda|\chi^n|^2(1 - |\chi^n|^2).$$

Hence, an application of the maximum principle gives:

$$|\chi^n| \le 1 \text{ on } \Omega^4. \tag{2.14}$$

In a way similar to that for obtaining (2.8), one derives

$$2\tilde{E}(\mathcal{A}^n, \chi^n) \ge \frac{1}{2}\|\nabla\chi^n\|^2_{L^2(\Omega^4)} - C_1\|\beta^n\|^2_{L^2(\Omega^4)} - C_2 \tag{2.15}$$

where (2.14) has been used, C_1, C_2 are constants independent of n, and $\mathcal{A}^n$ has the decomposition: $\mathcal{A}^n = \mathcal{A}^{n0} + \beta^n$ with $\mathcal{A}^{n0} \in \mathbf{R}^4$ and $\beta^n \in \overset{\circ}{H}(\Omega^4)$.

As before, one can see that $\{\beta^n\}$ is a bounded sequence in $\overset{\circ}{H}(\Omega^4)$. Consequently, in virtue of (2.14) and (2.15) we have the boundedness of $\{\chi^n\}$ in $H(\Omega^4)$.

From the above results, we may assume for simplicity that there is $(\tilde{A}, \tilde{\phi}) \in H(\Omega^4)$ so that (2.9a) holds and

$$\begin{aligned} \tilde{A} &= (s) - \lim_{n\to\infty} \mathcal{A}^n \text{ in } L^p(\Omega^4) \qquad (2 \le p < 4), \\ \tilde{\phi} &= (s) - \lim_{n\to\infty} \chi^n \text{ in } L^p(\Omega^4) \qquad (4 \le p < \infty), \end{aligned} \tag{2.16}$$

where we have used the compact embedding $H(\Omega^4) \to L^p(\Omega^4)$ $(1 \le p < 4)$, the Hölder inequality, and (2.14).

The inequality (2.11) and the definition of χ^n imply that $\{(\mathcal{A}^n, \chi^n)\}$ is also a minimizing sequence of the problem (2.6). From this fact, (2.9a), and (2.16) we immediately obtain the expected conclusion (2.10).

A gauge transformation induced by a phase change of the type (2.3) can always be chosen to make $|\tilde{A}^0| < \mu$ and keep $\tilde{E}$ invariant, thus a minimizer of the problem (2.6) is found. As a critical point of the functional $\tilde{E}$ in the open set

$$\mathcal{O}_M = \left\{ (A, \phi) \in H(\Omega^d) \mid \left| \int_{\Omega^d} A dx \right| < M \right\}$$

of $H(\Omega^d)$, $(\tilde{A}, \tilde{\phi})$ must satisfy the equations

$$(2.17) \qquad \begin{cases} D_j^{\tilde{A}} D_j^{\tilde{A}} \tilde{\phi} + \frac{\lambda}{2}(1 - |\tilde{\phi}|^2)\tilde{\phi} = 0, \\ \partial_k \partial_k \tilde{A}_j + \frac{i}{2}(\tilde{\phi}(D_j^{\tilde{A}}\tilde{\phi})^* - \tilde{\phi}^*(D_j^{\tilde{A}}\tilde{\phi})) = \frac{1}{2}\partial_k(F_{kj}^{\text{ex}} - F_{jk}^{\text{ex}}). \end{cases}$$

It then follows from the standard elliptic regularity arguments that $(\tilde{A}, \tilde{\phi})$ is smooth (and periodic) in $\mathbf{R}^d$. On applying the operator ∂_j on both sides of (2.17b), making the summation with respect to the index j, and using (2.17a) we obtain $\nabla^2(\nabla \cdot \tilde{A}) = 0$. Since $\tilde{A}$ is periodic, we have $\nabla \cdot \tilde{A} = \text{const.} = C$. Using the periodicity of $\tilde{A}$ again one gets: $T_1 \cdots T_2 C = \int_{\Omega^d} \nabla \cdot \tilde{A} dx = 0$ This proves $\tilde{A}$ is divergence-free. Hence, in particular, $(\tilde{A}, \tilde{\phi})$ is a smooth solution of Eqs.(2.1) on $\mathbf{R}^d$ with multiple periods $\{T_j\}_{j=1}^d$.

Let $K(\Omega^d) = \{A \in H(\Omega^d) | \partial_j A_j = 0\}$. Then it is readily verified that

$$(2.18) \qquad 2\int (\partial_j A_k)(\partial_j A_k) dx = \int F_{jk} F_{jk} dx, \quad A \in K(\Omega^d).$$

For any $(A, \phi) \in H(\Omega^d)$ we choose a periodic solution of the equation $\Delta u = \nabla \cdot A$ on Ω^d (the existence of such a solution is elementary as can be seen by the Fredholm alternatives) and then construct the gauge transformation

$$\phi \mapsto \phi' = e^{-iu}\phi, \quad A \mapsto A' = A - \nabla u.$$

Using (2.18) we obtain the desired comparison

$$E(\tilde{A}, \tilde{\phi}) = \tilde{E}(\tilde{A}, \tilde{\phi}) \leq \tilde{E}(A', \phi') = E(A', \phi') = E(A, \phi).$$

The theorem is proved.

From the proof of Theorem 2.1 we see that the following conclusion holds.

PROPOSITION 2.2. *If (A, ϕ) is a solution of Eqs. (2.1) on a torus T^d $(d = 2, 3, 4)$, then $|\phi| \leq 1$.*

3. Abrikosov's mixed state solutions with prescribed flux strength. Now we consider Eqs. (2.1) on $\mathbf{R}^2$. In order to simulate Abrikosov's mixed states (vortex solutions with a lattice structure) we will seek for solutions such that the field strength of the vector potential is periodic.

Let $\Omega^2 = [0,T_1]\times[0,T_2]\subset \mathbf{R}^2$ be a basic lattice cell. For a vector field $A=(A_j)$, set

$$A_j = \frac{1}{2}H\varepsilon_j^k x^k + \mathcal{A}_j, \quad \mathcal{F}_{jk} = \partial_j\mathcal{A}_k - \partial_k\mathcal{A}_j, \quad j,k=1,2 \tag{3.1}$$

where $\varepsilon_k^j = k-j$ and H is a constant. In the sequel, we shall consider the field configurations (A,ϕ) so that A has a decomposition of the form (3.1) with $\mathcal{A}$ periodic on $\mathbf{R}^2$:

$$\mathcal{A}(x) = \mathcal{A}(y) \quad \text{if} \quad x^j = y^j \mod (T_j), \quad j=1,2.$$

With the notation

$$H_{jk} = \frac{1}{2}H[\varepsilon_k^\ell\delta_j^\ell - \varepsilon_j^\ell\delta_k^\ell],$$

one has

$$F_{jk} = \partial_j A_k - \partial_k A_j = H_{jk} + \mathcal{F}_{jk},$$

and the (unit) flux can be decided through:

$$\Phi = \frac{1}{2}\int_{\Omega^2}\varepsilon_k^j F_{jk}dx = \int_{\Omega^2}H_{12}dx = HT_1T_2.$$

In terms of $(\mathcal{A},\phi)$, the (unit) Gibbs energy is given by:

$$I(\mathcal{A},\phi) = \int_{\Omega^2}dx\left\{\frac{1}{4}\mathcal{F}_{jk}\mathcal{F}_{jk} + \frac{1}{2}(D_j^A\phi)^*(D_j^A\phi) + \right. \tag{3.2}$$

$$\left. + \frac{\lambda}{8}(|\phi|^2-1)^2 + \frac{1}{2}\mathcal{F}_{jk}(H_{jk}-F_{jk}^{\text{ex}}) + \frac{1}{4}H_{jk}H_{jk} - \frac{1}{2}H_{jk}F_{jk}^{\text{ex}}\right\}$$

where $D_j^A\phi = \partial_j\phi - i(\frac{1}{2}H\varepsilon_j^k x^k + \mathcal{A}_j)\phi$.

It can easily be seen that the Euler-Lagrange equations are:

$$\begin{cases} D_j^A D_j^A\phi + \dfrac{\lambda}{2}(1-|\phi|^2)\phi = 0, \\ \partial_k\mathcal{F}_{kj} + \dfrac{i}{2}(\phi(D_j^A\phi)^* - \phi^*(D_j^A\phi)) = \dfrac{1}{2}\partial_k(F_{kj}^{\text{ex}} - F_{jk}^{\text{ex}}), \text{ in } \Omega^2; \\ \mathcal{A} \text{ satisfies the periodic boundary condition on } \partial\Omega^2, \\ D_j^A\phi|_{\partial\Omega^2} = 0, \ j=1,2. \end{cases} \tag{3.3}$$

As before, suppose F_{jk}^{ex} is smooth. Let us consider the minimization problem

$$m = \min\{I(\mathcal{A},\phi) \mid \mathcal{A}\in H(\Omega^2), \phi\in W^{1,2}(\Omega^2)\}. \tag{3.4}$$

A minimizer of (3.4) (say $(\tilde{\mathcal{A}},\tilde{\phi})$) will be a smooth solution of (3.3) provided $\tilde{\mathcal{A}}$ is in the Coulomb gauge $\partial_j\tilde{\mathcal{A}}_j = 0$.

THEOREM 3.1. *For any constant H, Eqs. (3.3) have a smooth solution (say $(\tilde{\mathcal{A}}, \tilde{\phi})$). This solution solves the minimization problem (3.4) and has the properties:*

$$\partial_j \tilde{\mathcal{A}}_j = 0, \quad \int_{\Omega^2} \tilde{\mathcal{A}} dx = 0.$$

Proof. For any $(\mathcal{A}, \phi) \in H(\Omega^2) \times W^{1,2}(\Omega^2)$ of finite energy, make the gauge transformation:

$$\mathcal{A} \mapsto \mathcal{A}' = \mathcal{A} + \nabla\eta, \qquad \phi \mapsto \phi' = e^{i\eta}\phi$$

where

$$\eta = -\mathcal{A}_j^0 x^j, \qquad \mathcal{A}^0 = (T_1 T_2)^{-1} \int_{\Omega^2} \mathcal{A} dx.$$

Then $\int_{\Omega^2} \mathcal{A}' dx = 0$ and hence (3.4) is equivalent to the minimization problem

$$m = \min\{I(\mathcal{A}, \phi) \mid \mathcal{A} \in \overset{\circ}{H}(\Omega^2), \quad \phi \in W^{1,2}(\Omega^2)\}.$$

As in Section 2, we may replace the gauge-invariant kinematic energy density term $\frac{1}{4}\mathcal{F}_{jk}\mathcal{F}_{jk}$ in (3.2) by the non-physical "energy" term $\frac{1}{2}(\partial_j \mathcal{A}_k)(\partial_j \mathcal{A}_k)$ to get the modified energy functional $\tilde{I}(\mathcal{A}, \phi)$. It can be shown (see the proof of Theorem 2.1) $\tilde{I}$ has a minimizer $(\tilde{\mathcal{A}}, \tilde{\phi}) \in \overset{\circ}{H}(\Omega^2) \times W^{1,2}(\Omega^2)$ which is the desired solution stated in the theorem.

Remark. With the notation in Theorem 3.1, $(\tilde{A}, \tilde{\phi}) \equiv ((\frac{1}{2} H \varepsilon_j^k x^k + \tilde{\mathcal{A}}_j), \tilde{\phi})$ solves the equations (2.1). This solution is obviously in the Coulomb gauge and the total flux passing through Ω^2 is given by $\Phi = H T_1 T_2$.

To see the physical meaning of the prescribed magnetic flux problem studied above, let us formulate a partial proof of the Meissner effect in superconductivity theory.

Let $(\mathcal{A}, \phi)$ be a smooth solution of (3.3). Following [2], from an integration by parts technique, one finds:

$$I(\mathcal{A}, \phi) = \frac{1}{2}\left(|H| T_1 T_2 - \int_{\Omega^2} F_{jk} F_{jk}^{\text{ex}} dx \right) \tag{3.5}$$

$$\frac{1}{2}\int_{\Omega^2} dx \left\{ \left| \mathcal{F}_{12} + H \pm \frac{1}{2}(|\phi|^2 - 1) \right|^2 + |D_1^A \phi \pm i D_2^A \phi|^2 + \frac{\lambda - 1}{4}(|\phi|^2 - 1)^2 \right\}$$

$$\pm \frac{1}{2}\int_{\Omega^2} dx \left\{ \varepsilon_k^j \left(i D_j^A \phi (D_k^A \phi)^* - \frac{1}{2} F_{jk} |\phi|^2 \right) \right\},$$

according to $H = \pm |H|$.

On the other hand,

$$\varepsilon_k^j \left(i D_j^A \phi (D_k^A \phi)^* - \frac{1}{2} F_{jk} |\phi|^2 \right) = \text{Im}\{\partial_j (\varepsilon_k^j (\phi^* D_k^A \phi))\};$$

so the third term on the right-hand-side of (3.5) vanishes in virtue of the boundary condition in (3.3).

Now choose F^{ex}_{jk} =constant and $F^{\text{ex}}_{jk} = -F^{\text{ex}}_{kj}$. With the notation $F^{\text{ex}}_{12} = H^{\text{ex}}$, (3.5) becomes

$$I(\mathcal{A},\phi) = T_1T_2\left(\frac{1}{2}|H| - HH^{\text{ex}}\right) \tag{3.6}$$

$$+\frac{1}{2}\int_{\Omega^2} dx\left\{\left|\mathcal{F}_{12} + H \pm \frac{1}{2}(|\phi|^2-1)\right|^2 + |D^{\mathcal{A}}_1\phi \pm iD^{\mathcal{A}}_2\phi|^2 + \frac{\lambda-1}{4}(|\phi|^2-1)^2\right\}.$$

Assume $\lambda \geq 1$.

If the external magnetic field is so weak that

$$|H^{\text{ex}}| < \frac{1}{2}, \tag{3.7}$$

then $I(\mathcal{A},\phi) \geq 0$ and this energy lower bound is saturated up to a gauge transformation only by the superconducting vacuum $\mathcal{A} = 0, H = 0, \phi = 1$. Hence $F_{jk} = 0$. This proves that the magnetic field is screened if the external field is sufficiently weak. Therefore the Meissner effect for type II superconducting materials follows. However, if (3.7) is violated, more sophisticated issues are likely to appear.

As an illustration, let us make the critical choice $\lambda = 1$ (the intermediate phase between type I and II superconductivity).

From (3.6), we have

$$I(\mathcal{A},\phi) \geq T_1T_2\left(\frac{1}{2}|H| - HH^{\text{ex}}\right) \tag{3.8}$$

and the above lower bound is attained if $(\mathcal{A},\phi)$ satisfies the following (Bogomol'nyi) equations *with sources*:

$$\begin{cases} D^{\mathcal{A}}_1\phi \pm iD^{\mathcal{A}}_2\phi + \frac{H}{2}(\pm x^1 + ix^2)\phi = 0, \\ \mathcal{F}_{12} \pm \frac{1}{2}(|\phi|^2-1) + H = 0, \qquad \text{in } \Omega^2, \\ \mathcal{A} \text{ satisfies the periodic boundary condition on } \partial\Omega^2, \\ D^{\mathcal{A}}_j\phi|_{\partial\Omega^2} = 0, \quad j = 1,2. \end{cases} \tag{3.9}$$

Consequently, if Eqs. (3.9) have a solution, then the problem of finding the least energy solutions of (3.3) in the case $\lambda = 1$ is equivalent to solving (3.9).

Let us assume Eqs. (3.9) have a solution for any prescribed value of H. For such a solution, of course, the equality in (3.8) holds. Thus we may make the following observations.

For $|H^{\text{ex}}| = 1/2$, the energy minimum $I_m = 0$ is attained at the vacuum solutions $A = 0, H = 0, |\phi| = 1$ as well as at the solutions of Eqs. (3.9) for any given H satisfying $\text{sgn}H = \text{sgn}H^{\text{ex}}$. This describes a "pattern selection" phenomenon: the direction of the excited flux coincides with that of the external field.

Finally, if $|H^{\text{ex}}| > 1/2$, arbitrarily low energy levels can be achieved by solutions of Eqs. (3.9) as far as $|H|$ is large enough and H verifies $\text{sgn}H = \text{sgn}H^{\text{ex}}$. This situation is akin to that for the vortex model on the full Euclidean plane $\mathbf{R}^2$ [15].

At this moment, the existence of a solution to Eqs. (3.9) remains to be an open question. Due to some physical reasons, it seems natural to conjecture that (3.9) have no solution if $|H|$ is sufficiently large.

4. **Solutions on $\mathbf{R}^4$.** In the presence of an external magnetic field F_{jk}^{ex} $(j,k = 1,2,3,4)$, the total Gibbs energy of the abelian Higgs model is:

$$E(A,\phi) = \int_{\mathbf{R}^4} dx \left\{ \frac{1}{4}F_{jk}F_{jk} + \frac{1}{2}(D_j^A\phi)^*(D_j^A\phi) + \frac{\lambda}{8}(|\phi|^2-1)^2 - \frac{1}{2}F_{jk}F_{jk}^{\text{ex}} \right\}.$$

Define the vector-valued function space

$$\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4) = \text{ the completion of the set } C_0^\infty(\mathbf{R}^4) \text{ under the norm } \|A\|^2_{\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4)} = \int_{\mathbf{R}^4}(\partial_j A_k)(\partial_j A_k)dx.$$

We will use the well-known embedding inequality (cf. [6]):

$$\|A\|_{L^4(\mathbf{R}^4)} \le C\|A\|_{\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4)}, \quad A \in \overset{\circ}{W}{}^{1,2}(\mathbf{R}^4). \tag{4.1}$$

With the notation

$$K(\mathbf{R}^4) = \{A \in \overset{\circ}{W}{}^{1,2}(\mathbf{R}^4) \mid \partial_j A_j = 0\} \subset \overset{\circ}{W}{}^{1,2}(\mathbf{R}^4)$$

and assuming $F_{jk}^{\text{ex}} \in C^\infty(\mathbf{R}^4)$ to simplify the statements of our results, we have

THEOREM 4.1. *For $F_{jk}^{\text{ex}} \in L^2(\mathbf{R}^4)$, Eqs. (2.1) over $\mathbf{R}^4$ have a smooth solution (A,ϕ) so that $A \in K(\mathbf{R}^4)$ and (A,ϕ) solves the minimization problem:*

$$m = \min\{E(A,\phi) \mid (A,\phi) \in K(\mathbf{R}^4) \times W^{1,2}_{\text{loc}}(\mathbf{R}^4)\}. \tag{4.2}$$

Proof. Note first that $F_{jk}^{\text{ex}} \in L^2(\mathbf{R}^4)$ ensures that m is a finite number.

As in [14], we shall solve (4.2) by obtaining a solution of the problem:

$$\tilde{m} = \min\{\tilde{E}(A,\phi) \mid (A,\phi) \in \overset{\circ}{W}{}^{1,2}(\mathbf{R}^4) \times W^{1,2}_{\text{loc}}(\mathbf{R}^4)\} \tag{4.3}$$

where $\tilde{E}$ is the modified non-physical energy functional defined by:

$$\tilde{E}(A,\phi) = \frac{1}{2}\|A\|^2_{\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4)} + \frac{1}{2}\int_{\mathbf{R}^4} dx \left\{ (D_j^A\phi)^*(D_j^A\phi) + \frac{\lambda}{4}(|\phi|^2-1)^2 - F_{jk}F_{jk}^{\text{ex}} \right\}.$$

Let $\{(A^n,\phi^n)\}$ be a minimization sequence of the problem (4.3) and $\tilde{E}_n = \tilde{E}(A^n,\phi^n)$, $n = 1,2,\cdots$. For simplicity, we assume $\tilde{E}_n < \infty$ for each n.

LEMMA 4.2. *There is a complex function $\psi^n \in W^{1,2}_{\text{loc}}(\mathbf{R}^4)$ such that $|\psi^n| \le 1$ a.e. and*

$$\tilde{E}(A^n, \psi^n) = \min\{\tilde{E}(A^n, \psi) \mid \psi \in W^{1,2}_{\text{loc}}(\mathbf{R}^4)\}. \tag{4.4}$$

Proof. For fixed $n \ge 1$, let $\{\chi^N\}$ be a minimizing sequence of the problem (4.4). We may assume $\tilde{E}(A^n, \chi^N) \ge \tilde{E}(A^n, \chi^{N+1})$, $N = 1, 2, \cdots$, and $\tilde{E}(A^n, \chi^1) < \infty$.

From the inequality

$$\left|D_j^{A^n} \chi^N\right|^2 \ge \frac{1}{2}\left|\partial_j \chi^N\right|^2 - \frac{3}{2}(|A^n|^4 + (|\chi^N|^2 - 1)^2) - 3|A^n|^2$$

and $|\chi^N|^4 \le 2(|\chi^N|^2 - 1)^2 + 1$, we see that, for any finite domain $\Omega \in \mathbf{R}^4$, $\{\nabla \chi^N\}$ and $\{\chi^N\}$ are bounded sequences in $L^2(\Omega)$ and $L^4(\Omega)$, respectively. For simplicity, we may assume $\{\chi^N\}$ is weakly convergent in $L^4(\Omega)$ and $W^{1,2}(\Omega)$. A standard diagonal subsequence argument shows that there exists $\psi^n \in W^{1,2}_{\text{loc}}(\mathbf{R}^4) \cap L^4_{\text{loc}}(\mathbf{R}^4) (= W^{1,2}_{\text{loc}}(\mathbf{R}^4))$ so that $\chi^N \xrightarrow{w} \psi^n$ in $L^4(\Omega)$ and $W^{1,2}(\Omega)$ for any bounded domain $\Omega \subset \mathbf{R}^4$. With this observation in mind, it is readily checked that ψ^n solves the minimization problem (4.4).

It remains to prove the pointwise bound $|\psi^n| \le 1$ a.e. on $\mathbf{R}^4$.

In fact, as a critical point of the functional

$$J(\psi) \equiv \tilde{E}(A^n, \psi), \quad \psi \in W^{1,2}_{\text{loc}}(\mathbf{R}^4),$$

ψ^n satisfies, for any $\psi \in W^{1,2}_0(\Omega)$ (Ω is a bounded domain in $\mathbf{R}^4$), the equation

$$\operatorname{Re} \int_{\mathbf{R}^4} dx \left\{ D_j^{A^n} \psi (D_j^{A^n} \psi^n)^* + \frac{\lambda}{2} \psi (|\psi^n|^2 - 1)(\psi^n)^* \right\} = 0. \tag{4.5}$$

We can now proceed as in [12,14].

Define a function $\eta \in C_0^\infty(\mathbf{R}^1)$ with the properties:

$$0 \le \eta \le 1, \quad \eta(s) = \begin{cases} 1, & |s| \le 1, \\ 0, & |s| \ge 2. \end{cases}$$

Then η induces a family of cutoff functions $\eta_\rho(x) = \eta(|x|/\rho)$, $x \in \mathbf{R}^4$, $\rho > 0$.

Set $\Omega_\rho = \{x \in \mathbf{R}^4 \mid |x| < \rho\}$ and suppose $\Omega_\rho^+ \equiv \{x \in \Omega_\rho \mid |\psi^n(x)| > 1\} \ne \emptyset$ for $\rho >$ some ρ_0. Define $\psi_\rho \in W^{1,2}_0(\Omega_{2\rho})$ ($\rho \ge \rho_0$) by the formula:

$$\psi_\rho(x) = \eta_\rho(x)(|\psi^n(x)| - 1)^+ \frac{\psi^n(x)}{|\psi^n(x)|}$$

with the notation $a^+ = \max\{a, 0\}(a \in \mathbf{R}^1)$. Define $f = \psi^n/|\psi^n|$ on $(\mathbf{R}^4)^+ \equiv \{x \in \mathbf{R}^4 \mid |\psi^n(x)| > 1\}$. Then $f^* f = 1$ and on $\Omega_{2\rho}^+$:

$$D_j^{A^n} \psi_\rho = \partial_j \eta_\rho (|\psi^n| - 1) f + ((\partial_j |\psi^n|) f + (|\psi^n| - 1) D_j^{A^n} f) \eta_\rho.$$

Therefore, on replacing ψ in (4.5) by ψ_ρ and using the above relation and $D_j^{A^n}\psi^n = (\partial_j|\psi^n|)f + |\psi^n|D_j^{A^n}f$ and $\mathrm{Re}[f(D_j^{A^n}f)^*] = 0$, we have

$$\int_{\Omega_{2\rho}^+} dx \left\{ |\nabla|\psi^n||^2\,\eta_\rho + (|\psi^n|-1)\partial_j\eta_\rho\cdot\partial_j|\psi^n| + |\psi^n|(|\psi^n|-1)\eta_\rho(D_j^{A^n}f)^*(D_j^{A^n}f) + \frac{\lambda}{2}(|\psi^n|-1)^2(|\psi^n|+1)|\psi^n|\eta_\rho \right\} = 0. \tag{4.6}$$

On the other hand, from the simple inequalities $(|\psi^n|-1) \le (|\psi^n|-1)^2$ (on $\Omega_{2\rho}^+$), $|\nabla\eta_\rho| < C/\rho$ ($C>0$ is a constant independent of $\rho>0$), and the useful comparison

$$|\nabla|\psi||^2 \le (D_j^A\psi)(D_j^A\psi)^*, \tag{4.7}$$

we get immediately:

$$\begin{aligned} &\|(|\psi^n|-1)(\nabla\eta_\rho\cdot\nabla|\psi^n|)\|_{L^1(\Omega_{2\rho}^+)} \\ &\le \|(|\psi^n|^2-1)\|_{L^2(\mathbf{R}^4)}\|(\nabla\eta_\rho\cdot\nabla|\psi^n|)\|_{L^2(\Omega_{2\rho}^+)} \\ &\le \frac{C_1}{\rho}\|(|\psi^n|^2-1)\|_{L^2(\mathbf{R}^4)}\sum_{j=1}^4\|D_j^{A^n}\psi^n\|_{L^2(\mathbf{R}^4)} \\ &\le \frac{1}{\rho}C_2(\tilde{E}_n,\lambda). \end{aligned} \tag{4.8}$$

On substituting (4.8) into (4.6) and letting $\rho\to\infty$ we obtain $\mathrm{mes}((\mathbf{R}^4)^+) = 0$. Lemma 4.2 follows.

Let us now continue the proof of the theorem.

From the property of the sequence $\{(A^n,\psi^n)\}$ obtained above, we see as before that there exists $(A,\phi)\in \mathring{W}(\mathbf{R}^4)\times W^{1,2}_{\mathrm{loc}}(\mathbf{R}^4)$ and a subsequence of $\{(A^n,\psi^n)\}$, which we still denote by $\{(A^n,\psi^n)\}$ for simplicity, so that for any bounded domain $\Omega\subset\mathbf{R}^4$ with sufficiently smooth boundary,

$$(s)-\lim_{n\to\infty} A^n = A \text{ in } L^p(\Omega),\quad 1\le p<4,$$

$$(s)-\lim_{n\to\infty} \psi^n = \phi \text{ in } L^p(\Omega),\quad 4\le p<\infty, \tag{4.9}$$

$$(w)-\lim_{n\to\infty} \psi^n = \phi \text{ in } W^{1,2}_{\mathrm{loc}}(\Omega),$$

where, in deriving (4.9a), the embedding inequality (4.1) has been used and the limit $A^n \xrightarrow{w} A$ in $\mathring{W}^{1,2}(\mathbf{R}^4)$ assumed.

In virtue of (4.9) (where (4.9b) is the most important limit for our purpose below), we can draw the conclusion:

$$\tilde{E}_\Omega(A,\phi) \le \liminf_{n\to\infty} \tilde{E}_\Omega(A^n,\psi^n),$$

where

$$\tilde{E}_\Omega(A,\phi) = \frac{1}{2}\int_\Omega dx \left\{ (\partial_j A_k)(\partial_j A_k) + (D_j^A\phi)(D_j^A\phi)^* + \frac{\lambda}{4}(|\phi|^2-1)^2 - F_{jk}F_{jk}^{\rm ex} \right\}.$$

On the other hand, $\forall \varepsilon > 0$, there is a bounded domain $\Omega_\varepsilon \subset \mathbf{R}^4$ such that

$$\int_{\mathbf{R}^4-\Omega} F_{jk}^{\rm ex}F_{jk}^{\rm ex}dx < \varepsilon, \quad \Omega \supset \Omega_\varepsilon.$$

Therefore, one obtains easily by using (4.3):

$$\tilde{E}_\Omega(A,\phi) \le \tilde{m} + \varepsilon M, \quad \Omega \supset \Omega_\varepsilon, \tag{4.10}$$

where $M \equiv \sup_n \|A^n\|_{\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4)}$. But $\|A\|_{\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4)} \le \liminf_{n\to\infty} \|A^n\|_{\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4)}$, hence (4.10) becomes

$$\frac{1}{2}\int_\Omega dx \left\{ (\partial_j A_k)(\partial_j A_k) + (D_j^A\phi)(D_j^A\phi)^* + \frac{\lambda}{4}(|\phi|^2-1)^2 \right\}$$
$$\le \tilde{m} + \int_{\mathbf{R}^4} F_{jk}F_{jk}^{ex} + 2\varepsilon M, \quad \Omega \supset \Omega_\varepsilon.$$

In the limit $\Omega \to \mathbf{R}^4$, we get, regarding that $\varepsilon > 0$ is arbitrary, the inequality $\tilde{E}(A,\phi) \le \tilde{m}$. Hence (A,ϕ) is a solution of the minimization problem (4.3).

As a critical point of the functional $\tilde{E}$ over the space $\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4) \times W^{1,2}_{\rm loc}(\mathbf{R}^4)$, (A,ϕ) is a solution of the elliptic equations (2.17) on $\mathbf{R}^4$. Hence (A,ϕ) is smooth. In virtue of (2.17a) we can calculate directly that $\Delta(\partial_j A_j) = 0$. But $\partial_j A_j \in L^2(\mathbf{R}^4)$, therefore $\partial_j A_j = 0$. So $A \in K(\mathbf{R}^4)$.

Finally, since the identity (2.18) holds in $K(\mathbf{R}^4)$, thus we can conclude that (A,ϕ) also solves the original minimization problem (4.2). Theorem 4.1 is proved.

The proof of Theorem 4.1 already shows:

PROPOSITION 4.3. *If (A,ϕ) is a finite energy solution of Eqs. (2.1) on $\mathbf{R}^4$, then $|\phi| \le 1$.*

5. The case of a general source current. In this case the gauge and the Higgs boson pair (A,ϕ) is coupled to the external source current $J^{\rm ex} = (J_j^{\rm ex})$ through the Lagrangian density:

$$\mathcal{L}(A,\phi) = \frac{1}{4}F_{jk}F_{jk} + \frac{1}{2}(D_j\phi)(D_j\phi)^* + \frac{\lambda}{8}(|\phi|^2-1) + A_jJ_j^{\rm ex}.$$

The corresponding Euler-Lagrange equations take the form:

$$\text{(5.1)} \qquad \begin{cases} D_j^A D_j^A \phi + \frac{\lambda}{2}(1-|\phi|^2)\phi = 0, \\ \partial_k F_{kj} + \frac{i}{2}(\phi(D_j^A\phi)^* - \phi^*(D_j^A\phi)) = J_j^{\text{ex}} \end{cases}$$

In virtue of Eq. (5.1a), we find that J^{ex} must satisfy the consistency condition:

$$\text{(5.2)} \qquad \partial_j J_j^{\text{ex}} = 0.$$

Condition (5.2) also ensures that the action $L = \int \mathcal{L} dx$ is gauge-invariant. For convenience, we again assume J^{ex} is smooth.

THEOREM 5.1. *Suppose J^{ex} satisfies the constraint (5.2).*

(a) For the model defined on a torus T^d ($d = 2, 3, 4$), Eqs. (5.1) have a smooth solution $(\tilde{A}, \tilde{\phi})$. Moreover, this solution is in the Coulomb gauge and solves the minimization problem

$$m = \min\{L(A,\phi) \mid (A,\phi) \in W^{1,2}(T^d)\}.$$

(b) For the model defined on an Euclidean space-time $\mathbf{R}^d$ ($d = 3, 4$) (the case $d = 3$ has been discussed in [14]), we assume

$$\text{(5.3)} \qquad J^{\text{ex}} \in L^{\frac{2d}{d+2}}(\mathbf{R}^d).$$

Then Eqs. (5.1) over $\mathbf{R}^d$ have a smooth solution $(\tilde{A}, \tilde{\phi})$. This solution is in the Coulomb gauge and solves the minimization problem

$$m = \min\{L(A,\phi) \mid (A,\phi) \in K(\mathbf{R}^d) \times W^{1,2}_{\text{loc}}(\mathbf{R}^d)\}.$$

The proof of this theorem is the same as those for the corresponding cases in Theorems 2.1 and 4.1 respectively. The only difference here is that we now require the condition (5.3) to ensure the lower boundedness of the action L in part (b) stated above. In fact, we have $L \geq -C(d)\|J^{\text{ex}}\|^2_{L^{2d/d+2}(\mathbf{R}^d)}$, where $C(d) > 0$ is a constant depending only on d.

6. A brief remark on asymptotic decay. As in [14], similar asymptotic decay results for the abelian Higgs models with sources on $\mathbf{R}^4$ can be proved, provided some higher order integrability conditions on the source terms are imposed, although we choose not to get into a detailed discussion of the topic here. Instead, we only remark that the solutions obtained in Theorem 4.1 or 5.1 have the property that the gauge potential will decay to zero at infinity. This may be a special feature of the solutions in the Coulomb gauge. The case $d = 3$ has been studied in [14]. In this section we shall establish the decay result for $d = 4$.

To simplify the notation, we define $g_k = D_k^A\phi$. On using the identity $D_k^A D_j^A \psi - D_j^A D_k^A \psi = -iF_{kj}\psi$, we find from (2.1) (hence the source is generated from an external magnetic field, say):

$$\begin{aligned} D_k^A D_k^A g_\ell &= -\frac{\lambda}{2}(1-|\phi|^2)g_\ell + \frac{\lambda+1}{2}|\phi|^2 g_\ell \\ &+ \frac{\lambda-1}{2}\phi^2 g_\ell^* - 2iF_{k\ell}g_k - \frac{i}{2}(\partial_k(F_{k\ell}^{\text{ex}} - F_{\ell k}^{\text{ex}}))\phi. \end{aligned} \tag{6.1}$$

Suppose now $F_{jk}^{\text{ex}} \in W^{1,2}(\mathbf{R}^4)$. We first show that $D_k^A g_\ell \in L^2(\mathbf{R}^4)$.

To this end, we multiply both sides of (6.1) by the test function $\eta_\rho^2 g_\ell^*$ and integrate the resulting equality by parts. We obtain:

$$\begin{aligned} \text{the left-hand-side} &= \int_{\mathbf{R}^4} \{\eta_\rho^2 g_\ell^* D_k^A D_k^A g_\ell\} dx \\ &= -\int_{\mathbf{R}^4} \eta_\rho^2 |D_k^A g_\ell|^2 dx - \frac{2}{\rho}\int_{\mathbf{R}^4} \partial_k \eta\left(\frac{x}{\rho}\right) g_\ell^*(\eta_\rho D_k^A g_\ell) dx, \\ |\text{the right-hand-side}| &\le \frac{3\lambda+2}{2}\|g\|_{L^2(\mathbf{R}^4)}^2 + 2\int_{\mathbf{R}^4} |F_{k\ell}||\eta_\rho^2 g_k g_\ell| dx \\ &+ \frac{1}{2}\int_{\mathbf{R}^4} (|F_{k\ell}^{\text{ex}}| + |F_{\ell k}^{\text{ex}}|)|g_\ell| dx, \end{aligned}$$

where we have used Proposition 4.3. Therefore, under the assumption that (A,ϕ) is of finite energy and from a simple interpolation inequality, one obtains the bound

$$\int_{\mathbf{R}^4} \eta_\rho^2 |D_k^A g_\ell|^2 dx \le C_1 + 2\int_{\mathbf{R}^4} |F_{k\ell}||\eta_\rho^2 g_k g_\ell| dx. \tag{6.2}$$

where $C_1 > 0$ is a constant depending on $E(A,\phi)$, $\|F_{jk}^{\text{ex}}\|_{W^{1,2}(\mathbf{R}^4)}$, and λ, but independent of $\rho \ge 1$.

From Eq. (2.17b) and the simple decomposition

$$\Delta \eta_\rho A_j = \eta_\rho \Delta A_j + 2\nabla\eta_\rho \cdot \nabla A_j + A_j \nabla^2 \eta_\rho$$

we easily obtain

$$\begin{aligned} \|\Delta \eta_\rho A_j\|_{L^2(\mathbf{R}^4)} &\le \|g\|_{L^2(\mathbf{R}^4)} + C_2\|A\|_{\overset{\circ}{W}{}^{1,2}(\mathbf{R}^4)} \\ &+ C_3\|A\|_{L^4(\mathbf{R}^4)}^2 + \|F^{\text{ex}}\|_{W^{1,2}(\mathbf{R}^4)} \end{aligned} \tag{6.3}$$

where $C_2, C_3 > 0$ are independent of $\rho \ge 1$

On the other hand, an integration by parts yields

$$
(6.4)\qquad \int_{\mathbf{R}^4} (\Delta\eta_\rho A_j)^2 dx = -\int_{\mathbf{R}^4} [(\nabla(\Delta\eta_\rho A_j)\cdot\nabla(\eta_\rho A_j)]dx
$$

$$
= \int_{\mathbf{R}^4} \sum_{k,\ell=1}^{4} (\partial_k\partial_\ell(\eta_\rho A_j))^2 dx \geq \int_{\Omega_\rho} \sum_{k,\ell=1}^{4} (\partial_k\partial_\ell A_j)^2 dx.
$$

On substituting the bound (6.3) into (6.4), observing (4.1), and letting $\rho \to \infty$, we find $\partial_\ell A_j \in W^{1,2}(\mathbf{R}^4)$. Using (4.1) again we get $\partial_\ell A_j \in L^4(\mathbf{R}^4)$.

After the above preparation, we can draw our attention back to (6.2). Since $F_{k\ell} \in L^4(\mathbf{R}^4)$, from the Hölder inequality,

$$
(6.5)\qquad \int_{\mathbf{R}^4} |F_{k\ell}|\eta_\rho^2 |g_k g_\ell| dx \leq \|F_{k\ell}\|_{L^4(\mathbf{R}^4)} \|g_k\|_{L^2(\mathbf{R}^4)} \|\eta_\rho g_\ell\|_{L^4(\mathbf{R}^4)}.
$$

In virtue of (4.1), the following inequality holds:

$$
(6.6)\qquad \|\eta_\rho g\|_{L^4(\mathbf{R}^4)} \leq C\|\nabla(\eta_\rho |g|)\|_{L^2(\mathbf{R}^4)}
$$

$$
\leq C\|(\eta_\rho \nabla |g|)\|_{L^2(\mathbf{R}^4)} + C_4 \|g\|_{L^2(\mathbf{R}^4)}
$$

where $C_4 > 0$ is independent of $\rho \geq 1$.

An easy calculation shows that

$$
(6.7)\qquad |\partial_k |g_\ell|\,| \leq |D_k^A g_\ell|.
$$

Inserting (6.7) into (6.6) and substituting the resulting inequality into (6.5), one finds, by virtue of (6.2),

$$
(6.8)\qquad \sum_{k,\ell=1}^{4} \|\eta_\rho D_k^A g_\ell\|_{L^2(\mathbf{R}^4)}^2 \leq C' + C'' \sum_{k,\ell=1}^{4} \|\eta_\rho D_k^A g_\ell\|_{L^2(\mathbf{R}^4)},
$$

where C', C'' are independent of $\rho \geq 1$. From (6.8) the fact $D_k^A g_\ell \in L^2(\mathbf{R}^4)$ follows readily. As a consequence, we get, from (6.7) and (4.1), $g_\ell \in L^4(\mathbf{R}^4)$.

A differentiation of (2.17b) gives us

$$
(6.9)\qquad \Delta(\partial_\ell A_j) = \frac{i}{2}(g_j g_\ell^* - g_j^* g_\ell) + \frac{i}{2}(\phi^*(D_\ell^A g_j) - \phi(D_\ell^A g_j)^*)
$$

$$
+ \frac{1}{2}\partial_\ell\partial_k(F_{kj}^{\mathrm{ex}} - F_{jk}^{\mathrm{ex}}).
$$

Assume now $F_{jk}^{\mathrm{ex}} \in W^{2,2}(\mathbf{R}^4)$. Since the right-hand-side of (6.9) is in $L^2(\mathbf{R}^4)$, from an inequality of the type (6.4) we can conclude that $(\partial_\ell A_j) \in W^{2,2}(\mathbf{R}^4)$. Hence $\partial_\ell\partial_k A_j \in L^4(\mathbf{R}^4)$. Therefore we have obtained $A \in W^{2,4}(\mathbf{R}^4)$. Finally, from the standard Sobolev embedding $W^{j+m,q}(\mathbf{R}^d) \to W^{j,p}(\mathbf{R}^d)$ $(pd/(d+pm) < q \leq p)$ we get $A \in W^{1,p}(\mathbf{R}^4)$ for all $4 \leq p < \infty$. In particular $A \to 0$ as $|x| \to \infty$.

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