NON-LINEAR STABILITY OF

ASYMPTOTIC SUCTION*

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ABSTRACT

The semigroup approach to the Navier-Stokes equation in halfspace is used to prove that the stability of the asymptotic suction velocity profile is determined by the eigenvalues of the classical Orr-Sommerfeld equation. The usual obstacle, namely, that the corresponding linear operator contains 0 in the spectrum is removed with the use of weighted spaces.

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1. Introduction

A flow over a plane $y = 0$ in $\mathbb{R}^3$ given by

$$U(x,y,z) = (1 - e^{-y}, -1/R, 0)$$

is called an asymptotic suction velocity profile [12]. $R > 0$ is the Reynolds number. U satisfies the Navier-Stokes equation

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p_0 + \frac{1}{R} \Delta v$$

$$\text{div } v = 0$$

with $p_0 = 0$. In the present paper it is proved that the stability of U for small perturbations which initially decay exponentially in the y direction and are periodic in the x and z direction is governed by the eigenvalues of the classical Orr-Sommerfeld equation [1,8,12]. For precise statements see Theorems 4,5, 9 and 15.

Rigorous non-linear stability analyses for flows in (essentially) unbounded domains are usually based on energy methods [3,4,5,13], and predict sufficient conditions for global stability [8]. Recently, Galdi & Rionero [6] proved a universal sufficient condition for global stability and it applies, in particular, to the asymptotic suction. However, much importance is attached to (conditional) stability analysis of some special flows over infinite flate plate [1,8,12]. In these cases the most widely accepted stability analyses are based on studies of the eigenvalues of the Orr-Sommerfeld equation [1,8,12], but no rigorous justification was known.

Let h_a denote the function $h_a(x,y,z) = e^{-ay}$ for $y \geq 0$ and $a \in \mathbb{C}$. Assuming that $\omega > 0$ and that

$$v = (1 - h_1 + h_\omega u_1 \ , \ - 1/R + h_\omega u_2 \ , \ h_\omega u_3) \ , \ p_o = h_\omega p$$

is a solution of the Navier-Stokes equation leads we find the following equations for $u = (u_1,u_2,u_3)$ and p

$$\frac{\partial u}{\partial t} - \frac{1}{R}\Delta u + \frac{2\omega - 1}{R}\frac{\partial u}{\partial y} + (1 - h_1)\frac{\partial u}{\partial x} + \frac{\omega - \omega^2}{R} u +$$

$$+ (h_1 u_2 + \frac{\partial p}{\partial x} , \frac{\partial p}{\partial y} - \omega p , \frac{\partial p}{\partial z}) + h_\omega (u_1 \frac{\partial u}{\partial x} + u_2(\frac{\partial u}{\partial y} - \omega u) + u_3 \frac{\partial u}{\partial z}) = 0 ,$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z} + \frac{\partial u_2}{\partial y} - \omega u_2 = 0 .$$

This set of equations (perturbation equations) will be studied in the present paper. As mentioned above, u and p are assumed to be periodic in x and z. In Section 2 the Fourier components for u and p are analyzed. The full problem is studied in Section 3. In Subsection 3.5 the corresponding abstract semilinear parabolic equation is introduced and analyzed. In Subsection 3.6 it is shown that all solutions of the abstract problem are infinitely differentiable in time in the classical sense. In order to preserve simplicity no attempt has been made to obtain classical spatial smoothness [2]. Therefore, the results presented in 3.5 apply to the perturbation equations, provided that the space derivatives are obtained by termwise differentiation of the Fourier series.

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2. Fourier Components

<u>2.1</u> <u>Preliminaries I.</u> Throughout $\underline{H}(\underline{H}^j)$ denotes the Hilbert space $L^2(0,\infty)$ (j-fold product of $L^2(0,\infty)$) and $\|\cdot\|$ represents either the norm in $\underline{H}$ or $\underline{H}^j$, depending on the context. The set of all complex valued functions which are absolutely continuous on $[0,a]$ for every $a > 0$ is denoted by $\underline{AC}$.

Several operators on $\underline{H}$ appear frequently and are defined as follows:

$$\underline{D}(T_0) = \{f \in \underline{AC} \cap \underline{H} \ | f' \in \underline{H}\} \ , \quad T_0 f = f' \quad ,$$
$$\underline{D}(T_1) = \{f \in \underline{D}(T_0) | f(0) = 0\} \ , \quad T_1 f = f' \ ,$$
$$\underline{D}(T) = \{f \in \underline{D}(T_1) | T_1 f \in \underline{D}(T_0)\} \ , \quad Tf = -f'' \ .$$

For $Re(z) > 0$ define $F(z)$, $G(z) \in \underline{B}(\underline{H})$ by

$$(F(z)g)(x) = \int_0^x e^{z(s - x)} g(s)ds \ ,$$

$$(G(z)g)(x) = \int_x^\infty e^{z(x - s)} g(s)ds \ .$$

In the obvious manner define operators $T_1^{(j)}$, $T^{(j)}: \underline{H}^j \to \underline{H}^j$ by

$$T_1^{(j)} = T_1 \oplus \ldots \oplus T_1 \quad \text{and} \quad T^{(j)} = T \oplus \ldots \oplus T \ .$$

Observe that $F(z) = (z + T_1)^{-1}$ and $G(z) = (z - T_0)^{-1}$ for

$\mathrm{Re}(z) > 0$. The function $h_a(x) = e^{-ax}$, $a \in \mathbb{C}$, $x \geq 0$ will be usually considered as a multiplication operator on $\underline{H}$ with its maximal domain.

Lemma 1. Suppose $\xi \in \mathbb{C}$, $\gamma \in \mathbb{R}$, $R > 0$ and $S = \frac{1}{R} T + \frac{2\gamma - 1}{R} T_1 + \xi$.

Then:

(a) $z \in \sigma(S)$ iff $R(\mathrm{Im}(z - \xi))^2 \leq (2\gamma - 1)^2 \mathrm{Re}(z - \xi)$ and $\mathrm{Re}(z) \geq \mathrm{Re}(\xi)$

(b) if $\gamma > 1/2$ then $\sigma_p(S) = \emptyset$

(c) if $\gamma < 1/2$ then $z \in \sigma(S) \setminus \sigma_p(S)$ iff $R(\mathrm{Im}(z - \xi))^2 = (2\gamma - 1)^2 \mathrm{Re}(z - \xi)$

(d) if $z \notin \sigma(S)$ and $t = \sqrt{(\gamma - 1/2)^2 - R(z - \xi)}$ with $\mathrm{Re}(t) \geq 0$ then $(S - z)^{-1} = RF(t - \gamma + 1/2)G(t + \gamma - 1/2)$

(e) if $\gamma = 1/2$, $\phi \in (0,\pi/2]$, $r > 0$, $a \leq \mathrm{Re}(\xi) - (|\mathrm{Im}(\xi)| \cos\phi + r)/\sin\phi$ and $|\arg(z - a)| \in [\phi,\pi]$ then $z \notin \sigma(S)$ and

$$\|(S - z)^{-1}\| < \frac{1}{r} ,$$

$$\|(S - z)^{-1}\| < \frac{1}{|z - a|} \quad \frac{|\mathrm{Im}(\xi)| + r}{r \sin\phi} ,$$

$$\|T(S - z)^{-1}\| < \frac{R}{\sin\phi} .$$

Proof. Conclusions a through d are obvious. Using the spectral resolution of T one can easily obtain

$$\|(T - z)^{-1}\| \leq \begin{cases} \dfrac{1}{|\mathrm{Im}(z)|} & \text{if } \mathrm{Re}(z) \geq 0 \\ \dfrac{1}{|z|} & \text{if } \mathrm{Re}(z) < 0 \end{cases}$$

and

$$\|T(T - z)^{-1}\| \leq \begin{cases} \dfrac{|z|}{|\mathrm{Im}(z)|} & \text{if } \mathrm{Re}(z) \geq 0 \\ 1 & \text{if } \mathrm{Re}(z) < 0 \,. \end{cases}$$

These bounds imply e .

From now on it will be assumed that

$$\alpha,\beta \in \mathbb{R} \;\; ; \;\; \omega,R \in (0,\infty) \;,$$

$$\lambda \equiv \sqrt{\alpha^2 + \beta^2} \in \{0\} \cup (\omega,\infty) \;,$$

$$S \equiv \frac{1}{R}\, T + \frac{2\omega - 1}{R}\, T_1 + \frac{\lambda^2 + \omega - \omega^2}{R} + i\alpha \qquad .$$

The following technical lemma is needed.

Lemma 2. Suppose that $\phi \in (0,\pi/2]$ and

$$a \leq \frac{\lambda^2}{2R} - \frac{5\lambda + 4}{\sin\phi} - \frac{17|\omega^2 - \omega| + 4}{R\sin^2\phi} .$$

If $|\arg(z - a)| \in [\phi,\pi]$ then $z \notin \sigma(S)$ and

$$\|(S - z)^{-1}\| < \frac{4R}{8R + \lambda^2 \sin\phi}$$

$$\|(S - z)^{-1}\| < \frac{1}{2(\lambda + 1)}$$

$$\|(S - z)^{-1}\| < \frac{4}{|z - a| \sin\phi}$$

$$\|(S - z)^{-1}\| < \frac{8}{|z| \sin\phi}$$

$$\|T(S - z)^{-1}\| < \frac{2R}{\sin\phi} .$$

Proof. Define

$$r = \frac{\lambda^2 \sin\phi}{2R} + 5\lambda + 4 - |\alpha| \cos\phi + \frac{17|\omega^2 - \omega| + 4}{R\sin\phi} + \frac{\omega - \omega^2}{R} \sin\phi ,$$

$$S_1 = \frac{1}{R} T + \frac{\lambda^2 + \omega - \omega^2}{R} + i\alpha - z .$$

Lemma 1 implies

$$\|T_1 S_1^{-1}\| < \|S_1^{-1}\|^{1/2} \; \|TS_1^{-1}\|^{1/2} < \left(\frac{R}{r \sin \phi} \right)^{1/2} \quad .$$

So that

$$\left\| \frac{2\omega - 1}{R} T_1 S_1^{-1} \right\| < 1/2 \ ,$$

$$(S - z)^{-1} = S_1^{-1} \left(1 + \frac{2\omega - 1}{R} T_1 S_1^{-1}\right)^{-1} \quad .$$

Bounds on S_1 as given by Lemma 1 imply Lemma 2.

2.2 Pressure.

Theorem 1. For every $v = (v_1, v_2, v_3) \in \underline{H}^3$ there exist a unique $u = (u_1, u_2, u_3) \in \underline{H}^3$ with $u_2 \in \underline{D}(T_1)$ and a unique $p \in \underline{D}(T_o)$ such that

$$v_1 = u_1 + i\alpha p \ ,$$

$$v_2 = u_2 + (T_o - \omega)p \ ,$$

$$v_3 = u_3 + i\beta p \ ,$$

$$i\alpha u_1 + i\beta u_3 + (T_1 - \omega)u_2 = 0 \ .$$

Moreover, if $\lambda = 0$ then $u_1 = v_1$, $u_2 = 0$, $u_3 = v_3$ and $p = -G(\omega)v_2$; and if $\lambda > \omega$ then

$$p = F(\lambda - \omega)G(\lambda + \omega)v_0 - (1/\lambda)G(\lambda + \omega)v_0 - (1/\lambda)F(\lambda - \omega)(v_0 - \lambda v_2) ,$$

$$u_1 = v_1 - i\alpha p ,$$

$$u_2 = \lambda F(\lambda - \omega)G(\lambda + \omega)v_0 - F(\lambda - \omega)(v_0 - \lambda v_2) ,$$

$$u_3 = v_3 - i\beta p ,$$

where $v_0 = \lambda v_2 + i\alpha v_1 + i\beta v_3$.

Proof. Do the obvious thing.

Let p, u and v be as in Theorem 1. Define $\hat{P}:\underline{H}^3 \to \underline{H}^3$ and $\hat{\Pi}:\underline{H}^3 \to \underline{H}$ by

$$\hat{P}v = u ,$$

$$\hat{\Pi}v = p .$$

Let $\underline{N}$ be the range of $\hat{P}$, i.e.

$$\underline{N} = \{u \in \underline{H}^3 \mid u = (u_1,u_2,u_3) ,\ u_2 \in \underline{D}(T_1) \text{ and } i\alpha u_1 + i\beta u_3 + (T_1 - \omega)u_2 = 0\} .$$

Clearly, $\underline{N}$ is a Hilbert space.

Theorem 2. a) If $\lambda = 0$ then $\|\hat{P}\| = 1$ and $\|\hat{\Pi}\| = 1/\omega$.

b) If $\lambda > \omega$ then $\|\hat{P}\| < \frac{5\lambda^2}{\lambda^2 - \omega^2}$ and $\|\hat{\Pi}\| < \frac{4\lambda}{\lambda^2 - \omega^2}$.

Proof. Note that $\|F(z)\| = \|G(z)\| = 1/\mathrm{Re}(z)$ for $\mathrm{Re}(z) > 0$. This fact and Theorem 1 imply a . Let p, u, v and v_0 be as in Theorem 1 and assume that $\lambda > \omega$. Define $a \equiv \sqrt{\|v_1\|^2 + \|v_3\|^2}$, so that $\|v_0\| < \lambda(\|v_2\| + a)$ and $\|v_0 - \lambda v_2\| < \lambda a$. Hence

$$\|p\| < \frac{\lambda}{\lambda^2 - \omega^2}(\|v_2\| + a) + \frac{1}{\lambda + \omega}(\|v_2\| + a) + \frac{1}{\lambda - \omega} a$$

$$< \frac{\|v\|}{\lambda^2 - \omega^2}\sqrt{(2\lambda - \omega)^2 + 9\lambda^2}$$

and,

$$\|u_1\|^2 + \|u_3\|^2 < (\|v_1\| + \|\alpha p\|)^2 + (\|v_3\| + \|\beta p\|)^2$$

$$< (a + \lambda p)^2$$

$$< \left(\frac{\lambda^2\|v\|}{\lambda^2 - \omega^2}\right)^2\left(\left(2 - \frac{\omega}{\lambda}\right)^2 + \left(4 - \left(\frac{\omega}{\lambda}\right)^2\right)^2\right)$$

and,

$$\|u_2\| < \frac{\lambda^2}{\lambda^2 - \omega^2}\|v_2\| + \frac{2\lambda^2 + \omega\lambda}{\lambda^2 - \omega^2} a$$

$$< \frac{\lambda^2\|v\|}{\lambda^2 - \omega^2}\sqrt{1 + \left(2 + \frac{\omega}{\lambda}\right)^2} .$$

These bounds give part b .

2.3 The linear operator. In accordance with the perturbation equations define $\hat{A}:\underline{H}^3 \to \underline{H}^3$ by

$$(\hat{A}u)_1 = Su_1 - i\alpha h_1 u_1 + h_1 u_2$$

$$(\hat{A}u)_2 = Su_2 - i\alpha h_1 u_2$$

$$(\hat{A}u)_3 = Su_3 - i\alpha h_1 u_3$$

for $u = (u_1, u_2, u_3) \in \underline{D}(\hat{A}) = \underline{D}(T) \times \underline{D}(T) \times \underline{D}(T)$. Define $\hat{A}_p:\underline{N} \to \underline{N}$ by

$$\hat{A}_p u = \hat{P}\hat{A} u$$

for $u \in \underline{D}(\hat{A}_p) = \underline{D}(\hat{A}) \cap \underline{N}$.

Theorem 3. If $\alpha = 0$ then $\sigma_p(\hat{A}_p) = \sigma_p(S)$ and $\sigma(\hat{A}_p) = \sigma(S)$.

Theorem 4. If $\alpha \neq 0$ then

a) $\sigma(S) \setminus \sigma_p(S) \subset \sigma(\hat{A}_p)$

b) if $z \in \sigma(\hat{A}_p) \setminus \sigma(S)$ then z is an isolated eigenvalue of $\hat{A}_p$,

$$-\frac{2}{(1 + 2\lambda)^2} < \frac{\mathrm{Im}(z)}{\alpha} < 1 \quad \text{and}$$

$$\mathrm{Re}(z) > \max\{\frac{\lambda^2 + \gamma - \gamma^2}{R} - \frac{|\alpha|}{1+2\lambda}\left(\frac{2\lambda}{\lambda + \gamma + 1} + \frac{1}{\lambda - \gamma + 1}\right) | \gamma \in \mathbb{R} ,$$

$$|\gamma - 1/2| < |\omega - 1/2|\} .$$

Theorem 5. If $\omega \in (0,1)$ and $\mathrm{Re}(z) < 0$ then $z \in \sigma(\hat{A}_p)$ iff there exists $\phi \in L^2(0,\infty)$ such that

1) $\phi'''' - 2\lambda^2\phi'' + \lambda^4\phi = -\phi''' + \lambda^2\phi' + R(i\alpha - i\alpha h_1 - z)(\phi'' - \lambda^2\phi) + {}$
$+ i\alpha R h_1 \phi$

2) $\phi(0) = \phi'(0) = 0$, $\phi \not\equiv 0$

3) $\int_0^\infty e^x |\lambda\phi^{(j)}(x) + \phi^{(j+1)}(x)|^2 dx < \infty$ for $j = 0,1,2$.

Theorem 6. If $\phi \in (0,\frac{\pi}{2}]$,

$$a < \frac{\lambda^2}{2R} - \frac{5\lambda + 4}{\sin\phi} - \frac{17|\omega - \omega^2| + 4}{R\sin^2\phi}$$

and $|\arg(z - a)| \in [\phi,\pi]$ then $z \notin \sigma(\hat{A}_p)$ and

$$\|(\hat{A}_p - z)^{-1}\| < \frac{16R}{8R + \lambda^2\sin\phi} \left|\frac{\lambda + \omega}{\lambda - \omega}\right| ,$$

$$\|(\hat{A}_p - z)^{-1}\| < \frac{16}{|z - a|\sin\phi} \left|\frac{\lambda + \omega}{\lambda - \omega}\right| ,$$

$$\|(\hat{A}_p - z)^{-1}\| \le \frac{32}{|z| \sin \phi} \frac{\lambda + \omega}{|\lambda - \omega|} ,$$

$$\|T^{(3)}(\hat{A}_p - z)^{-1}\| \le \frac{22R}{\sin \phi} \frac{\lambda + \omega}{|\lambda - \omega|} .$$

Theorems 3-6 will be proved in the following Subsection (2.4) for the case $\lambda = 0$; and in Subsection 2.5 for $\lambda > \omega$. The techniques used are similar to those in [11] . The essential observation is made in Lemma 4, Subsection 2.5. Much more could be said about the spectrum, see [10,11] .

Theorem 7. If $z \in \mathbb{C}$, $t > 0$, $\phi \in (0,\pi/2)$ and

$$a \le \frac{\lambda^2}{2R} - \frac{5\lambda + 4}{\sin \phi} - \frac{17|\omega - \omega^2| + 4}{R \sin^2 \phi}$$

then

$$\|e^{-\hat{A}_p t}\| \le \frac{14}{\sin^2\phi \cos \phi} \frac{\lambda + \omega}{|\lambda - \omega|} e^{-at} ,$$

$$\|(\hat{A}_p + z)e^{-\hat{A}_p t}\| \le \frac{|z| + 11}{\sin \phi \cos \phi} \frac{\lambda + \omega}{|\lambda - \omega|} \frac{1}{t} e^{-at} .$$

Proof. Let $a_1 < a$ and $\phi_1 \in (\phi,\pi/2)$. For $x \in \mathbb{R}$ define

$\gamma(x) = |x|\cos \phi_1 - ix \sin\phi_1 + a_1$, so that

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-\mathrm{Re}(t\gamma(x))} |\gamma'(x)| dx = \frac{1}{\pi t \cos \phi_1} \quad e^{-a_1 t} \quad .$$

By Theorem 6

$$\|(\gamma(x) - \hat{A}_p)^{-1}\| < \frac{\lambda + \omega}{|\lambda - \omega|} \frac{16}{\sin \phi} \quad \frac{1}{(a - a_1)\sin \phi_1} \quad .$$

Hence [7,9]

$$\|e^{-\hat{A}_p t}\| < \frac{\lambda + \omega}{|\lambda - \omega|} \frac{16}{\pi \sin \phi \sin \phi_1 \cos \phi_1} \quad \frac{1}{(a - a_1)t} e^{-a_1 t} \quad .$$

Now let $(a - a_1)t = 1$ and $\phi_1 \to \phi$.

By Theorem 6

$$\|(z + \gamma(x))(\gamma(x) - \hat{A}_p)^{-1}\| < \frac{\lambda + \omega}{|\lambda - \omega|} \left(2|z| + \frac{32}{\sin \phi}\right)$$

and hence

$$\|(\hat{A}_p + z)e^{-\hat{A}_p t}\| < \frac{\lambda + \omega}{|\lambda - \omega|} \quad \frac{|z| + 11}{\sin \phi \cos \phi_1} \quad \frac{1}{t} e^{-a_1 t} \quad .$$

2.4 Case $\lambda = 0$. Observe that in the case $\lambda = 0$

$$\underline{N} = \{v = (v_1, 0, v_3) | v_1, v_3 \in \underline{H}\}$$

and

$$(\hat{A}u)_1 = Su_1 \quad ,$$

$$(\hat{A}u)_2 = 0 \quad ,$$

$$(\hat{A}u)_3 = Su_3$$

for $u \in \underline{D}(\hat{A}_p)$. Thus, Theorem 3 holds. Lemma 2 implies Theorem 6. If $\omega \in (0,1)$ and $\text{Re}(z) < 0$ then $z \notin \sigma(\hat{A}_p)$ by Lemma 1. It is easy to see that if $\text{Re}(z) < 0$ then there is no $\phi \in L^2(0,\infty)$ for which conditions 1, 2 and 3 of Theorem 5 hold.

2.5 Case $\lambda > \omega$. Define $W:\underline{N} \to \underline{H}^2$ by

$$(Wu)_1 = \beta u_1 - \alpha u_3$$

$$(Wu)_2 = \lambda u_2 - i\alpha u_1 - i\beta u_3$$

where $u = (u_1, u_2, u_3) \in \underline{N}$.

Lemma 3. $\|W\| = \lambda$, W^{-1} exists and $\|W^{-1}\| = \sqrt{\lambda^2 + \omega^2}/(\lambda(\lambda - \omega))$. If $w = (w_1, w_2) \in \underline{H}^2$ then

$$(W^{-1}w)_1 = \frac{i\alpha w_2 + \beta w_1}{\lambda^2} - \frac{i\alpha}{\lambda} F(\lambda - \omega)w_2$$

$$(W^{-1}w)_2 = F(\lambda - \omega)w_2$$

$$(W^{-1}w)_3 = \frac{i\beta w_2 - \alpha w_1}{\lambda^2} - \frac{i\beta}{\lambda} F(\lambda - \omega)w_2 \quad .$$

Furthermore, $W(\underline{D}(\hat{A}_p)) = \underline{D}(T) \times \underline{D}(T)$ and if $w \in \underline{D}(T) \times \underline{D}(T)$ then

$$\|T^{(3)}W^{-1}w\| \le \frac{\sqrt{\lambda^2 + \omega^2}}{\lambda(\lambda - \omega)} \|T^{(2)}w\| + \sqrt{\frac{2}{\lambda - \omega}} \|T^{(2)}w\|^{3/4} \|w\|^{1/4} .$$

Proof. Suppose that $u \in \underline{N}$ and $w = Wu$. Then

$$\|w\|^2 = \lambda^2\|u\|^2 - 2\lambda\omega\|u_2\|^2$$

$$\lambda^2\|u\|^2 = \|w\|^2 + 2\lambda\omega \|F(\lambda - \omega)w_2\|^2 .$$

If $w \in \underline{D}(T) \times \underline{D}(T)$ then

$$T^{(3)}W^{-1}w - W^{-1}T^{(2)}w = (-\frac{i\alpha}{\lambda} v, v, -\frac{i\beta}{\lambda} v)$$

where $v = (TF(\lambda - \omega) - F(\lambda - \omega)T)w_2 = -w_2'(0)h_{\lambda - \omega}$. By Hölders's inequality $|w_2'(0)| \le \sqrt{2} \|Tw_2\|^{3/4}\|w_2\|^{1/4}$ and therefore

$$\|T^{(3)}W^{-1}w - W^{-1}T^{(2)}w\| \le \sqrt{\frac{2}{\lambda - \omega}} \|Tw_2\|^{3/4}\|w_2\|^{1/4} \quad .$$

Define $D:\underline{H}^2 \to \underline{H}^2$ by

$$(Dw)_1 = (S - i\alpha h_1)w_1 + \beta h_1 F(\lambda - \omega)w_2$$

$$(Dw)_2 = (S - i\alpha h_1 + 2i\alpha\lambda\ G(\lambda + \omega)h_1 F(\lambda - \omega) - i\alpha h_1 F(\lambda - \omega))w_2$$

for $w = (w_1, w_2) \in \underline{D}(D) = \underline{D}(T) \times \underline{D}(T)$.

<u>Lemma 4.</u> $\sigma_p(\hat{A}_p) = \sigma_p(D)$, $\sigma(\hat{A}_p) = \sigma(D)$ and $(\hat{A}_p - z)^{-1} = W^{-1}(D - z)^{-1}W$ for $z \notin \sigma(D)$.

<u>Proof.</u> By Lemma 3 it is enough to show that $W\hat{A}_p u = DW u$ for all $u \in \underline{D}(\hat{A}_p)$. Let $u \in \underline{D}(\hat{A}_p)$ and $v = \hat{A}u$. Then

$$(\hat{A}_p u)_1 = v_1 - i\alpha p = (S - i\alpha h_1)u_1 + h_1 u_2 - i\alpha p ,$$

$$(\hat{A}_p u)_2 = v_2 - (T_o - \omega)p = (S - i\alpha h_1)u_2 - (T_o - \omega)p ,$$

$$(\hat{A}_p u)_3 = v_3 - i\beta p = (S - i\alpha h_1)u_3 - i\beta p ,$$

for p as in Theorem 1. $u_2 = F(\lambda - \omega)(Wu)_2$ by Lemma 3 and therefore

$$(W\hat{A}_p u)_1 = (DWu)_1$$

$$(W\hat{A}_p u)_2 = (S - i\alpha h_1)(Wu)_2 - i\alpha h_1 F(\lambda - \omega)(Wu)_2 - \lambda(\lambda - \omega + T_o)p .$$

A simple computation of $(\lambda - \omega + T_0)p$ finishes the proof.

Define $B_{11}, B_{12}, B_{22} \in \underline{B}(\underline{H})$ by

$$B_{11} = -i\alpha h_1$$

$$B_{12} = \beta h_1 F(\lambda - \omega)$$

$$B_{22} = -i\alpha h_1 + 2i\alpha\lambda\, G(\lambda + \omega) h_1 F(\lambda - \omega) - i\alpha h_1 F(\lambda - \omega)$$

$$= -i\alpha h_1 + \frac{2i\alpha\lambda}{1 + 2\lambda} h_1\, G(\lambda + \omega + 1) - \frac{i\alpha}{1 + 2\lambda} F(\lambda - \omega + 1) h_1 .$$

Denote

$$B_M = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

and

$$S^{(2)} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \quad \underline{D}(S^{(2)}) = \underline{D}(T) \times \underline{D}(T) .$$

Therefore $D = S^{(2)} + B_M$.

Lemma 5.

a) $\sigma(S)\setminus\sigma_p(S) \subset \sigma(D)$.

b) If $z \in \sigma(D)\setminus\sigma(S)$ then z is an isolated eigenvalue of D .

c) $\sigma_p(S + B_{11}) \cup (\sigma_p(S + B_{22})\setminus\sigma(S)) \subset \sigma_p(D) \subset \sigma_p(S + B_{11}) \cup \sigma_p(S + B_{22})$.

Proof. B_M is bounded and $S^{(2)}$- compact since $h_1 \in L^\infty(0,\infty) \cap L^2(0,\infty)$ [9] . This implies a and b [9] . It is easy to verify part c .

If $\alpha = 0$ it is easy to see that $\sigma(D) = \sigma(S)$ and $\sigma_p(D) = \sigma_p(S)$ (also by the Lemma 5). This proves Theorem 3. Lemmas 4 and 5 also imply Theorem 4, except for the bounds on the eigenvalues.

Lemma 6. Suppose that $\gamma \in \mathbb{R}$, $|\gamma - 1/2| < |\omega - 1/2|$ and $z \notin \sigma(S)$.

a) If $z \in \sigma_p(S + B_{22})$ then there exists $g \in \underline{D}(T)$, $g \neq 0$ such that

$$zg = (\frac{1}{R} T + \frac{2\gamma - 1}{R} T_1 + \frac{\lambda^2 + \gamma - \gamma^2}{R} + i\alpha - i\alpha h_1 +$$

$$\frac{2i\alpha\lambda}{1 + 2\lambda} h_1 G(\lambda + \gamma + 1) - \frac{i\alpha}{1 + 2\lambda} F(\lambda - \gamma + 1)h_1)g \quad .$$

b) If $z \in \sigma_p(S + B_{11})$ then there exists $g \in \underline{D}(T)$, $g \neq 0$ such that

$$zg = (\frac{1}{R} T + \frac{2\gamma - 1}{R} T_1 + \frac{\lambda^2 + \gamma - \gamma^2}{R} + i\alpha - i\alpha h_1)g \ ,$$

c) If $1 + \gamma - \omega > 0$ then the converses of parts a and b hold.

Proof. Define $\xi = \sqrt{1/4 + \lambda^2 + R(i\alpha - z)}$ and note that $Re(\xi) > |\omega - 1/2|$.

Observe also that if $Re(s) > 0$, $Re(s + a) > 0$, $f \in \underline{H}$ and $h_a f \in \underline{H}$ then $h_a F(s)f = F(s + a)h_a f$ and $h_a G(s + a)f = G(s)h_a f$.

Suppose that $z \in \sigma_p(S + B_{22})$. Lemma 1 implies that there exists $f \in \underline{D}(T)$, $f \neq 0$ such that

$$f = -RF(\xi - \omega + 1/2)G(\xi + \omega - 1/2)h_1\psi$$

where

$$\psi = (-i\alpha + \frac{2i\alpha\lambda}{1 + 2\lambda} G(\lambda + \omega + 1) - \frac{i\alpha}{1 + 2\lambda} F(\lambda - \omega))f .$$

This implies that

$$f = -Rh_{\gamma - \omega}F(\xi - \gamma + 1/2)G(\xi + \gamma - 1/2)h_{1 - \gamma + \omega}\psi .$$

If $g \equiv h_{\omega - \gamma}f$ then

$$g = -RF(\xi - \gamma + 1/2)G(\xi + \gamma - 1/2)(-i\alpha h_1 + \frac{2i\alpha\lambda}{1 + 2\lambda} h_1 G(\lambda + \gamma + 1) -$$

$$\frac{i\alpha}{1 + 2\lambda} F(\lambda - \gamma + 1)h_1)g$$

and this proves part a . If $1 + \gamma - \omega > 0$ then one can go backwards in a similar way. The same proof applies for part b and its converse.

The following lemma completes the proof of Theorem 4.

Lemma 7.

a) If $z \in \sigma_p(S + B_{22}) \setminus \sigma(S)$ then $\alpha \neq 0$,

$$-\frac{2}{(1+2\lambda)^2} < \frac{\operatorname{Im}(z)}{\alpha} < 1 \ ,$$

$$\operatorname{Re}(z) > \max\{\frac{\lambda^2+\gamma-\gamma^2}{R} - \frac{|\alpha|}{1+2\lambda}\left(\frac{2\lambda}{1+\lambda+\gamma} + \frac{1}{\lambda+1-\gamma}\right) | \gamma \in \mathbb{R} \ ,$$

$$|\gamma - 1/2| < |\omega - 1/2|\} \ .$$

b) If $z \in \sigma_p(S + B_{11}) \setminus \sigma(S)$ then $\alpha \neq 0$,

$$0 < \frac{\operatorname{Im}(z)}{\alpha} < 1$$

$$\operatorname{Re}(z) > \frac{\lambda^2 + 1/4}{R} \ .$$

Proof. Suppose that $\gamma \in \mathbb{R}$, $|\gamma - 1/2| < |\omega - 1/2|$ and $z \in \sigma_p(S + B_{22}) \setminus \sigma(S)$. By Lemma 6 there is $g \in \underline{D}(T)$, $g \neq 0$ such that

$$\left(z - i\alpha - \frac{\lambda^2+\gamma-\gamma^2}{R}\right)\|g\|^2 = \frac{1}{R}\|T_1 g\|^2 + \frac{2\gamma - 1}{R}(T_1 g, g) - i\alpha(h_1 g, g) +$$

$$\frac{2i\alpha\lambda}{1+2\lambda}(h_1 G(\lambda+\gamma+1)g, g) - \frac{i\alpha}{1+2\lambda}(F(\lambda-\gamma+1)h_1 g, g) \ .$$

Therefore

$$\left(\operatorname{Re}(z) - \frac{\lambda^2+\gamma-\gamma^2}{R}\right)\|g\|^2 > -\frac{2|\alpha|\lambda}{1+2\lambda}\frac{\|g\|^2}{\lambda+\gamma+1} - \frac{|\alpha|}{1+2\lambda}\frac{\|g\|^2}{\lambda-\gamma+1} \ .$$

By choosing $\gamma = 1/2$ one obtains

$$(\frac{Im(z)}{\alpha} - 1)\|g\|^2 = -(h_1 g,g) + \frac{2\lambda}{1+2\lambda} Re((h_1 G(\lambda + \frac{3}{2})g,g)) -$$

$$\frac{1}{1+2\lambda} Re((F(\lambda + 1/2)h_1 g,g)) \quad .$$

If $f = h_{1/2} g$ then

$$\frac{Im(z)}{\alpha} \|g\|^2 = ((1-h_1)g,g) + \frac{2\lambda(\lambda+1)}{1+2\lambda} \|F(\lambda+1)f\|^2 -$$

$$\frac{1}{1+2\lambda} Re((F(\lambda + 1/2)h_1 g,g)) > \frac{-2}{(1+2\lambda)^2} \|g\|^2$$

and

$$(\frac{Im(z)}{\alpha} - 1)\|g\|^2 = -\|f\|^2 + \frac{2\lambda(\lambda+1)}{1+2\lambda} \|F(\lambda+1)f\|^2 -$$

$$\frac{\lambda}{1+2\lambda} \|F(\lambda)f\|^2 < (-1 + \frac{2\lambda}{1+2\lambda}\frac{1}{\lambda+1})\|f\|^2 < 0 \ .$$

Lemma 8. Suppose that $\omega < \frac{3}{2}$ and $z \notin \sigma(S)$. Then $z \in \sigma_p(S + B_{22})$ iff there exists $\phi \in L^2(0,\infty)$ such that

1) $\phi'''' - 2\lambda^2\phi'' + \lambda^4\phi = -\phi''' + \lambda^2\phi' + R(i\alpha - i\alpha h_1 - z)(\phi'' - \lambda^2\phi) + i\alpha R h_1 \phi$.

2) $\phi(0) = \phi'(0) = 0$, $\phi \not\equiv 0$

3) $\int_0^\infty e^x |\lambda\phi^{(j)}(x) + \phi^{(j+1)}(x)|^2 dx < \infty$ for $j = 0,1,2$.

Proof. Suppose that $z \in \sigma_p(S + B_{22})$. Let g be as in Lemma 6a with $\gamma = 1/2$. Define $\phi = F(\lambda)h_{1/2}g$. A straightforward computation shows that ϕ satisfies the above conditions. The converse statement follows from Lemma 6 in a similar way.

Thus, Theorem 5 is proved. Theorem 6 follows from Lemmas 3 and 4 and from the following observation.

Lemma 9. Suppose that $\phi \in (0,\frac{\pi}{2}]$, $a < \frac{\lambda^2}{2R} - \frac{5\lambda + 4}{\sin\phi} - \frac{17|\omega - \omega^2| + 4}{R\sin^2\phi}$ and $|\arg(z - a)| \in [\phi,\pi]$. Then $z \notin \sigma(D)$ and

$$\|(D - z)^{-1}\| < \frac{16R}{8R + \lambda^2 \sin\phi}$$

$$\|(D - z)^{-1}\| < \frac{16}{|z - a|\sin\phi}$$

$$\|(D - z)^{-1}\| < \frac{32}{|z|\sin\phi}$$

$$\|T(D - z)^{-1}\| < \frac{8R}{\sin\phi} \quad .$$

Proof. Recall that $D = S^{(2)} + B_M$ and note that

$$\|B_M\|^2 < \|B_{11}\|^2 + \|B_{22}\|^2 + \|B_{12}\|^2$$

$$< \alpha^2 + \beta^2 + (\lambda + \frac{\lambda}{2\lambda + 1}\frac{2\lambda}{\lambda + \omega + 1} + \frac{\lambda}{2\lambda + 1}\frac{1}{\lambda - \omega + 1})^2$$

$$< \lambda^2 + (\lambda + 1)^2$$

$$< (\frac{3}{2}(\lambda + 1))^2 .$$

By Lemma 2

$$\|B_M(S^{(2)} - z)^{-1}\| \le \|B_M\| \ \|(S - z)^{-1}\| \le \frac{3}{4}$$

so that

$$\|(1 + B_M(S^{(2)} - z)^{-1})^{-1}\| \le 4 .$$

3. Solution

3.1 Preliminaries II. Fix $\alpha_0, \beta_0 \in (0,\infty)$ and let $\Omega = [0,2\pi/\alpha_0] \times [0,\infty) \times [0,2\pi/\beta_0]$. Denote by $|||\cdot|||$ the usual norm on either $\underline{H}_0 = L^2(\Omega)$ or $\underline{H}^3{}_0$. $L^\infty(\Omega)$ and $(L^\infty(\Omega))^3$ norms will be denoted by $|||\cdot|||_\infty$, with convention that $|||u|||_\infty = \max_i \{ |||u_i|||_\infty\}$ for $u = (u_1,u_2,u_3) \in (L^\infty(\Omega))^3$.

For $n,m \in \mathbb{Z}$ and $x,z \in \mathbb{R}$ let $\phi_{nm}(x,z) = e^{in\alpha_0 x + im\beta_0 z}$. Recall that if $f_{nm} \in \underline{H}$ for $n,m \in \mathbb{Z}$ and $\sum_{nm} \|f_{nm}\|^2 < \infty$ then $\sum_{nm} f_{nm}(y)\phi_{nm}(x,z)$ defines an $f \in \underline{H}_0$. Moreover, every $f \in \underline{H}_0$ can be uniquely expressed in that way. Thus, one can define $Q^{(1)}_{nm} \in \underline{B}(\underline{H}_0,\underline{H})$ for $n,m \in \mathbb{Z}$ by $Q^{(1)}_{nm}f = f_{nm}$. Observe that for every $f \in \underline{H}_0$

$$|||f|||^2 = \frac{(2\pi)^2}{\alpha_0\beta_0} \sum_{nm} \|Q^{(1)}_{nm}f\|^2 .$$

Define $Q_{nm} \in \underline{B}(\underline{H}^3{}_0,\underline{H}^3)$ for $n,m \in \mathbb{Z}$ by

$$Q_{nm}f = (Q^{(1)}_{nm}f_1 \ , \ Q^{(1)}_{nm}f_2 \ , \ Q^{(1)}_{nm}f_3)$$

for $f = (f_1, f_2, f_3) \in \underline{H}^3_0$.

Choose any $\omega > 0$ such that $\omega < \alpha_1 \equiv \min\{\alpha_0, \beta_0\}$. $R \in (0,\infty)$ is the Reynolds number. Fix any $n,m \in \mathbb{Z}$ and let $\alpha = n\alpha_0$, $\beta = m\beta_0$. Now, for these numbers α, β, ω, R define

$$\lambda_{nm} = \sqrt{\alpha^2 + \beta^2} \in \{0\} \cup (\omega,\infty)$$

$\underline{N}_{nm} = \underline{N}$ as in 2.2

$P_{nm} = \hat{P}$ as in 2.2

$\Pi_{nm} = \hat{\Pi}$ as in 2.2

$A_{nm} = \hat{A}$ as in 2.3

$A_{pnm} = \hat{A}_p$ as in 2.3

$$\Xi_{nm}(\phi) = \frac{\lambda^2_{nm}}{2R} - \frac{5\lambda_{nm} + 4}{\sin\phi} - \frac{17|\omega - \omega^2| + 4}{R\sin^2\phi} \quad \text{for } \phi \in (0,\tfrac{\pi}{2}] .$$

<u>3.2 Pressure</u>. Define ("gradient") $C_0 : \underline{H}_0 \to \underline{H}^3_0$ by

1) $f \in \underline{D}(C_0)$ iff $Q^{(1)}_{nm}f \in \underline{D}(T_0)$ for all $n,m \in \mathbb{Z}$ and

$$\sum_{nm} \|T_0 Q^{(1)}_{nm}f\|^2 + (n^2 + m^2)\|Q^{(1)}_{nm}f\|^2 < \infty .$$

2) $Q_{nm}C_0 f = (in\alpha_0 Q^{(1)}_{nm}f \ , \ (T_0 - \omega)Q^{(1)}_{nm}f \ , \ im\beta_0 Q^{(1)}_{nm}f)$ for all $n,m \in \mathbb{Z}$ and all $f \in \underline{D}(C_0)$.

Define ("divergence") $C_1 : \underline{H}^3{}_0 \to \underline{H}_0$ by

1) $f = (f_1, f_2, f_3) \in \underline{D}(C_1)$ iff $Q_{nm}^{(1)} f_2 \in \underline{D}(T_1)$ for all $n,m \in \mathbb{Z}$

and $\sum_{nm} \| in\alpha_0 Q_{nm}^{(1)} f_1 + (T_1 - \omega) Q_{nm}^{(1)} f_2 + im\beta_0 Q_{nm}^{(1)} f_3 \|^2 < \infty$.

2) $Q_{nm}^{(1)} C_1 f = in\alpha_0 Q_{nm}^{(1)} f_1 + (T_1 - \omega) Q_{nm}^{(1)} f_2 + im\beta_0 Q_{nm}^{(1)} f_3$

for all $f \in \underline{D}(C_1)$ and all $n,m \in \mathbb{Z}$.

The basic working space will be $X = \{f \in \underline{D}(C_1) | C_1 f = 0\}$, equipped with the $\underline{H}^3{}_0$ norm. Clearly, if $f \in \underline{H}^3{}_0$ then $f \in X$ iff $Q_{nm} f \in \underline{N}_{nm}$ for all $n,m \in \mathbb{Z}$.

Theorem 8. For every $v \in \underline{H}^3{}_0$ there exists a unique $u \in X$ and a unique $p \in \underline{D}(C_0)$ such that

$$v = u + C_0 p \text{ .}$$

Moreover, $Q_{nm} u = P_{nm} Q_{nm} v$ and $Q_{nm}^{(1)} p = \Pi_{nm} Q_{nm} v$ for all $n,m \in \mathbb{Z}$.

Furthermore, $|||u||| < \dfrac{5\alpha_1^{\ 2}}{\alpha_1^2 - \omega^2} |||v|||$ and $|||p||| < |||v||| \max\{\dfrac{1}{\omega}, \dfrac{4\alpha_1}{\alpha_1^2 - \omega^2}\}$.

Proof. See Theorems 1 and 2.

Let u, v and p be as in the theorem above. Define $P \in \underline{B}(\underline{H}^3{}_0)$ and $\Pi \in \underline{B}(\underline{H}^3{}_0, \underline{H}_0)$ by $Pv = u$ and $\Pi v = p$. Observe that

$Q_{nm} P = P_{nm} Q_{nm}$ and $Q_{nm}^{(1)} \Pi = \Pi_{nm} Q_{nm}$ for all $n,m \in \mathbb{Z}$.

3.3 The linearized operator. Define $A:\underline{H}^3_0 \to \underline{H}^3_0$ by

1) $u \in \underline{D}(A)$ iff $Q_{nm}u \in \underline{D}(T^{(3)})$ for all $n,m \in \mathbb{Z}$ and
$\sum_{nm} \|T^{(3)}Q_{nm}u\|^2 + (n^2 + m^2)^2\|Q_{nm}u\|^2 < \infty$.

2) $Q_{nm}Au = A_{nm}Q_{nm}u$ for all $n,m \in \mathbb{Z}$ and all $u \in \underline{D}(A)$.

Define A_p: $X \to X$ by $\underline{D}(A_p) = \underline{D}(A) \cap X$ and $A_p u = PAu$ for all $u \in \underline{D}(A_p)$. Observe that for all $u \in \underline{D}(A_p)$ and all $n,m \in \mathbb{Z}$ $Q_{nm}u \in \underline{D}(A_{pnm})$ and $Q_{nm}A_p u = A_{pnm}\, Q_{nm}u$.

Theorem 9.

a) $\sigma_p(A_p) = \bigcup_{nm} \sigma_p(A_{pnm})$, $\sigma(A_p) = \bigcup_{nm} \sigma(A_{pnm})$.

b) If $u \in X$ and $z \notin \sigma(A_p)$ then $Q_{nm}(A_p - z)^{-1}u = (A_{pnm} - z)^{-1}Q_{nm}u$ for all $n,m \in \mathbb{Z}$.

c) If $z \notin \sigma(A_p)$ then $A(A_p - z)^{-1} \in \underline{B}(X,\underline{H}^3_0)$.

d) If $\phi \in (0,\frac{\pi}{2}]$, $a < \min_{nm}\{\Xi_{nm}(\phi)\}$ and $|\arg(z - a)| \in [\phi,\pi]$

then $z \notin \sigma(A_p)$,

$$\|(A_p - z)^{-1}\| \leq 2\,\frac{\alpha_1 + \omega}{\alpha_1 - \omega},$$

$$\|(A_p - z)^{-1}\| \leq \frac{16}{|z - a|\,\sin\phi}\;\frac{\alpha_1 + \omega}{\alpha_1 - \omega}.$$

Proof. It is easy to see that $\sigma_p(A_p) = \bigcup_{nm} \sigma_p(A_{pnm})$ and that $\bigcup_{nm} \sigma(A_{pnm}) \subset \sigma(A_p)$. Fix $z \notin \bigcup_{nm} \sigma(A_{pnm})$. By Theorem 6 there exists $c \in (0,\infty)$ such that $(1 + n^2 + m^2)\|(A_{pnm} - z)^{-1}\| < c$ and $\|T^{(3)}(A_{pnm} - z)^{-1}\| < c$ for all $n,m \in \mathbb{Z}$. This implies that $\sup_{nm}\{\|A_{nm}(A_{pnm} - z)^{-1}\|\} < \infty$. Thus, if $A_1 : X \to X$ is defined by $Q_{nm}A_1 v = (A_{pnm} - z)^{-1}Q_{nm}v$, $v \in X$, $n,m \in \mathbb{Z}$ then $A_1 \in \underline{B}(X)$, $AA_1 \in \underline{B}(X,\underline{H}^3{}_o)$ and $A_1 v \in \underline{D}(A_p)$ for all $v \in X$. It is easy to see that $A_1(A_p - z) \subset (A_p - z)A_1 = 1$ and hence $A_1 = (A_p - z)^{-1}$. This proves conclusions a,b and c. Part d follows from b and Theorem 6.

Theorem 10. If $t > 0$, $\gamma > 0$, $a < \inf \mathrm{Re}\, \sigma(A_p)$, $n \in \mathbb{Z}$, $m \in \mathbb{Z}$ and $u \in X$ then $Q_{nm}e^{-A_p t}u = e^{-A_{pnm}t}Q_{nm}u$ and $Q_{nm}(A_p - a)^{-\gamma}u = (A_{pnm} - a)^{-\gamma}Q_{nm}u$.

Proof. Define $Q_{pnm} \in \underline{B}(X,\underline{N}_{nm})$ by $Q_{pnm}v = Q_{nm}v$ for $v \in X$. Then, for a suitable path in $\mathbb{C}$ [7,9], one has

$$Q_{pnm}e^{-A_p t} = \frac{1}{2\pi i} Q_{pnm} \int (z - A_p)^{-1}e^{-tz}dz$$

$$= \frac{1}{2\pi i} \int (z - A_{pnm})^{-1}e^{-tz}Q_{pnm}dz$$

$$= e^{-A_{pnm}t}\, Q_{pnm} .$$

Similarly [7] ,

$$Q_{pnm}(A_p - a)^{-\gamma} = \frac{1}{\Gamma(\gamma)} Q_{pnm} \int_0^\infty s^{\gamma - 1} e^{-(A_p - a)s} ds$$

$$= \frac{1}{\Gamma(\gamma)} \int_0^\infty s^{\gamma - 1} e^{-(A_{pnm} - a)s} Q_{pnm} ds$$

$$= (A_{pnm} - a)^{-\gamma} Q_{pnm} .$$

3.4 The nonlinear operator. A similar version of the following theorem can be found in [2] .

Theorem 11. If $\gamma > \frac{3}{4}$ and $a < \inf \operatorname{Re}\sigma(A_p)$ then there exists $c \in (0,\infty)$ such that for all $v \in X$

$$\|(A_p - a)^{-\gamma} v\|_\infty < c\|v\| .$$

Proof. Define a norm on $(L^\infty(0,\infty))^3$ by $\|u\|_\infty = \max_i \{\|u_i\|_\infty\}$ for $u = (u_1,u_2,u_3) \in (L^\infty(0,\infty))^3$. Note that if $f \in \underline{D}(T)$ then $\|f\|_\infty < \sqrt{2}\, \|Tf\|^{1/4} \|f\|^{3/4}$. Fix $\phi \in (0,\pi/2)$ and $s < \min_{nm} \{ \Xi_{nm}(\phi)\}$. Then

$$\|u\|_\infty < \sqrt{2}\, \|T^{(3)}(A_{pnm} - s)^{-1}(A_{pnm} - s)u\|^{1/4} \|u\|^{3/4}$$

$$< \sqrt{2}\, \left(\frac{22R}{\sin\phi}\ \frac{\alpha_1 + \omega}{\alpha_1 - \omega}\right)^{1/4} \|(A_{pnm} - s)u\|^{1/4} \|u\|^{3/4}$$

$$\equiv c_0 \|(A_{pnm} - s)u\|^{1/4}\ \|u\|^{3/4}$$

for all $n,m \in \mathbb{Z}$ and all $u \in \underline{D}(A_{pnm})$. Now, following [7] , let I_{nm} be the identity map from $\underline{N}_{nm}$ into $(L^{\infty}(0,\infty))^3$. Clearly, I_{nm} is a closed linear operator and $I_{nm}e^{-A_{pnm}t}u$ is a continuous function of t for $t > 0$ and $u \in \underline{N}_{nm}$. Hence for every $u \in \underline{N}_{nm}$ the following holds

$$\int_0^{\infty} t^{\gamma - 1} \| I_{nm} e^{-(A_{pnm} - s)t} u \|_{\infty} dt$$

$$< c_0 \int_0^{\infty} t^{\gamma - 1} \| (A_{pnm} - s) e^{-(A_{pnm} - s)t} u \|^{1/4} \| e^{-(A_{pnm} - s)t} u \|^{3/4} dt$$

$$< \|u\| c_0 \int_0^{\infty} t^{\gamma - 1} \left(\frac{|s| + 11}{t \sin\phi \cos\phi}\right)^{1/4} \left(\frac{14}{\sin^2\phi \cos\phi}\right)^{3/4} \frac{\alpha_1 + \omega}{\alpha_1 - \omega} e^{(s - \Xi_{nm}(\phi))t} dt$$

$$\equiv c_1 \|u\| \left(\Xi_{nm}(\phi) - s\right)^{1/4 - \gamma} .$$

The last inequality is implied by Theorem 7. Therefore

$$\| I_{nm}(A_{pnm} - s)^{-\gamma} u \|_{\infty} < c_2 (\Xi_{nm}(\phi) - s)^{1/4 - \gamma} \|u\|$$

for all $n,m \in \mathbb{Z}$ and all $u \in \underline{N}_{nm}$. Therefore, for every $v \in X$

$$|\!|\!| (A_p - s)^{-\gamma} v |\!|\!|_{\infty} < \sum_{nm} \| I_{nm} Q_{nm} (A_p - s)^{-\gamma} v \|_{\infty}$$

$$= \sum_{nm} \| I_{nm} (A_{pnm} - s)^{-\gamma} Q_{nm} v \|_{\infty}$$

$$< c_2 \sum_{nm} (\Xi_{nm}(\phi) - s)^{1/4 - \gamma} \| Q_{nm} v \|$$

$$< c_3 |\!|\!| v |\!|\!| < \infty .$$

The following observation concludes the proof [7]:

$$|||(A_p - a)^{-\gamma}v|||_\infty = |||(A_p - s)^{-\gamma}(A_p - s)^{\gamma}(A_p - a)^{-\gamma} v|||_\infty \leq c\, |||v||| \ .$$

Define $B_j : \underline{H}^3{}_o \to \underline{H}^3{}_o$ for $j = 1,2$ and 3 by

$$\underline{D}(B_1) = \{u \in \underline{H}^3{}_o \,|\, \sum_{nm} n^2 \|Q_{nm}u\|^2 < \infty\} \ , \ Q_{nm}B_1u = in\alpha_0 Q_{nm}u \ ,$$

$$\underline{D}(B_2) = \{u \in \underline{H}^3{}_o \,|\, Q_{nm}u \in \underline{D}(T_1^{(3)}) \ , \ \sum_{nm} \|T_1^{(3)}Q_{nm}u\|^2 < \infty\} \ ,$$

$$Q_{nm}B_2u = T_1^{(3)}Q_{nm}u \ ,$$

$$\underline{D}(B_3) = \{u \in \underline{H}^3{}_o \,|\, \sum_{nm} m^2 \|Q_{nm}u\|^2 < \infty\} \ , \ Q_{nm}B_3u = im\beta_0 Q_{nm}u \ .$$

The following theorem is similar to the preceeding theorem and its proof is based on the same ideas [2,7].

Theorem 12. If $\gamma > 1/2$ and $a < \inf \mathrm{Re}\,\sigma(A)$ then there is a $c \in (0,\infty)$ such that for all $v \in X$ and $j = 1,2,3$

$$|||B_j(A_p - a)^{-\gamma}v||| \leq c\, |||v||| \ .$$

Proof. Fix $\phi \in (0,\frac{\pi}{2})$ and $s < \min_{nm} \{ \Xi_{nm}(\phi)\}$. Theorem 7 implies that for all $n,m \in \mathbb{Z}$

$$\|(A_{pnm} - s)^{-\gamma}\| \leq \frac{1}{\Gamma(\gamma)} \int_0^\infty t^{\gamma - 1} \|e^{-(A_{pnm} - s)t}\| dt$$

$$\leq \frac{\alpha_1 + \omega}{\alpha_1 - \omega} \ \frac{14}{\sin^2\phi \cos\phi} \ (\Xi_{nm}(\phi) - s)^{-\gamma} \ .$$

This proves the theorem for $j = 1$ and 3 (and also for $\gamma = 1/2$). Observe that for $n,m \in \mathbb{Z}$ and $t > 0$ the function $t \to T_1^{(3)} e^{-A_{pmn}t}$ is continuous. Theorems 6 and 7 imply that

$$\| T_1^{(3)} e^{-(A_{pnm} - s)t} \| \le \| T^{(3)} e^{-(A_{pnm} - s)t} \|^{1/2} \| e^{-(A_{pnm} - s)t} \|^{1/2}$$

$$\le \left(\frac{22R}{\sin \phi} \; \frac{\alpha_1 + \omega}{\alpha_1 - \omega} \right)^{1/2} \| (A_{pnm} - s) e^{-(A_{pnm} - s)t} \|^{1/2} \| e^{-(A_{pnm} - s)t} \|^{1/2}$$

$$\le c_1 \, t^{-1/2} \, e^{-(\Xi_{nm}(\phi) - s)t}$$

where c_1 does not depend on $n,m \in \mathbb{Z}$. Therefore, for all $n,m \in \mathbb{Z}$

$$\| T_1^{(3)} (A_{pnm} - s)^{-\gamma} \| \le \frac{c_1 \Gamma(\gamma - 1/2)}{\Gamma(\gamma)} \; (\Xi_{nm}(\phi) - s)^{1/2 - \gamma}$$

which proves the theorem for $j = 2$.

Fix any $a < \inf \mathrm{Re}\, \sigma(A)$ and define [7] for $\gamma \ge 0$ a Banach space X^γ with norm $\| \cdot \|_\gamma$ by $X^\gamma = \underline{D}((A_p - a)^\gamma)$ and $\| x \|_\gamma = ||| (A_p - a)^\gamma x |||$.

For $u = (u_1, u_2, u_3) \in X^\gamma$, $\gamma > \frac{3}{4}$ and $v \in X^\delta$, $\delta > 1/2$ define $B(u,v) \in \underline{H}^3_0$ by

$$B(u,v) = (h_\omega u_1) B_1 v + (h_\omega u_2)(B_2 - \omega) v + (h_\omega u_3) B_3 v$$

and let $B_p(u,v) = PB(u,v)$. Clearly, B is a bilinear operator and for every $\gamma > \frac{3}{4}$ and $\delta > 1/2$ there is b such that $||| B(u,v) ||| \le b \| u \|_\gamma \| v \|_\delta$ for all $u \in X^\gamma$ and all $v \in X^\delta$.

3.5 Stability. For $t_0 \in (0,\infty]$ define a set $\underline{S}(t_0)$ by $x \in \underline{S}(t_0)$ iff

i) $x \in C([0,t_0),X)$

ii) $x(t) \in \underline{D}(A)$ and $\frac{dx}{dt}(t)$ exists in X for all $t \in (0,t_0)$

iii) for all $t \in (0,t_0)$ there exists $p(t) \in \underline{D}(C_0)$ such that
$\frac{dx}{dt} + Ax(t) + B(x(t),x(t)) + C_0 p(t) = 0$

iv) $B_p(x(t),x(t))$ is locally Hölder continuous function of $t \in (0,t_0)$ and $\int_0^t \|B_p(x(s), x(s))\| \, ds < \infty$ for some $t > 0$.

By Theorem 8, iii can be replaced by

iii') $\frac{dx}{dt}(t) + A_p x(t) + B_p(x(t),x(t)) = 0$ for all $t \in (0,t_0)$.

Therefore, the results of D. Henry [7] are directly applicable.

Lemma 10.[7] If $x \in \underline{S}(t_0)$ then $x \in C((0,t_0),X^\delta)$ for all $\delta \in [0,1]$ and for all $t \in [0,t_0)$

$$x(t) = e^{-A_p t}x(0) - \int_0^t e^{-A_p(t - s)}B_p(x(s),x(s))ds .$$

Define $\underline{S}^\gamma(t_0)$ for $\gamma \in (\frac{3}{4},1)$ and $t_0 \in (0,\infty]$ by

$$\underline{S}^\gamma(t_0) = \underline{S}(t_0) \cap C([0,t_0),X^\gamma).$$

Theorem 13.[7] If $\gamma \in (\frac{3}{4},1)$ and $x_0 \in X^\gamma$ then there is a $t_0 > 0$ and a unique $x \in \underline{S}^\gamma(t_0)$ such that $x(0) = x_0$.

Theorem 14.[7] If $t_0 \in (0,\infty)$, $x \in \underline{S}(t_0)$ then either $\sup\{\|x(t)\|_\gamma \mid t_0/2 < t < t_0\} = \infty$ for all $\gamma > \frac{3}{4}$ or there is a $t_1 > t_0$ and a $y \in \underline{S}(t_1)$ such that $y(t) = x(t)$ for all $0 \leq t < t_0$.

Remark 1. In [7] it is assumed that $x(0) \in X^\delta$ for some $\delta \in (\frac{3}{4},1)$. This assumption can be easily removed by choosing an interior point for a new starting point.

Theorem 15.[7] If $\inf \operatorname{Re} \sigma(A_p) > a > 0$ and $\gamma \in (\frac{3}{4},1)$ then there exist $c_1, c_2 \in (0,\infty)$ such that for all $x_0 \in X^\gamma$ with $\|x_0\|_\gamma \leq c_1$ there is a unique $x \in \underline{S}^\gamma(\infty)$ with $x(0) = x_0$ and

$$\|x(t)\|_\gamma \leq c_2\|x_0\|_\gamma e^{-at} \quad \text{for} \quad t \geq 0 .$$

Remark 2. Observe that under the above assumptions $|||x(t)|||_\infty \leq c_3\|x_0\|_\gamma e^{-at}$ and $|||x(t)||| \leq c_4\|x_0\|_\gamma e^{-at}$ for some c_3 and c_4 and for all $t \geq 0$. If, in addition, $x_0 \in \underline{D}(A)$ then all $\|x_0\|_\gamma$ can be replaced with $|||A_p x_0|||$.

Remark 3. If $\omega \in (0,1)$ then the sign (+,0,-) of $\inf \operatorname{Re}\sigma(A_p)$ is determined by the classical Orr-Sommerfeld equation (Theorems 4,5 and 9).

Remark 4. If $\inf \operatorname{Re} \sigma(A_p) < 0$ then 0 is not stable [7] .

3.6 Smoothness

Theorem 16.[7] If $x \in \underline{S}^{\gamma}(t_0)$ and $t_1 \in (0,t_0)$ then there exist $c \in (0,\infty)$ and $\beta \in (0,1)$ such that

$$|||x(t) - x(0)|||_{\infty} \leq ct^{\beta} \quad \text{for} \quad t \in [0,t_1] .$$

The following theorem is also expected [2] .

Theorem 17. If $x \in \underline{S}(t_0)$ then

$$x,\ Ax \in C^{\infty}((0,t_0),\underline{H}^3{}_0)$$

$$x \in C^{\infty}((0,t_0),(L^{\infty}(\Omega))^3)$$

$$p \in C^{\infty}((0,t_0),\underline{H}_0)$$

where p is as in the definition of $\underline{S}$ (Subsection 3.5).

Proof. Suppose that $x \in \underline{S}(t_0) \cap C^n((0,t_0),X^{\gamma})$ for some $n \geq 0$ and some $\gamma \in (\frac{3}{4},1)$. By Lemma 10 this assumption is satisfied for $n = 0$. Define $g(t) = -B_p(x(t),x(t))$ for $0 < t < t_0$.

Fix any $0 < \tau < \tau_1 < \tau_2 < t_0$. There exists $c \in (0,\infty)$ such that for all $t,s \in [\tau_1,\tau_2]$

$$|||g^{(n)}(t) - g^{(n)}(s)||| \leq c(|t - s| + \|x^{(n)}(s) - x^{(n)}(t)\|_{\gamma})$$

$$x^{(n)}(t) = f(t) + \int_{\tau}^{t} e^{-A_p(t - u)} g^{(n)}(u)du$$

$$f(t) = (-A_p)^n e^{-A_p(t - \tau)} x(\tau) + (-A_p)^{n - 1} e^{-A_p(t - \tau)} g(\tau) + \dots$$
$$+ e^{-A_p(t - \tau)} g^{(n - 1)}(\tau) .$$

There is a $d \in (0,\infty)$ such that

$$\|f'(t)\|_\gamma < d \quad \text{for} \quad t \in [\tau_1,\tau_2]$$

$$|||g^{(n)}(t)||| < d \quad \text{for} \quad t \in [\tau,\tau_2]$$

$$\|e^{-A_p t}\|_\gamma < dt^{-\gamma} \quad \text{for} \quad t \in (0,\tau_2]$$

$$\|(e^{-A_p h}-1)e^{-A_p t}\|_\gamma < dht^{-\gamma - 1} \quad \text{for} \quad h,t \in (0,\tau_2] .$$

Suppose that $\tau_1 < t < t + h < \tau_2$. Then

$$x^{(n)}(t + h) - x^{(n)}(t) = f(t + h) - f(t) +$$
$$+ \int_\tau^{\tau_1} (e^{-A_p h} - 1)e^{-A_p(t - u)} g^{(n)}(u)du$$
$$+ \int_{\tau_1}^{\tau_1 + h} e^{-A_p(t + h - u)} g^{(n)}(u)du +$$
$$+ \int_{\tau_1}^{t} e^{-A_p(t - u)} (g^{(n)}(u + h) - g^{(n)}(u))du$$

so that

$$\|x^{(n)}(t + h) - x^{(n)}(t)\|_\gamma < h(d + \frac{1}{\gamma} d^2(t - \tau_1)^{-\gamma} + d^2(t - \tau_1)^{-\gamma}) +$$
$$+ d\int_{\tau_i}^{t} (t - u)^{-\gamma} \; |||g^{(n)}(u + h) - g^{(n)}(u)||| \, du$$

and therefore

$$|||g^{(n)}(t + h) - g^{(n)}(t)||| <$$

$$c_1 h(t - \tau_1)^{-\gamma} + cd \int_{\tau_1}^{t} (t - u)^{-\gamma} ||| g^{(n)}(u + h) - g^{(n)}(u)||| du$$

for some constant c_1 (independent of h and t). This is a modified Gronwall's inequality [7], therefore there is $c_2 \in (0,\infty)$ such that for all $\tau_1 < t < t + h < \tau_2$

$$|||g^{(n)}(t + h) - g^{(n)}(t)||| < c_2 h(t - \tau_1)^{-\gamma}$$

and hence [7] the mapping

$$t \to \int_{\tau_1}^{t} e^{-A_p(t - s)} g^{(n)}(s)ds$$

is in $C^1((\tau_1,\tau_2),X^\gamma)$. Therefore $x^{(n)} \in C^1((\tau_1,\tau_2),X^\gamma)$.

By the induction principle $x \in C^\infty((0,t_0),X^\gamma)$. By Theorem 11 $x \in C^\infty((0,t_0), (L^\infty(\Omega))^3)$. It is clear that $x,x',g \in C^\infty((0,t_0),X)$, hence $A_p x = g - x' \in C^\infty((0,t_0),X)$. By Theorem 9 $Ax \in C^\infty((0,t_0),\underline{H}^3{}_0)$. The following observation completes the proof

$$p = \Pi((A_p - A)x + B_p(x,x) - B(x,x)) .$$

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