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Abstract. An informative new proof is given for the theorem of Nowakowski that determines for all n and k the minimum size of a cutset for an element A with $|A| = k$ of the Boolean algebra B_n of all subsets of $\{1, \ldots, n\}$, ordered by inclusion. An inequality is obtained for cutsets for A that is reminiscent of Lubell's inequality for antichains in B_n. A new result that is provided by this approach is a list of all minimum cutsets for A.

A familiar notion from the theory of network flows, the cutset, has a natural analogue in the theory of ordered sets (posets). A *cutset* in a poset $(P, \leq)$ is a subset $K \subseteq P$ that meets every maximal chain in P. For instance, in the Boolean lattice B_n of all subsets of the n-set $[n] = \{1, \ldots, n\}$, ordered by inclusion, the collection $\binom{[n]}{k}$ of k-subsets is a cutset for every k. Studies relating cutsets to antichains appear in [Gr,RZ]. Lih [Li] asked for the maximum size of a cutset K in B_n that is minimal with respect to inclusion, i.e., for all $a \in K$, $K - \{a\}$ is not a cutset. A construction has recently been found that produces a minimal cutset K_n in B_n such that $|K_n| \sim 2^n$ as $n \to \infty$, i.e., K_n contains almost all elements of B_n [FGK].

Bell and Ginsburg [BG] introduced the related notion of a cutset for an element. A collection $K \subseteq P$ is a *cutset for* x, where $x \in P$, if no element of K is comparable to x and if $K \cup \{x\}$ is a cutset in P. Other studies of this notion include [SW,GRS].

Two examples of cutsets for x that we shall require here are denoted by $L(x)$ and $U(x)$. $L(x)$ is the collection of all $y \in P$ such that $y \not\leq x$ and y covers some $z < x$. Thus $L(x)$ is the shadow of the ideal of all elements below x, excluding x itself. Dually, $U(x)$ denotes the shadow of the filter of all elements above x, excluding x, i.e., $U(x)$ contains all $y \not\geq x$ such that y is covered by some $z > x$. For comparison, we mention another example that is important for ranked posets P: The set of elements of the same rank as x excluding x is a cutset for x.

Nowakowski [N] determined the minimum size $C(a, n)$ of a cutset for $A \in B_n$, where A is any subset of $[n]$ with $|A| = a$. He showed that with just two exceptions,

$$C(a, n) = \min\{|L(A)|, |U(A)|\}.$$

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The key to his proof is an elaborate inductive construction, for even $n \geq 8$, of a collection of disjoint saturated chains of subsets unrelated to $A = [n/2]$, where the set of bottoms of the chains is $L(A)$ and the set of tops is $U(A)$.

In this note we take a very different approach that is reminiscent of Lubell's proof of Sperner's Theorem [Lu] and of more recent two-part Sperner theorems [FGOS,EK]. This new method provides a natural proof for all cases including the anomalous ones for small n. An inequality is obtained, (2) below, that is of interest in itself. From it one may derive not just the value $C(a,n)$ itself but also a list of *all* minimum cutsets for $A \in B_n$, something the chain construction cannot provide. Conversely, the existence of a collection of disjoint chains as above follows from our reproof that $C(n/2,n) = |L(A)| = |U(A)|$ for even $n \geq 8$ by applying the appropriate directed graph version of Menger's Theorem. This was observed by Nowakowski [N]. However, an explicit construction is not obtained this way. It would be very nice to find a simpler and more natural chain construction than that in [N]. Here is our result.

THEOREM. *The minimum size $C(a,n)$ of a cutset for $A \in B_n$ with $|A| = a$ is given as follows:*

$$C(2,4) = 5$$
$$C(3,6) = 9$$

Otherwise,

$$C(a,n) = \begin{cases} |L(A)| = (2^a - 1)(n-a), & \text{if } a \leq \frac{n}{2} \\ |U(A)| = (2^{n-a} - 1)a, & \text{if } a \geq \frac{n}{2} \end{cases}.$$

The extremal families K are as follows: If $a < \frac{n}{2}$ then $K = L(A)$, except for $(a,n) = (2,5)$, when $K = L(A)$ or $K = \binom{[5]}{2} - A$.

If $a > \frac{n}{2}$ then $K = U(A)$, except for $(a,n) = (3,5)$, when $K = U(A)$ or $K = \binom{[5]}{3} - A$.

If $a = \frac{n}{2}$, then

$$K = \begin{cases} L(A) = U(A), & \text{if } n = 2 \\ \binom{[n]}{a} - A, & \text{if } n = 4,6 \\ L(A) \text{or } U(A), & \text{if } n \geq 8. \end{cases}$$

Proof. Let $A = \{x_1, \ldots x_a\} \subseteq [n]$, $b = n - a$, and $B = [n] - A = \{y_1, \ldots, y_b\}$. To each permutation α of A there corresponds a chain A_α of subsets of A:

$$A_\alpha = \{\emptyset, \{\alpha(1)\}, \{\alpha(1), \alpha(2)\}, \ldots, A\}.$$

Here we abbreviated $\alpha(x_i)$ by $\alpha(i)$. Likewise, to each permutation β of B corresponds a chain B_β of subsets of B:

$$B_\beta = \{\emptyset, \{\beta(1)\}, \{\beta(1), \beta(2)\}, \ldots, B\}.$$

Given permutations α and β as above, define the *rectangle* of subsets

$$R(\alpha,\beta) = \{S \cup T : S \in A_\alpha, T \in B_\beta\},$$

which we represent in Fig. 1.

A subset $U \subseteq [n]$ belongs to rectangles $R(\alpha,\beta)$ such that $U \cap A \in A_\alpha$ and $U \cap B \in B_\beta$. It follows that the number of rectangles $R(\alpha,\beta)$ containing U is

$$|U \cap A|!(a - |U \cap A|)!|U \cap B|!(b - |U \cap B|)!$$

Now suppose $K \subseteq B_n$ and $U \in K$. By evenly spreading out the contribution of U over all rectangles that contain it, we obtain the following sum.

$$(1) \qquad |K| = \sum_{\alpha,\beta} \sum_{U \in R(\alpha,\beta)\cap K} \frac{1}{|U \cap A|!(a - |U \cap A|)!|U \cap B|!(b - |U \cap B|)!}$$

Each of the $n!$ maximal chains of subsets in B_n is contained in precisely one of the rectangles $R(\alpha,\beta)$: It is the rectangle where α and β describe the order in which elements of A and B, respectively, are added to the chain. Since there are $a!b!$ choices for (α,β), we obtain this fundamental inequality for $|K|$:

$$(2) \qquad |K| \geq a!b! \min_{(\alpha,\beta)} \sum_{U \in R(\alpha,\beta)\cap K} \binom{a}{|U \cap A|}\binom{b}{|U \cap B|}.$$

Inequality (2) holds for any $K \subseteq B_n$.

Assume henceforth that K is a cutset with respect to A. We may assume $a \leq \frac{n}{2}$, i.e., $a \leq b$, since the case $a > \frac{n}{2}$ is the dual of $a < \frac{n}{2}$. We shall obtain a lower bound for the right side in (2) by determining the best-possible lower bound for a single rectangle $R(\alpha,\beta)$ of the summation $\sum \binom{a}{|U\cap A|}\binom{b}{|U\cap B|}$ over a collection of squares (sets in $R(\alpha,\beta)$) that blocks chains in $R(\alpha,\beta)$. It turns out that this bound can always be realized by a cutset K for A, so that $|K| = C(a,n)$, while the conditions for equality for a single rectangle lead to the list of all minimum cutsets for A provided by the theorem.

A collection K of subsets unrelated to A is a cutset for A if and only if for all (α,β), $K \cap R(\alpha,\beta)$ intersects every maximal chain in $R(\alpha,\beta)$ except the chain through A. Referring to Fig. 1, this means that $K \cap R(\alpha,\beta)$ intersects every path from the bottom left square ($\emptyset \cup \emptyset = \emptyset$) to the top right square ($A \cup B = [n]$) such that the path only moves right or up at each step, except that the path through A along the left and top borders is avoided. We now determine the minimum weight for $K \cap R(\alpha,\beta)$ such that this property holds. We may as well assume that α and β are identity permutations, written $(\alpha,\beta) = (id, id)$, so that $\alpha(i) = x_i$ and $\beta(j) = y_j$. For all $i \in [a]$, a cutset K for A must contain some set in the *hook* H_i that consists of the sets $\{x_1,\ldots,x_{a-i}\} \cup \{y_1,\ldots,y_j\}$, $1 \leq j \leq i$, and the sets

$\{x_1,\ldots,x_{a-j}\} \cup \{y_1,\ldots,y_i\}$, $1 \le j \le i$, since the sets $\{x_1,\ldots,x_k\}$, $0 \le k \le a-i$, and $A \cup \{y_1,\ldots,y_k\}$, $i \le k \le b$, can be added to H_i to form a maximal chain in $R(id,id)$ (see Fig. 2).

Let the ends and middle of hook H_i be denoted as follows:

$$L_i = \{x_1,\ldots,x_{a-i}\} \cup \{y_1\}, \text{ the left end}$$
$$T_i = \{x_1,\ldots,x_{a-1}\} \cup \{y_1,\ldots,y_i\}, \text{ the top end}$$
$$M_i = \{x_1,\ldots,x_{a-i}\} \cup \{y_1,\ldots,y_i\}, \text{ the middle}$$

If one starts from the left end of H_i and goes over to the middle, the weights $\binom{a}{|U\cap A|}\binom{b}{|U\cap B|}$ are given by $\binom{a}{i}$ times $\binom{b}{1},\binom{b}{2},\ldots,\binom{b}{i}$, while if one goes from the top end down to the middle, the weights are $\binom{b}{i}$ times $\binom{a}{1}$, $\binom{a}{2},\ldots,\binom{a}{i}$. It can be checked readily that the minimum weight on hook H_i occurs at precisely these places:

$$(3)\qquad \begin{cases} L_i & \text{if } a < b-1 \text{ or if } a = b-1 \text{ and } i < a \\ L_i, M_i & \text{if } a = b-1 \text{ and } i = a \\ L_i, T_i & \text{if } a = b \text{ and } i < a-1 \\ L_i, M_i, T_i & \text{if } a = b \text{ and } i = a-1 \\ M_i & \text{if } a = b \text{ and } i = a \end{cases}$$

First consider the case that $a < b-1$. Then for all i, $1 \le i \le a$, the minimum weight on H_i occurs at L_i, so that by (2), any cutset K for A satisfies

$$|K| \ge (\binom{a}{1} + \cdots + \binom{a}{a})\binom{b}{1} = (2^a - 1)b = |L(A)|,$$

a bound which is sharp because $L(A)$ is a cutset for A. By applying similar reasoning to every rectangle $R(\alpha,\beta)$ in the summation (1), it follows that $L(A)$ is the unique minimum-sized cutset for A.

Next suppose that $a = b-1$. We see from (3) that L_i has minimum weight in every hook H_i, so that as above $|K| \ge |L(A)|$. Thus we again have $C(a,n) = |L(A)|$, and $L(A)$ is extremal. The only other way to attain the bound $|L(A)|$ using $R(id,id)$ is to select L_i for $1 \le i \le a-1$ along with M_a.

For $a = 1$ this is no change because $L_1 = M_1$. For $a = 2$, if K is a minimum cutset for A, then by (1) and (3), we must use L_1 in every rectangle, that is, K contains all $U \subseteq [5]$ with $|U \cap A = |U \cap B| = 1$. It also follows that the remaining 3 sets U in K must satisfy $|U \cap A| = 0$ and $|U \cap B| = 1$ or 2. It is easily verified that these three sets must have the same size in order to meet all maximal chains in [n], so that either $K = L(A)$ or $K = \binom{[5]}{2} - A$. For $a \ge 3$, the collection $\{L_1, L_2, \ldots, L_{a-1}, M_a\}$ can be avoided by the chain in $R(id, id)$ formed by adding elements in the order $y_1, y_2, x_1, x_2, \ldots, x_a, y_3, y_4, \ldots, y_b$. It follows that the only minimum cutset for A is $L(A)$.

It remains to consider the case $a = b$. If $a = 1$ there is no choice but $K = L(A)$. If $a = 2$, then (1), (2), (3) imply that a minimum K contains all U with $|U \cap A| = |U \cap B| = 1$ or with $|U \cap A| = 0$ and $|U \cap B| = 2$, i.e., $K = \binom{[4]}{2} - A$. If $a = 3$, the sum in (2) is minimized by selecting L_1 in H_1, L_2 or M_2 in H_2, and M_3 in H_3. However, the chain with order $y_1, y_2, x_1, x_2, x_3, y_3$ is in $R(id, id)$ and avoids $\{L_1, L_2, M_3\}$. Therefore one must select $\{L_1, M_2, M_3\}$. The same remark applies to every $R(\alpha, \beta)$ in view of (1), which means that $\binom{[6]}{3} - A$ is the unique minimum cutset for A.

Finally, we consider $a = b \geq 4$. By applying (2) and (3) we learn that for a minimum cutset K for A, $|K|$ is at least the value obtained by selecting L_1 from H_1, one of L_i and T_i from H_i for $2 \leq i \leq a-2$, one of L_{a-1}, M_{a-1}, and T_{a-1} from H_{a-1}, and M_a from H_a. Now suppose $|K|$ attains this value. Assume without loss of generality that $L_2 \in K$; the argument is symmetric if instead $T_2 \in K$. Then the chain $y_1, y_2, x_1, x_2, \ldots, x_a, y_3, y_4, \ldots, y_b$ avoids the remaining listed possible sets for K in $R(id, id)$. This is a contradiction since K is a cutset.

Therefore, K must contain an element that is not minimum weight among the elements on its hook. The next smallest possible value for $|K|$ is obtained by replacing M_a, which has weight 1, by one of the four sets of weight a in H_a. These are L_a, R_a, and the two elements next to M_a. Now suppose $|K \cap R(id, id)|$ attains this new lower bound. First we may suppose by symmetry that $L_2 \in K$. Then the chain listed above avoids all other possible sets in K except L_a, so K must contain L_a. Then the selections $L_3, \ldots, L_{a-1} \in K$ are forced. Applying similar reasoning to every rectangle, it follows that $|K| \geq |L(A)|$. Since this bound is attained by $L(A)$, it is sharp. Now we seek all K that meet the bound. In $R(id, id)$, if one supposes similarly that $T_2 \in K$, rather than $L_2 \in K$ as above, it follows that $T_3, \ldots, T_a \in K$. Referring back to (1), we have that for every $R(\alpha, \beta)$, $K \cap R(\alpha, \beta)$ either contains all left ends or all top ends of hooks. However, a mixture of these two options is impossible, so $K \cap R(\alpha, \beta)$ contains the same type of ends for all (α, β). It follows that K is either $L(A)$ or $U(A)$, which completes the proof of the last case. □

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REFERENCES

[BG] M. Bell and J. Ginsburg, *Compact spaces and spaces of maximal complete subgraphs*, Trans. Amer. Math Soc., 283 (1984), pp. 329–338.

[EK] P. L. Erdős and G. O. H. Katona, *Convex hulls of more-part Sperner families*, Graphs and Combin., 2 (1986), pp. 123-134.

[FGK] Z. Füredi, J. R. Griggs, and D. J. Kleitman, *A minimal cutset of the Boolean lattice with almost all members*, IMA Preprint Series No. 421 (1988). submitted.

[GRS] J. Ginsburg, I. Rival, and B. Sands, *Antichains and finite sets that meet all maximal chains*. to appear.

[Li] Ko-Wei Lih, *Problems in finite extremal set theory*, in *Designs and finite geometries (Kyoto,1986), Sûrikaisekikenkyûsho Kôkyûroku No. 607 (1987)*, pp. 1–4.
[Lu] D. Lubell, *A short proof of Sperner's theorem*, J. Combinatorial Th., 1 (1966), p. 299.
[N] R. Nowakowski, *Cutsets of Boolean lattices*, Discrete Math., 63 (1987), pp. 233–242.
[RZ] I. Rival and N. Zaguia, *Crosscuts and cutsets*, Order, 1 (1984), pp. 235–247.
[SW] N. Sauer and R. Woodrow, *Finite cutsets and finite antichains*, Order, 1 (1984), pp. 35–46.

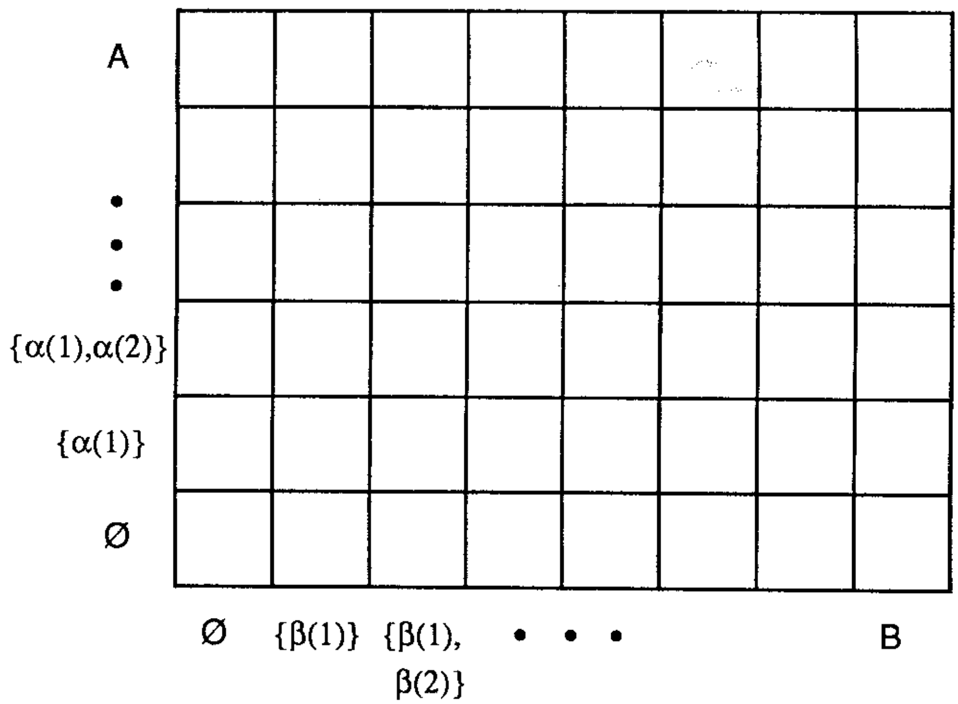

Fig. 1. The Rectangle R (α,β)

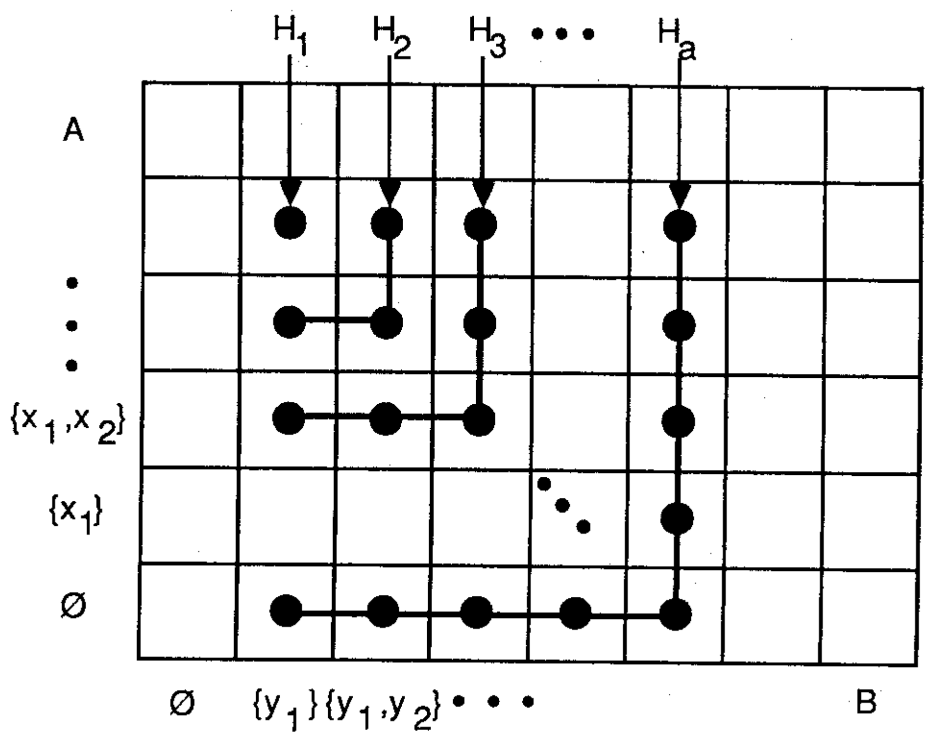

Fig. 2. Hooks in R(id,id)