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LAURENT HABSIEGER*

Abstract. Inspired by the proof of the irrationality of $\zeta(2)$ and $\zeta(3)$, Alladi and Robinson used Legendre polynomials to obtain some irrationality measures for numbers of the form $\log(1+z)$. We extend these results by studying arithmetic and asymptotic properties of sequences related to Gegenbauer polynomials. This yields some irrationality measures for ${}_2F_1\left(\begin{matrix}1, & 1/2 \\ \frac{\alpha+3}{2} & \end{matrix}; x\right)$, under some suitable conditions on α and x. We also present a possible generalisation to numbers of the form

$$\frac{\Gamma(1+\beta)\Gamma(\frac{1+\alpha}{2})}{\Gamma(1+\beta+\frac{1+\alpha}{2})}{}_2F_1\left(\begin{matrix}\frac{1+\beta}{2}, & 1-\frac{\beta}{2} \\ 1+\beta+\frac{1+\alpha}{2} & \end{matrix}; x\right).$$

0. Introduction. In 1978 Apery found an amazing proof of the irrationality of $\zeta(3)$. It was based on the following sequences of rational numbers. Let (a_n) and (b_n) be defined by the recurrence relation

$$(n+1)^3 u_{n+1} - [(n+1)^3 + n^3 + 4(2n+1)^3]u_n + n^3 u_{n-1} = 0$$

and the initial conditions $(a_0, a_1) = (1, 5)$ and $(b_0, b_1) = (0, 6)$. Then $\frac{b_n}{a_n} - \zeta(3) = O(n^{-3}a_n^{-2})$, as n goes to infinity.

Moreover, for every integer n, the rational numbers a_n and $2d_n^3 b_n$ are rational integers, where d_n denotes the lowest common multiple of $1, \dots, n$. Combining these good arithmetic and asymptotic properties, one can find the following irrationality measure for $\zeta(3)$: for all $\epsilon > 0$ and all $q > q_0(\epsilon)$, we have

$$|\zeta(3) - \frac{p}{q}| > \frac{1}{q^{\theta+\epsilon}} \text{ where } \quad \theta = 13.61782\dots$$

Similarly, he found an analogous result for $\zeta(2)$: For all $\epsilon > 0$ and all $q > q_0(\epsilon)$, we have

$$|\zeta(2) - \frac{p}{q}| > \frac{1}{q^{\theta'+\epsilon}}, \text{ where } \theta' = 11.85078\dots,$$

by using the sequences defined by the recurrence relation

$$(n+1)^2 u_{n+1} - (11n^2 + 11n + 3)u_n - n^2 u_{n-1} = 0$$

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and the initial conditions $(a'_0, a'_1) = (1,3)$ and $(b'_0, b'_1) = (0,5)$.

As Wirsing pointed out, this is related to a known irrationality proof of log 2, which uses the sequences defined by the recurrence relation

$$(n+1)u_{n+1} - 3(2n+1)u_n + nu_{n-1} = 0$$

and the initial conditions $(a_0, a_1) = (1,3)$ and $(b_0, b_1) = (0,1)$. Simultaneously, Beukers [3] noticed that

$$2(b_n - a_n\zeta(3)) = \int_0^1\int_0^1 P_n(x)P_n(y)\frac{\log xy}{1-xy}dxdy$$
$$\text{and} \quad b'_n - a'_n\zeta(2) = \int_0^1\int_0^1 P_n(x)(1-y)^n\frac{dxdy}{1-xy},$$

where $P_n(x)$ is a Legendre polynomial given by the formula $P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}[x^n(1-x)^n]$.

This integral version was used by Alladi and Robinson [1] to find irrationality measures for $ln(1+r)$, where r is a small rational number and for $\frac{\pi}{\sqrt{3}}$. Their proof was related to the sequences defined by the recurrence relation

$$(n+1)u_{n+1} - (2n+1)(2x+1)u_n + nu_{n-1} = 0$$

and the initial conditions $(a_0, a_1) = (1, 2x+1)$ and $(b_0, b_1) = (0,1)$. By taking x in the set of rational numbers, they obtain some irrationality measures for number of the form $\log(1+r)$; but, to get similar results for $\frac{\pi}{\sqrt{3}}$, they have to work in the ring of Gaussian integers, which seems artificial. So we decided to investigate the sequence defined by the recurrence relation

$$(n+1)u_{n+1} - a(2n+1)a_n + bnu_{n-1} = 0$$

and the initial condition $(a_0, a_1) = (1, a)$. It is not hard to show that a necessary and sufficient condition for (a_n) integral is $b = a^2 - 4c$ for some integer c. Then a_n/a^n may be expressed as a polynomial in c/a^2, with integral cofficients. Putting $x = 1 - 4c/a^2$ we are lead to investigate the sequences defined by the recurrence relation

$$(n+1)u_{n+1} - (2n+1+\alpha)u_n + x(n+\alpha)u_{n-1} = 0.$$

The choice $\alpha = 0$ corresponds to the results of Alladi and Robinson and gives irrationality measures for $\frac{1}{\sqrt{x}}\arg th\sqrt{x}$ and $\frac{1}{\sqrt{-x}}\operatorname{arctg}\sqrt{-x}$, according to the sign of x.

In the first part we will introduce the notations and the background that will be used in what follows. Part 2 will be devoted to the study of the asyptotic behavior of the sequence, and Part 3 to their arithmetic behavior. Combining the results from these two parts will produce irrationality results in Part 4. Part 5 will give one possible generalization.

Before starting this study, I would like to thank Richard Askey for the fruitful discussions we had at the Institute and Dennis Stanton for advice and the constant encouragement he provided me.

1. Notation and remarks. In what follows, (u_n) will denote a sequence satisfying the recurrence relation

$$\text{(R)} \qquad (n+1)u_{n+1} - (2n+1+\alpha)u_n + x(n+\alpha)u_{n-1} = 0.$$

Moreover (a_n) (resp. (b_n)) will denote the sequence defined by (R) and the initial conditions $(a_0, a_1) = (1, 1+\alpha))$ (resp $(b_0, b_1) = (0,1)$). The generating function of a sequence (u_n) will be denoted by $y(t) := \sum_{n\geq 0} u_n t^n$. The generating function of the sequene (a_n) (resp. (b_n)) will be denoted by $f(t)$ (resp. $g(t)$). One can note that "(u_n) verifies (R)" is equivalent to "y verifies $(\mathfrak{D})$", where $\mathfrak{D}$ is the differential equation

$$(\mathfrak{D}) \qquad (1-2t+xt^2)y'(t) + (1+\alpha)(tx-1)y(t) = y'(0) - (1+\alpha)y(0).$$

The following proposition gives explicit formulas for the sequences (a_n) and (b_n), and their generating functions. We adopt the standard notation $(t)_k$ to denote $\prod_{i=0}^{k-1}(t+i)$.

Proposition 1. *For all $n \in N$, we have:*

$$a_n(\alpha) = a_n = \sum_k \frac{(\frac{1+\alpha}{2})_k}{k!} \frac{(2k-1+\alpha)_{n-2k}}{(n-2k)!}(1-x)^k,$$

$$f_\alpha(t) = f(t) = (1-2t+xt^2)^{\frac{-1-\alpha}{2}},$$

$$b_n(\alpha) = b_n = \sum_k a_k(\alpha)\frac{a_{n-1-k}(-\alpha)}{n-k},$$

$$g_\alpha(t) = g(t) = f_\alpha(t)\int_0^t f_{-\alpha}(u)du.$$

Proof. Let us denote by a'_n the right-hand side of the first equality. Trivially we have $(a'_0, a'_1) = (1, 1+\alpha)$. Moreover (a'_n) satisfies (R):

$$\begin{aligned}
&(n+1)a'_{n+1} - (2n+1+\alpha)a'_n + x(n+\alpha)a'_{n-1} \\
&= \sum_k \frac{(\frac{1+\alpha}{2})_k}{k!} \frac{(2k+1+\alpha)_{n-2k}}{(n+1-2k)!}(1-x)^k \\
&\times[(n+1)(n+1+\alpha) - (n+1-2k)(2n+1+\alpha) + (n+1-2k)(n-2k) \\
&\qquad - k\frac{(2k+\alpha)(2k+\alpha-1)}{k+\frac{\alpha-1}{2}}] \\
&= 0.
\end{aligned}$$

The generating function $f_\alpha(t)$ and the function $(1-2t+xt^2)^{\frac{-1-\alpha}{2}}$ satisfy the same first order differential equation (for in this case $y'(0) = (1+\alpha)y(0)$) and take the same value at 0, so they are equal. To find another solution of $\mathcal{D}$, let us write $y(t) = c(t)f_\alpha(t)$. Then we have to solve

$$(1-2t+xt^2)c'(t)f_\alpha(t) = c.$$

So $c'(t) = cf_{-\alpha}(t)$ and thus $f_\alpha(t)\int_0^t f_{-\alpha}(u)du$ is another solution. By looking at the values of this function and its derivative at 0, one can find it is in fact the function $g_\alpha(t)$, the corresponding formula for b_n can easily be deduced from this. □

Remarks. When $\alpha = 0$ the formulas for a_n and b_n become:

$$a_n = \sum_k \frac{n!}{(n-2k)!k!^2}\left(\frac{1-x}{4}\right)^k \quad \text{and} \quad b_n = \sum_k a_k \frac{a_{n-1-k}}{n-k}.$$

In the proof of the irrationality of log 2 suggested by Alfred van der Poorten [10], we get the same formula relating b_n to a_n but the summation formula for a_n is different: $a_n = \sum_k \frac{(n+k)!}{(n-k)!k!^2}$. This explains partially why Alladi and Robinson's generalization gave different results from those presented here. However these two summation formulas are equivalent, thanks to the quadratic transformations of the hypergeometric function.

The proof of the summation formula for b_n given above uses in an essential way generating functions and the differential equation $(\mathcal{D})$. It would be interesting to find a direct proof of it, using properties of the sequence (a_n), or to find an analogous explicit formula for (b_n)

2. Asymptotic behavior of the sequences. Now let us take x in $\mathbf{C}\backslash[1,+\infty[$ and let us choose $\sqrt{1-x}$ with a positive real part. Then the equation $xt^2 - 2t + 1 = 0$ has two roots λ_i and λ_s of different norms ($|\lambda_i| < |\lambda_s|$):

$$\lambda_i = (1+\sqrt{1-x})^{-1} \quad \text{and} \quad \lambda_s = (1-\sqrt{1-x})^{-1}.$$

Let also restrict α to have modulus less than one. The asymptotic behavior of the sequences defined by (R) is given by the following proposition.

Proposition 2. *When n goes to infinity, the following results hold:*

i) *there exists a constant c such that $u_n = cn^{\frac{\alpha-1}{2}}\lambda_i^{-n}(1+O(\frac{1}{n}))$.*

ii) *the sequence b_n/a_n converges to a limit ℓ and satisfies*

$$\ell - \frac{b_n}{a_n} = \sum_{m\geq n} \frac{(1+\alpha)_m}{(m+1)!} \times \frac{x^m}{a_m a_{m+1}}.$$

iii) *We have $\ell = \frac{1}{1+\alpha}{}_2F_1\left(\begin{matrix}1, & 1/2 \\ & \frac{\alpha+3}{2}\end{matrix};x\right)$, where ${}_2F_1\left(\begin{matrix}a, & b \\ & c\end{matrix};x\right) = \sum_{n\geq 0}\frac{(a)_n(b)_n}{n!(c)_n}x^n$ denotes the usual hypergeometric function.*

Proof. i) Let us put $v_n = (n+1)^{\frac{1-\alpha}{2}} \lambda_i^n u_n$ and $V_n = \binom{v_n}{v_{n+1}}$. Then we get a recurrence relation $V_{n+1} = M_n V_n$, where M_m is the matrix

$$\begin{pmatrix} 0 & 1 \\ -x\lambda_i^2 \frac{(n+3)^{\frac{1-\alpha}{2}}(n+1+\alpha)(1+n)^{\frac{\alpha-1}{2}}}{n+2} & \lambda_i (n+3)^{\frac{1-\alpha}{2}}(2n+3+\alpha)(n+2)^{\frac{\alpha-8}{2}} \end{pmatrix}.$$

It is a straightforward calculation to see that $M_n = M + O(\frac{1}{n^2})$ when n goes to infinity, where M is the matrix $\begin{pmatrix} 0 & 1 \\ -x\lambda_i^2 & 2\lambda_i \end{pmatrix}$. The matrix M has for eigenvalues 1 and $\frac{\lambda_i}{\lambda_s}$. Since λ_i/λ_s has modulus less than one, the product $\prod_{n=O}^{\infty} M_n = P$ converges to the matrix of a projection and the first assertion holds.

ii) Moreover the proof given above shows that $ln|u_n|$ tends to λ_i if $PV_0 \neq 0$, and $ln(u_n)$ tends to λ_s if $PV_0 = 0$ and $V_0 \neq 0$. Since the radius of convergence of f is clearly λ_i^{-1} by Proposition 1, the constant defined by i) for (a_n) is non-zero and so the sequence b_n/a_n converges as n goes to infinity. Moreover for every positive integer n we have:

$$\begin{aligned} b_{n+1}a_n - a_{n+1}b_n &= \frac{x(n+\alpha)}{n+1}(b_n a_{n-1} - a_n b_{n-1}) \\ &= \frac{x^n \Gamma(n+1+\alpha)}{\Gamma(n+2)\Gamma(\alpha+1)}(b_1 a_0 - a_1 b_0). \end{aligned}$$

So that we get

$$\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} = \frac{x^n(1+\alpha)n}{(n+1)! a_n a_{n+1}}.$$

Summing from n to infinity yields to

$$\ell - \frac{b_n}{a_n} = \sum_{m \geq n} \frac{(1+\alpha)_m}{(m+1)!} \times \frac{x^m}{a_m a_{m+1}},$$

where ℓ denotes the limit of the sequence b_n/a_n. The function $g - \ell f$ must have no singularity at λ_i and thus

$$\begin{aligned} \ell &= \lim_{z \to \lambda_i} \frac{g(z)}{f(z)} = \int_0^{\lambda_i} f_{-\alpha}(t)dt \\ &= \frac{1}{1+\sqrt{1-x}} \int_0^1 \left[(1-u)(1 - \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}u)\right]^{\frac{\alpha-1}{2}} du \\ &= \frac{2}{\alpha+1} \times \frac{1}{1+\sqrt{1-x}} {}_2F_1 \left(\begin{matrix} 1 \quad , \quad \frac{1-\alpha}{2} \\ 1 + \frac{\alpha+1}{2} \end{matrix} ; \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} \right) \text{ by } [5, p.114, (1)], \\ &= \frac{1}{\alpha+1} \times \frac{1}{\sqrt{1-x}} \quad {}_2F_1, \left(\begin{matrix} 1/2 \quad , \quad \frac{\alpha+1}{2} \\ 1 + \frac{\alpha+1}{2} \end{matrix} ; \frac{x}{x-1} \right) \text{ by } [5, p.11, (11)], \\ &= \frac{1}{1+\alpha} {}_2F_1 \left(\begin{matrix} 1, \quad 1/2 \\ \frac{\alpha+3}{2} \end{matrix} ; x \right) \quad \text{by } [5, p.105, (3)]. \end{aligned}$$

□

Remarks. As Askey [2] has pointed out, the relevant tool is the asymptotic behavior of orthogonal polynomials. Indeed the polynomials introduced by Beukers are $L_n(2x+1)$, where L_n is the nth Legendre polynomial. The recurrence relation for L_n is

$$(n+1)L_n(x) = (2n+1)xL_n(x) - nL_{n-1}(x).$$

The Legendre polynomials are special cases ($\lambda = \frac{1}{2}$) of the Gegenbauer polynomials that satisfy the recurrence relation (cf. [6], p. 175):

$$(n+1)C_{n+1}^{\lambda}(x) = 2(n+\lambda)xC_n^{\lambda}(x) - (n+2\lambda-1)C_{n-1}^{\lambda}(x).$$

As a matter of fact our sequence (a_n) is related to the Gegenbauer polynomials by the formula

$$a_n = x^{n/2}C_n^{\frac{1+a}{2}}(x^{-1/2}).$$

In the same way, our sequence (b_n) is related to Gegenbauer polynomials of the second kind. Thus another way to get the asymptotic behavior of these sequences is to apply general results on orthogonal polynomials. Such theorems may be found in [9].

3. Arithmetic Behavior of the Sequences (a_n) and (b_n). In order to control the denominator of the coefficient of $(1-x)^k$ in the expansion of a_n given by the Proposition 1 we need the following Lemma.

LEMMA 1. *Let p be a prime. Define $\chi_p : \mathbf{N} \to \mathbf{N}$ by $\chi_p(n)$ is the sum of the digits of n written in base p. Then for every positive integer r and every integer s we have*

$$\prod_{p|r} p^{\frac{n-\chi_p(n)}{p-1}} \times \prod_{k=1}^{n} \frac{kr+s}{k} \in \mathbf{Z},$$

for each nonnegative integer n.

Proof. Let us consider the canonical map $\varphi : \mathbf{Z}^n \to \left(\prod_{\alpha>0} \mathbf{Z}/p^\alpha\mathbf{Z}\right)^n$. Then the power of p in $n!$ is equal to the number of zero classes in the range of $\{1,\ldots,n\}$ by φ. Moreover for each positive integer α the range of $\{1,\ldots,n\}$ contains $[\frac{n}{p^\alpha}]$ complete copies of $\mathbf{Z}/p^\alpha\mathbf{Z}$, where $[x]$ denotes the integer part of x. Defining $ord_p(m)$ to be the power of p in the prime number factorization of the integer m, we get

$$\operatorname{ord}_p n! = \sum_{\alpha>0} \left[\frac{n}{p^\alpha}\right] = \frac{n-\chi_p(n)}{p-1},$$

as shown by a straightforward computation.

Now for p not dividing r the map $k \to kr+s$ induces on isomorphism from $\mathbf{Z}/p^\alpha\mathbf{Z}$ onto $\mathbf{Z}/p^\alpha\mathbf{Z}$, for each positive integer α. Thus the image of φ on $\{k+s,\ldots,kn+s\}$ will also contain $[\frac{n}{p^x}]$ complete copies of $\mathbf{Z}/p^\alpha\mathbf{Z}$. In particular there will be at least $[\frac{n}{p^\alpha}]$ zero classes in the component $(\mathbf{Z}/p^\alpha\mathbf{Z})^n$ and the lemma is proved. □

We are now able to estimate the size of the denominators of a_n and b_n. This will be stated in the following Proposition.

PROPOSITION 3. *Let δ be a denominator of $1+\alpha$ and define ϵ to be 1 if δ is even and 2 if δ is odd. Let also δ' be a denominator of $\frac{(1-x)\delta}{2\epsilon}$ and let us put $\omega_p = \mathrm{ord}_p\,\delta'$ if p does not divide δ and $\omega_p = ord_p\delta' - \frac{1}{p-1}$ if p divides δ. Finally let us define $c_n = \delta^n \prod_{p|\delta} p^{[\frac{n}{p-1}]} \prod_{p|\delta'} p^{[\omega_p \cdot \frac{n}{2}]}$. Then $c_n a_n$ and $c_n d_n b_n$ are integers, where d_n denotes the lowest common multiple of $1,\ldots,n$.*

Proof. Thanks to the previous lemma and the definitions the three following quantitites are integers:

$$\frac{(\frac{1+\alpha}{2})k}{k!} \times (2\delta)^k \times \prod_{p|2\delta} p^{\frac{k-\chi_p(k)}{p-1}},$$

$$\frac{(2k+1+\alpha)n-2k}{(n-2k)!} \times \delta^{n-2k} \times \prod_{p|\delta} p^{\frac{2n-2k-\chi_p(n-2k)}{p-1}},$$

$$(1-x)^k \times \left(\frac{\delta\delta'}{2\epsilon}\right)^k.$$

Thus if one multiplies the summand in Proposition 1 by $\delta^n \delta^{1k} \epsilon^{-\chi_2(k)} \prod_{p|\delta} p^{\frac{n-k-\chi_p(k)-\chi_p(n-2k)}{p-1}}$ the result is an integer. This last number divides c_n for every integer k in $\{0,\ldots,[\frac{n}{2}]\}$ and therefore $a_n c_n$ is an integer. Now for any integer k in $\{0,\ldots,n-1\}$, we know that $c_k c_{n-1-k}$ divides c_n and $n-k$ divides d_n. Noticing that the transformation α gives $-\alpha$ does not change the definition of c_n, we can see from Proposition 1 that $c_n d_n b_n$ is an integer. □

We are now able to give an irrationality measure for the limit ℓ described in Proposition 2.

4. Irrationality results. Let us put $\Delta = \delta \prod_{p|\delta} p^{\frac{1}{p-1}} \prod_{p|\delta'} p^{\frac{\omega_p}{2}}$. Then we have the following statement.

PROPOSITION 4. *With the above notations there exist integers sequences (p_n) and (q_n) such that:*

$$ln|q_n| \sim n[ln\left|\frac{\Delta}{\lambda i}\right| + 1],$$

$$ln|lq_n - p_n| \sim n[ln\left|\frac{\Delta}{\lambda s}\right| + 1].$$

Proof. Let us put $q_n = a_n c_n d_n$ and $p_n = b_n c_n d_n$. When n goes to infinity, we get the following equivalences:

$ln|a_n| \sim nln|\lambda i^{-1}|$ by Proposition 2, i),

$ln|\ell - b_n/a_n| \sim nln|x\lambda_i^2|$ by Proposition 2, ii),

$ln|c_n| \sim nln|\Delta|$ by the definition of c_n,

$ln|d_n| \sim n$ by the prime number theorem.

Combining these results gives the Proposition. □

For the purpose of the theorem we need a technical lemma that can be found in [1].

LEMMA 2. *Let θ be a non-zero real number. Suppose there exist $k_0 > 0$, $l_0 > \frac{1}{2}$, $Q > 1$ and $E > 1$ such that for all n there are p_n, q_n in $\mathbb{Z}$ with $|q_n| < k_0 Q^n$ and $|q_n\theta - p_n| \leq l_0 E^{-n}$. Suppose further that $p_n q_{n+1} \neq p_{n+1} q_n$ for all n. Then for any $p, q \in \mathbb{Z}$, $q \neq 0$,*

$$|\theta - \frac{p}{q}| > \frac{c}{|q|^{\frac{\log QE}{\log E}}}, \qquad \text{where } c = \frac{1}{2k_0} Q^{(-2+\frac{\log 2l_0}{\log E})}.$$

We are now able to state the

THEOREM. *With the above notation, ℓ is irrational if $x \neq 0$ and $ln|\frac{\Delta}{\lambda_s}| + 1 < 0$, and an irrationality measure is*

$$\mu = 1 - \frac{ln|\frac{\Delta}{\lambda_s}| + 1}{ln|\frac{\Delta}{\lambda_i}| + 1} = \frac{ln|\frac{\lambda_s}{\lambda_i}|}{1 + ln|\frac{\Delta}{\lambda_i}|} .$$

Proof. Here we choose $Q = \frac{\Delta}{|\lambda_i|} e^{1+\epsilon}$ and $E = \frac{|\lambda_s|}{\Delta} e^{-1-\epsilon}$, which is an authorized choice by Proposition 4, when $\epsilon > 0$. By Proposition 2 ii), we have $p_{n+1}q_n \neq p_n q_{n+1}$ for every n and then for any $p, q \in \mathbb{Z}$, $q \neq 0$,

$$|\ell - \frac{p}{q}| > c(\epsilon) q^{-\mu(\epsilon)},$$

with $\mu(\epsilon) = 1 - \frac{ln|\frac{\Delta}{\lambda_i}|+1+\epsilon}{ln|\frac{\Delta}{\lambda_s}|+1+\epsilon}$. The theorem follows. □

Let us take $\alpha = 0$. We find then the following result, partially discovered by Chudnovsky [4] and proved recently by Huttner. ([18], Corollaire 1):

COROLLARY. *For $|x| \in]0, 1[$, let δ' be a denominator of $\frac{1-x}{4}$. If $\sqrt{\delta'}(1 - \sqrt{1-x})e < 1$, then $\frac{Argth\sqrt{x}}{\sqrt{x}}$ if $x > 0$ (resp. $\frac{Arctg\sqrt{-x}}{\sqrt{-x}}$ if $x < 0$) is irrational, with the irrationality measure $\mu = 1 - \frac{ln(\sqrt{\delta'}(1-\sqrt{1-x}))+1}{ln(\sqrt{\delta'}(1+\sqrt{1-x}))+1}$.*

Proof. Indeed when $\alpha = 0$, the limit ℓ is ${}_2F_1\begin{pmatrix} 1 & , & 1/2 \\ & & 3/2 \end{pmatrix}; x\Big)$, that is

$$\ell = \sum_{n \geq 0} \frac{x^n}{2n+1} = \begin{cases} \frac{1}{\sqrt{x}} \operatorname{Argth} \sqrt{x} & \text{if} \quad x > 0, \\ \frac{1}{\sqrt{-x}} \operatorname{Arctg} \sqrt{-x} & \text{if} \quad x < 0. \end{cases}$$

As $\delta = 1$, we get $\Delta = \sqrt{\delta'}$ and the corollary holds. □

Remarks. Huttner's method is to solve a Padé approximating problem. He finds explicit polynomials $P^{(n)}(x)$ of degree n and $Q^{(p)}(x)$ of degree p, and the analytic function $R_{n+p+1}(x)$ such that the following relation is satisfied in a neighborhood of 0:

$$P^{(n)}(x)_2F_1\begin{pmatrix} a+1, & b \\ c+1 & ;x\end{pmatrix} + Q^{(p)}(x)_2F_1\begin{pmatrix} a, & b \\ c & ;x\end{pmatrix} = x^{n+p+1}R_{n+p+1}(x),$$

with $R_{n+p+1}(0) \neq 0$. When none of the numbers $a, b, c, c-a, c-b, a-b, c-a-b$ is an integer, he deduces [7] irrationality results for the Gauss quotient ${}_2F_1\begin{pmatrix} a+1, & b \\ c+1 & ;x\end{pmatrix} \Big/ {}_2F_1\begin{pmatrix} a, & b \\ c & ;x\end{pmatrix}$. He also gives results for some values of ${}_2F_1\begin{pmatrix} 1, & 1/k \\ 1+1/k & ;x\end{pmatrix}$ (cf [8]). Thus our methods intersect only for the case ${}_2F_1\begin{pmatrix} 1, & 1/2 \\ 3/2 & ;x\end{pmatrix}$ ($\alpha = 0$ in this paper) and give the same results, which is not very surprising since they are based on the same approximations.

The formulas used to describe the asymptotic behavior of the sequences (Proposition 1 and 2) are closed and thus may be used to get explicit approximations. Once again, when $\alpha = 0$, we would recover the explicit results of Huttner [8].

5. A Further Generalization. In order to get more general sequences, we may try to generalize the differential equation $(\mathcal{D})$, which is essentially equivalent to the recurrence relation $(\mathcal{R})$. First let us notice that $(\mathcal{D})$ may be written as $(y/f_\alpha)' = Cf_{-\alpha}$. So a natural generalisation is to replace the constant C by a function of t. To keep good arithmetical properties for the sequences requires special properties for the function $c(t)$, namely $\int_0^x t^n c(t)dt$ is a rational number for any nonnegative integer n, with an "easy" denominator. A good choice for $C(t)$ appears then to be Ct^β, for β rational. We then have the following

THEOREM. *Let β be a rational number in $]0,1[$ with denominator s. Let us put $d = \left(\sum_{\substack{j=1 \\ (j,s)=1}}^{s} \frac{1}{j}\right) \frac{s}{\phi(s)}$, where ϕ is Euler's function. Let α be a rational in $]-1,+1[$, with denominator δ and let ϵ be equal to 1 (resp 2) if δ is even (resp. odd). Let x be a non-zero rational number, and δ' a denominator of $\frac{(1-x)\delta}{2\epsilon}$. Let us put*

$$\Delta = \delta\sqrt{\delta'} \prod_{\substack{p|\delta \\ p\nmid\delta'}} p^{\frac{1}{p-1}} \prod_{\substack{p|\delta \\ p|\delta'}} p^{\frac{1}{2(p-1)}}.$$

If $ln|\Delta(1-\sqrt{1-x})|+d < 0$ then $\int_0^{(1+\sqrt{1-x})^{-1}} t^\beta f_{-\alpha}(t)dt$ is irrational and an irrationality measure is $\mu = 1 - \frac{ln|\Delta(1-\sqrt{1-x})|+d}{ln|\Delta(1+\sqrt{1-x}|+d}$

Proof. Since the technics used are similar to those in the previous sections, we will sketch the proof. This time we have

$$f_\alpha(t) = \sum_{n\geq 0} a_n t^n = (1-2t+xt^2)^{\frac{-1-\alpha}{2}}$$

$$g_\alpha(t) = f_\alpha(t) \int_0^t u^\beta f_{-\alpha}(u) du = \sum_{n \geq 0} b_n t^n.$$

Thus we have the expansion

$$b_n = \sum_k a_{n-1-k}(-\alpha) \frac{a_k(\alpha)}{\beta + 1 + k}$$

and a common denominator of $\{\frac{1}{\beta+1+k}; 0 \leq k \leq n-1\}$ will be $d'_n = l.c.m.\{sk + r : 1 \leq k \leq n\}$, if $\beta = r/s$. From Lemma 1 of Alladi and Robinson [1] we have the estimate $d'_n \sim nd$ as n goes to infinity. So, putting $p'_n = a_n c_n d'_n$ and $q'_n = b_n c_n d'_n$ (where c_n is defined as in §4), we get two sequences of integers (p'_n) and (q'_n) such that

$$ln|q'_n| \sim n[ln|\frac{\Delta}{\lambda_i}| + d]$$
$$ln|q'_n \ell' - p'_n| \sim n[ln|\frac{\Delta}{\lambda s}| + d],$$

where ℓ' is the limit of the sequence a_n/b_n. The differential equation satisfied by f_α and g_α is still of order 2:

$$z' - \beta z = 0, \quad \text{with} \quad z = (y f_\alpha^{-1})' f_{-\alpha}^{-1} = (1 - 2t + xt^2)y' + (1 + \alpha)(tx - 1)y$$

Since (f_α, g_α) is linearly independent this is a basis of solutions. So there exist a linear combination $\lambda f_\alpha + \mu g_\alpha$ that has no singularity at λ_i. A possible choice for λ and μ is $(\int_0^{\lambda_i} u^\beta f_{-x}(u) du, 1)$. Now $\ell' f_\alpha + g_\alpha$ has also no singularity at λ_i, thanks to the estimate of $ln|a_n \ell' - b_n|$. Thus $\ell' = \int_0^{\lambda_i} u^\beta f_{-\alpha}(u) du$ and theorem is proved, by applying again the Lemma 2. □

As in §2 we can use some quadratic transformations of the hypergeometric function to find the following expression for the limit:

$$\ell = \frac{\Gamma(1+\beta)\Gamma(\frac{1+\alpha}{2})}{\Gamma(1+\beta+\frac{1+\alpha}{2})} {}_2F_1\left(\begin{matrix} \frac{1+\beta}{2}, & 1 - \frac{\beta}{2} \\ 1 + \beta + \frac{1+\alpha}{2} & \end{matrix}; x\right).$$

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