A FREE BOUNDARY PROBLEM FOR A HAMILTON-JACOBI EQUATION ARISING IN IONS ETCHING

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A Free Boundary Problem for a Hamilton-Jacobi Equation Arising in Ions Etching

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§1 Introduction

When certain material occupying the space $\{y \mid y < y(x,t)\}$ is etched at its surface by bombardment with beam of ions with constant flux density, the evolution of the bombarded surface $y = y(x,t)$ is described by the equation

$$y_t + f(y_x) = 0 \quad , \tag{1.1}$$

where $f(p)$ is called the **sputtering function**; f is positive, continuous and uniformly bounded for $-\infty < p < \infty$ (see [8] for details). The initial shape of the surface is specified by the equation

$$y(x,0) = h(x) \quad . \tag{1.2}$$

If we differentiate (1.1) in x and set $y_x = p$, then the problem reduces to solving the conservation law

$$p_t + (f(p))_x = 0 \tag{1.3}$$

with the initial condition

$$p(x,0) = p_0(x) \equiv h'(x) \quad . \tag{1.4}$$

In this paper, we study a model arising in the fabrication of a semiconductor device by ion etching. In this model, we have two different materials with their respective sputtering functions $f_i (i = 1,2)$; one material is on the top of the other as shown in Figure 1, and their common material boundary is given by $y = g(x)$.

Let us denote by (I) the region in the (x,t) plane where the upper matrerial is located, and by (II) the region in the (x,t) plane where the lower material is located. (I) and (II) will be called the **upper material region** and the **lower material region**, respectively.

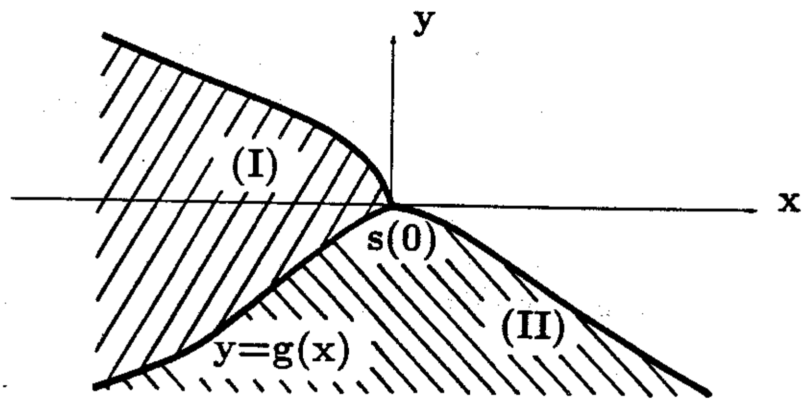

Figure 1

Assuming that $g(x)$ is monotone increasing and that $h(x)$ is monotone decreasing, we anticipate that the regions (I) and (II) will be given by

$$(\mathrm{I}) = \{(x,t) \mid x < s(t), t > 0\}\ ,\ \ (\mathrm{II}) = \{(x,t) \mid x > s(t), t > 0\} \tag{1.5}$$

where $\Gamma:\ x = s(t)$ is a curve, called the **free boundary**. Thus the etched surface $y = y(x,t)$ satisfies

$$y_t + f_1(y_x) = 0 \quad \text{for} \quad t > 0, x < s(t) \tag{1.6}$$

$$y_t + f_2(y_x) = 0 \quad \text{for} \quad t > 0, x > s(t) \tag{1.7}$$

$$y(x,0) = h(x) \quad \text{for} \quad -\infty < x < \infty \ . \tag{1.8}$$

Since the common material boundary is given by $y = g(x)$, we should have

$$y(s(t),t) = g(s(t)) \quad \text{for} \quad t > 0\ . \tag{1.9}$$

The problem can be formulated in terms of conservation laws by differentiating (1.6) (1.7) in x. However, it seems to be more convenient to work with the Hamilton-Jacobi equations (1.6) (1.7) and seek a viscosity solution (as defined by Crandall, Lions [2]). It will be shown that in general a viscosity solution is equivalent to the entropy solution of the corresponding conservation law in the case that the solution is piecewise smooth; this fact is crucial in proving the existence of a solution to the problem (1.6)–(1.9). The precise concept of a solution of (1.6)–(1.9) will be given in §2; in §§3–4 we prove existence and in §5 we prove uniqueness.

§2 Definition of the viscosity solution for (1.6)–(1.9)

Recall (cf. [3]) that a viscosity solution for the problem (1.1) on the open set U is a continuous function $u(x,t)$ such that for every $(x_0,t_0) \in U$

$$\tau + f(p) \leq 0 \quad \text{for } (p,\tau) \in D^+u(x_0,t_0) \tag{2.1}$$
$$\tau + f(p) \geq 0 \quad \text{for } (p,\tau) \in D^-u(x_0,t_0) \tag{2.2}$$

where $D^+u(x_0,t_0)$ is the superdifferential, consisting of the points (p,τ) such that

$$\limsup_{(x,t)\to(x_0,t_0)} \frac{u(x,t)-u(x_0,t_0)-p(x-x_0)-\tau(t-t_0)}{\sqrt{(x-x_0)^2+(t-t_0)^2}} \leq 0 \quad , \tag{2.3}$$

and $D^-u(x_0,t_0)$ is the subdifferential, consisting of the points (p,τ) such that

$$\liminf_{(x,t)\to(x_0,t_0)} \frac{u(x,t)-u(x_0,t_0)-p(x-x_0)-\tau(t-t_0)}{\sqrt{(x-x_0)^2+(t-t_0)^2}} \geq 0 \quad . \tag{2.4}$$

Recall (cf. [4]) also that for the problem (1.3), a piecewise continuous entropy solution with jump discontinuity on the smooth curve $x = s(t)$ is a function $p(x,t)$ satisfying (1.3) in the distribution sense and the entropy condition

$$\frac{f(p^+)-f(p)}{p^+-p} \leq s'(t) \leq \frac{f(p^-)-f(p)}{p^--p} \quad \text{for all } p \text{ between } p^+ \text{ and } p^- , \tag{2.5}$$

where

$$p^+ = p^+(s(t),t) \equiv p(s(t)+0,t) \tag{2.6}$$
$$p^- = p^-(s(t),t) \equiv p(s(t)-0,t) \tag{2.7}$$

LEMMA 2.1. *Suppose that the curve* Γ*:* $x = s(t)$ $(0 < t < T)$ *is continuous and piecewise* C^1*. Let* U *be an open set containing* Γ*. Set*

$$U^- = \{(x,t) \in U \mid 0 < t < T, x < s(t)\} \tag{2.8}$$
$$U^+ = \{(x,t) \in U \mid 0 < t < T, x > s(t)\} \quad . \tag{2.9}$$

Suppose that $u \in C^1(\overline{U^-}) \cap C^1(\overline{U^+}) \cap C(U)$*,* u_x *is discontinuous with jump discontinuity on* Γ*, and* u *satisfies in the classical sense*

$$u_t + f(u_x) = 0 \quad \textit{for } (x,t) \in U^- \cup U^+ \quad , \tag{2.10}$$

where f *is a given continuous function for all* $p \in R$*. Then* u *is a viscosity solution in* U *of (1.1) if and only if the entropy condition (2.5) is satisfied for* $p(x,t) = u_x(x,t)$ *at all continuity points of* $s'(t)$*.*

Proof : First assume that $s(t)$ is C^1. Take $(x_0, t_0) \in \Gamma$, i.e., $x_0 = s(t_0)$. We let $Du^{\pm}(x_0,t_0) = (u_x^{\pm}(x_0,t_0), u_t^{\pm}(x_0,t_0))$, and denote by $p_{_T}Du^{\pm}(x_0,t_0) \equiv (p_0, \tau_0)$ the projection of the vectors $Du^{\pm}(x_0,t_0)$ to the tangent line of Γ at (x_0,t_0). Then according to [2, Theorem I.10], u is a viscosity solution of (1.1) in U if and only if for every $(x_0,t_0) \in \Gamma$,

$$(\tau_0 + \xi n_t) + f(p_0 + \xi n_x) \le 0 \quad \text{for } Du^-(x_0,t_0)\cdot\vec{n} \le \xi \le Du^+(x_0,t_0)\cdot\vec{n} \tag{2.11}$$

$$(\tau_0 + \xi n_t) + f(p_0 + \xi n_x) \ge 0 \quad \text{for } Du^-(x_0,t_0)\cdot\vec{n} \ge \xi \ge Du^+(x_0,t_0)\cdot\vec{n} \tag{2.12}$$

where $\vec{n} = (n_x, n_t)$ is the normal vector of Γ at (x_0,t_0), given by

$$\vec{n} = (n_x, n_t) = \frac{1}{\sqrt{1+(s'(t_0))^2}}(1, -s'(t_0)) \quad .$$

Since u is C^1 up to the boundary Γ from each side, we have

$$\begin{aligned} \frac{d}{dt}u(s(t),t)\Big|_{t=t_0} &= u_x^+(x_0,t_0)s'(t_0) + u_t^+(x_0,t_0) \\ &= u_x^-(x_0,t_0)s'(t_0) + u_t^-(x_0,t_0) \quad . \end{aligned} \tag{2.13}$$

Hence the projection of the vectors $Du^{\pm}(x_0,t_0)$ to the tangent line of Γ at (x_0,t_0) is given by

$$\begin{aligned} p_{_T}Du^+(x_0,t_0) &= p_{_T}Du^-(x_0,t_0) = Du^{\pm}(x_0,t_0) - [\vec{n}\cdot Du^{\pm}(x_0,t_0)]\vec{n} \\ = \frac{u_x^+(x_0,t_0)s'(t_0) + u_t^+(x_0,t_0)}{1+(s'(t_0))^2}&(s'(t_0),\ 1) = \frac{u_x^-(x_0,t_0)s'(t_0) + u_t^-(x_0,t_0)}{1+(s'(t_0))^2}(s'(t_0),\ 1) \quad . \end{aligned} \tag{2.14}$$

Therefore for any $\xi \in R$,

$$p_{_T}Du^{\pm}(x_0,t_0) + \xi\vec{n} = (p, \tau) \tag{2.15}$$

where

$$\begin{aligned} p &= \frac{1}{1+(s'(t_0))^2}\left\{[u_x^+(x_0,t_0)s'(t_0) + u_t^+(x_0,t_0)]s'(t_0) + \xi\sqrt{1+(s'(t_0))^2}\right\} \\ &= \frac{1}{1+(s'(t_0))^2}\left\{[u_x^-(x_0,t_0)s'(t_0) + u_t^-(x_0,t_0)]s'(t_0) + \xi\sqrt{1+(s'(t_0))^2}\right\} \end{aligned} \tag{2.16}$$

$$\begin{aligned} \tau &= \frac{1}{1+(s'(t_0))^2}\left\{[u_x^+(x_0,t_0)s'(t_0) + u_t^+(x_0,t_0)] - s'(t_0)\xi\sqrt{1+(s'(t_0))^2}\right\} \\ &= \frac{1}{1+(s'(t_0))^2}\left\{[u_x^-(x_0,t_0)s'(t_0) + u_t^-(x_0,t_0)] - s'(t_0)\xi\sqrt{1+(s'(t_0))^2}\right\} \\ &= u_t^+(x_0,t_0) + s'(t_0)(u_x^+(x_0,t_0) - p) = u_t^-(x_0,t_0) + s'(t_0)(u_x^-(x_0,t_0) - p) \quad . \end{aligned} \tag{2.17}$$

By (2.16), it is easy to show that the inequality $Du^-(x_0,t_0)\cdot\vec{n} \le \xi \le Du^+(x_0,t_0)\cdot\vec{n}$ is equivlent to the inequality $u_x^-(x_0,t_0) \le p \le u_x^+(x_0,t_0)$. Similarly, the inequality

$Du^-(x_0,t_0)\cdot\vec{n} \geq \xi \geq Du^+(x_0,t_0)\cdot\vec{n}$ is equivlent to the inequality $u_x^-(x_0,t_0) \geq p \geq u_x^+(x_0,t_0)$. Thus, u is a viscosity solution in U if and only if

$$\tau + f(p) \;\geq\; 0 \quad \text{for } u_x^-(x_0,t_0) \leq p \leq u_x^+(x_0,t_0) \quad , \tag{2.18}$$
$$\tau + f(p) \;\leq\; 0 \quad \text{for } u_x^+(x_0,t_0) \leq p \leq u_x^-(x_0,t_0) \quad . \tag{2.19}$$

Note that by (2.10),

$$u_t^+(x_0,t_0) + f(u_x^+(x_0,t_0)) \;=\; 0 \quad , \tag{2.20}$$
$$u_t^-(x_0,t_0) + f(u_x^-(x_0,t_0)) \;=\; 0 \quad . \tag{2.21}$$

Substituting τ, p from (2.16) (2.17) into (2.18) and using the relations (2.20) (2.21), we get that (2.18) is equivalent to

$$s'(t_0)(u_x^+(x_0,t_0) - p) - [f(u_x^+(x_0,t_0)) - f(p)] \;\geq\; 0 \quad \text{for } p^- \leq p \leq p^+ \quad , \tag{2.22}$$
$$s'(t_0)(u_x^-(x_0,t_0) - p) - [f(u_x^-(x_0,t_0)) - f(p)] \;\geq\; 0 \quad \text{for } p^- \leq p \leq p^+ \quad , \tag{2.23}$$

provided $p^- \equiv u_x^-(x_0,t_0) < p^+ \equiv u_x^-(x_0,t_0)$. These inequalities are equivalent to the entropy condition (2.5) in the case $p^- < p^+$. The same procedure applied to (2.19) gives the equivalence between (2.19) and the entropy condition (2.5) in the case $p^- > p^+$. Thus the lemma is proved for the case that $s(t)$ is C^1.

The general case when $s(t)$ is piecewise C^1 follows immediately from the above special case and [3, Lemma 4.1], which implies that if u is a viscosity solution of (1.1) in $(a,b)\times(t_0,t_1) \cup (a,b)\times(t_1,t_2) \cup \cdots \cup (a,b)\times(t_{N-1},t_N)$ $(t_0 < t_1 < \cdots < t_N)$, then u is a viscosity solution of (1.1) in $(a,b)\times(t_0,t_N)$. □

Consider the equations (1.6)–(1.9) where we assume that

$$f_1,\ f_2 \in C(R) \quad , \tag{2.24}$$

$$h \in C(R)\ ,\ h(0) = 0\ ,\ h \text{ is strictly decreasing for } x \in (-\infty, 0] \quad , \tag{2.25}$$

$$g \in C(-\infty,0]\ ,\ g(0) = 0\ ,\ g \text{ is strictly increasing} \quad . \tag{2.26}$$

DEFINITION 2.2. Suppose that $y \in C(R\times[0,T])$, $s \in C([0,T])$ and $s(0) = 0$. The pair (y,s) will be called a viscosity solution of (1.6)–(1.9) if

i) y satisfies

$$y(x,t) > g(x) \quad \text{for } t \in [0,T]\ ,\ -\infty < x < s(t) \tag{2.27}$$
$$y(x,t) < g(x) \quad \text{for } t \in [0,T]\ ,\ s(t) < x \leq 0 \quad , \tag{2.28}$$

(and hence

$$y(s(t),t) = g(s(t)) \quad \text{for } t \in [0,T] \quad) \quad . \tag{2.29}$$

ii) The equations

$$y_t + f_1(y_x) = 0 \quad \text{for } t \in (0,T)\ ,\ -\infty < x < s(t) \quad , \tag{2.30}$$

$$y_t + f_2(y_x) = 0 \quad \text{for} \quad t \in (0,T)\ ,\ s(t) < x < \infty \tag{2.31}$$

are satisfied in the viscosity sense in their respective regions.

iii) y satisfies the initial condition

$$y(x,0) = h(x) \quad \text{for} \quad -\infty < x < \infty \quad . \tag{2.32}$$

iv) On the free boundary Γ,

$$\tau + f_1(p) \le 0 \quad \text{for} \quad (p,\tau) \in D^+_{\Gamma\_} y(s(t),t) \tag{2.33}$$

where $D^+_{\Gamma\_} y(s(t_0),t_0)$ consists of all points (p,τ) such that

$$\limsup_{(x,t)\to(x_0,t_0),x\le s(t)} \frac{y(x,t) - y(x_0,t_0) - p(x-x_0) - \tau(t-t_0)}{\sqrt{(x-x_0)^2 + (t-t_0)}} \le 0 \quad , \tag{2.34}$$

here $x_0 = s(t_0)$.

Note that if s is C^1 and u is C^1 up to the left side of Γ: $x = s(t)$, then $D^+_{\Gamma\_} y(s(t_0),t_0)$ consists of points (p,τ) such that $-\infty < p \le p^-$ and $\tau = y_t^-(x_0,t_0) + s'(t_0)(y_x^-(x_0,t_0) - p)$, and (2.33) is then equivalent to the condition

$$s'(t) \le \frac{f(p^-) - f(p)}{p^- - p} \quad \text{for all} \quad p \in (-\infty, p^-) \tag{2.35}$$

where $p^- = \lim_{x\to s(t)-0} u_x(x,t)$ (it corresponds to the case $p^+ = -\infty$ in Lemma 2.1).

The condition (2.35) says that characteristics from the upper material region will intersect Γ, and hence all information about Γ is from the upper material region alone. This meets the requirement of the mathematical setting of the physical problem as indicated in Ross [8]. Using definition 2.2, we shall establish, later on, existence and uniqueness first in the upper material region and then in the lower material region.

§3 Upper material problem

To prove the existence of a viscosity solution for the upper material, we use a change of variables as suggested by Ross [8]. For simplicity we assume that

$$g(x) = x \quad . \tag{3.1}$$

Suppose that the solution $y(x,t)$ of (1.6)–(1.9) is strictly monotone decreasing in x for $-\infty < x \le s(t)$. Suppose also that $\lim_{x\to-\infty} y(x,t) = +\infty$. Then we can change variables, viewing x as a function of y and t; the function $x(y,t)$ is clearly well defined for all $s(t) \le y < \infty$ and by differentiating the relation $y(x(y,t),t) = y$ formally, we get

$$x_t(y,t) - x_y(y,t) f_1\left(\frac{1}{x_y(y,t)}\right) = 0 \quad \text{for } t > 0, s(t) < y < \infty \quad . \tag{3.2}$$

Consider the equation

$$x_t(y,t) + \Phi(x_y(y,t)) = 0 \quad \text{for } t > 0, -\infty < y < \infty \tag{3.3}$$

where

$$\Phi(q) = \begin{cases} -q f_1(1/q) & \text{for } q < 0 \\ 0 & \text{for } q \ge 0 \end{cases} \quad . \tag{3.4}$$

Note that for $y > s(t)$ equation (3.3) coincides with equation (3.2). The motivation for setting $\Phi(q) = 0$ for $q \ge 0$ comes from the fact no ions are incident on the surface of the upper material for $q \ge 0$ (as explained in [8], see the Figure 2).

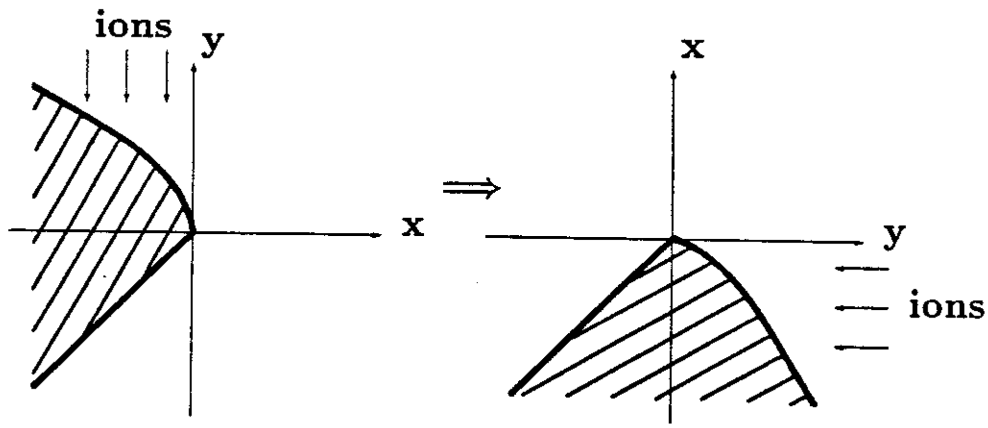

Figure 2

The upper material is initially occupying the space $\{(x,y) \mid x \le y \le h(x),\ -\infty < x \le 0\}$, and in the new coordinates it is given by $\{(y,x) \mid x \le u_0(y),\ -\infty < y < \infty\}$, where

$$u_0(y) = \begin{cases} y & \text{for } y \le 0 \\ h^{-1}(y) & \text{for } y > 0 \end{cases} \quad . \tag{3.5}$$

Thus a reasonable initial condition for the probelm (3.3) is

$$x(y,0) = u_0(y) \quad \text{for } -\infty < y < \infty. \tag{3.6}$$

Let us use $u(y,t)$ instead of $x(y,t)$, and rewrite (3.3) (3.6) as follows:

$$u_t + \Phi(u_y) = 0 \qquad \text{for } t > 0, -\infty < y < \infty \tag{3.7}$$

$$u(y,0) = u_0(y) \quad \text{for } -\infty < y < \infty. \tag{3.8}$$

The existence and uniqueness of a viscosity solution of (3.7) (3.8) follows from [1] (in fact, the result in [1] is much more general). In order to change the variables back to our original problem, we have to prove the following: For each $t > 0$,

(i) there exists a unique maximum of $u(y,t)$ attained on $\{(y,t) \mid u(y,t) = y\}$ and

(ii) $u(y,t)$ is strictly decreasing when $y > s(t)$, where $s(t)$ is the maximum point of $u(\cdot,t)$.

The exact formulation of (i) and (ii) is stated in Lemmas 3.2 and 3.3 below. We need several lemmas.

We begin with a general function $\Phi \in C(R)$, such that

$$\Phi(q) = 0 \quad \text{for } q \geq 0, \quad \Phi(q) > 0 \quad \text{for } q < 0 \quad , \tag{3.9}$$

and such that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|\Phi'(q)| \leq C_\epsilon \quad \text{for } q \leq -\epsilon \quad . \tag{3.10}$$

Take any initial data $u_0 \in C(R)$ (as in Figure 2) such that

$$u_0(y) = y \quad \text{for } y \leq 0 \quad , \tag{3.11}$$

and such that

$$-\frac{1}{c} \leq u_0'(y) \leq -c \quad \text{for all } y > 0 \quad \text{and some } c > 0 \quad . \tag{3.12}$$

It is clear that the upper convex hull of Φ on the interval $[-c/2, 1]$ must contain a line segment connecting $(1,0)$ and $(-q_0, \Phi(-q_0))$, with $0 < q_0 \leq c/2$. Set

$$\begin{aligned} \tilde{\Phi}(q) &= \Phi(q) \quad \text{for } q \leq -q_0 \\ &= \frac{\Phi(-q_0)}{1+q_0}(1-q) \quad \text{for } -q_0 < q < 1 \quad . \\ &= 0 \quad \text{for } q \geq 1 \end{aligned} \tag{3.13}$$

Then

$$\|\tilde{\Phi}'\|_{L^\infty(R)} \leq C_* \quad (C_* = C_\epsilon \text{ for } \epsilon = \frac{1}{q_0}) \quad .$$

We shall solve the conservation law (with $\tilde{\Phi}$) by approximation as in [4]. Choose a sequence of piecewise linear continuous functions $\Phi_n(q)$ such that

$$\Phi_n(q) = \tilde{\Phi}(q) \quad \text{for } -q_0 \leq q < \infty \tag{3.14}$$

$$\Phi_n(q) \to \tilde{\Phi}(q) \text{ uniformly in any compact set as } n \to \infty \tag{3.15}$$

$$\|\Phi_n'\|_{L^\infty(R)} \leq 2C_* \quad . \tag{3.16}$$

Choose also a sequence of piecewise linear functions $u_0^{(n)}(y)$ such that

$$u_0^{(n)}(y) = u_0(y) \quad \text{for } y \leq 0 \tag{3.17}$$

$$u_0^{(n)}(y) \to u_0(y) \text{ uniformly in any compact set as } n \to \infty \tag{3.18}$$

$$-\frac{2}{c} \leq \frac{d}{dy}u_0^{(n)}(y) \leq -\frac{c}{2} \leq -q_0 \quad \text{for } y \geq 0 \quad . \tag{3.19}$$

For fixed n, consider the conservation law

$$q_t + (\Phi_n(q))_y \;=\; 0 \qquad \text{for } \; t > 0, -\infty < y < \infty \quad , \tag{3.20}$$

$$q(y,0) \;=\; \frac{d}{dy}u_0^{(n)}(y) \quad \text{for } -\infty < y < \infty \quad . \tag{3.21}$$

Denote the (unique) entropy solution in $(-\infty,\infty)\times[0,\infty)$ by q_n . Our aim is to prove that

$$q_n(y,t) = 1 \quad \text{for } \; y < s_n(t),\; t > 0 \tag{3.22}$$

$$-C \leq q_n(y,t) \leq -q_0 < 0 \quad \text{for } \; y > s_n(t), t > 0 \tag{3.23}$$

$$-\frac{1}{\alpha} \leq s_n'(t) \leq -\alpha < 0 \quad \text{for } \; t > 0 \quad , \tag{3.24}$$

where $y = s_n(t)$ is the curve along which $q_n(y,t)$ changes sign, the constants C, q_0, α are indepedent of n.

Due to the special form of the function Φ and the initial value, we can derive the estimates (3.22)–(3.24) by following the construction of the entropy solution given in Lemmas 3.1 and 3.2 of Dafermos [4].

Suppose that

$$(q_n^{(1)}, \Phi_n(q_n^{(1)})), \ldots, (q_n^{(s-1)}, \Phi_n(q_n^{(s-1)})), (q_n^{(s)}, \Phi_n(q_n^{(s)}))$$

is the set of all vertices of the graph of Φ_n, where

$$-\infty < q_n^{(1)} < q_n^{(2)} < \ldots < q_n^{(s-1)} < q_n^{(s)} < \infty \quad .$$

Then

$$q_n^{(s)} = 1,\; q_n^{(s-1)} \leq -q_0,\; \frac{\Phi_n(q_n^{(s)}) - \Phi_n(q_n^{(s-1)})}{q_n^{(s)} - q_n^{(s-1)}} = -\frac{\Phi(-q_0)}{1+q_0} \quad . \tag{3.25}$$

We also have

$$\frac{d}{dy}u_0^{(n)}(y) = \begin{cases} 1 & \text{for } -\infty < y \leq 0 \\ v_n^{(1)} & \text{for } 0 < y \leq y_1 \\ \vdots & \\ v_n^{(m)} & \text{for } y_{m-1} < y < \infty \end{cases} \tag{3.26}$$

where

$$-\frac{2}{c} \leq v_n^{(i)} \leq -q_0 < 0 \qquad (1 \leq i \leq m) \quad . \tag{3.27}$$

As in [4], set

$$J = \{q_n^{(1)}, ..., q_n^{(s)}\} \cup \{1, v_n^{(1)}, ..., v_n^{(m)}\} \quad . \tag{3.28}$$

From the construction in Lemmas 3.1 and 3.2 of [4], it is clear that the entropy solution q_n of (3.20) (3.21) satisfies

$$\begin{aligned} q_n(y,t) &= 1 \quad \text{for } y \le s_0 t, 0 \le t < \tau & (3.29)\\ q_n(y,t) &\le -q_0 \quad \text{for } y > s_0 t, 0 \le t < \tau & (3.30)\end{aligned}$$

where $\tau > 0$ is small enough so that no interaction of shocks will occur, and by (3.13) (3.14) (3.16) (3.25),

$$s_0 \in \left[-2C_*, -\frac{\Phi(-q_0)}{1+q_0}\right] \quad . \tag{3.31}$$

Define

$$u_n(y,t) = \begin{cases} y & \text{for } y \le s_0 t, 0 \le t < \tau \\ s_0 t + \int_{s_0 t}^{y} q_n(\xi,t) d\xi & \text{for } y > s_0 t, 0 \le t < \tau \end{cases} \quad . \tag{3.32}$$

Then in $\{(y,t) \mid 0 < t < \tau\}$, $(u_n(y,t))_y$ is piecewise constant and $u_n(y,t)$ is Lipschitz continuous and piecewise smooth. By virtue of (3.29) (3.30), $u_n(\cdot,t)$ attains its strict maximum at $s(t) = s_0 t$.

For $y < s_0 t$, u_n satisfies

$$(u_n)_t + \Phi_n((u_n)_y) = 0 + \Phi_n(1) = 0 \tag{3.33}$$

in the classical sense. Since the Rankine-Hugoniot condition is satisfied for q_n, we have

$$s_0 = \frac{\Phi_n(1) - \Phi_n(q_n^+(s_0 t, t))}{1 - q_n^+(s_0 t, t)} \quad \text{(notice that } q_n^+(s_0 t, t) \le q_n^{(s-1)}) \quad . \tag{3.34}$$

Thus, in the interior of the region where $(u_n)_y \equiv q_n^+(s_0 t, t)$, we get from (3.32) (3.34) and (3.20) that

$$\begin{aligned} (u_n)_t &= s_0 - s_0 q_n^+(s_0 t, t) \\ &= \Phi_n(1) - \Phi_n(q_n^+(s_0 t, t)) \\ &= -\Phi_n(q_n^+(s_0 t, t)) = -\Phi_n((u_n)_y) \quad . \end{aligned} \tag{3.35}$$

Because there are only finitely many constant states for $(u_n)_y$ (with values in J), we obtain by repeating the above calculation with a slight modification that the equation

$$(u_n)_t + \Phi_n((u_n)_y) = 0 \ \text{ for } 0 < t < \tau \tag{3.36}$$

is satisfied in the classical sense in the interior of each region where $(u_n)_y$ is constant.

Since q_n is an entropy solution, by Lemma 2.1, we deduce that u_n is a viscosity solution of (3.36). It is obvious that the initial condition for u_n is satisfied.

We are ready to prove

LEMMA 3.1. *Under the assumptions (3.9)–(3.19), there exists a (unique) Lipschitz continuous function* $u_n(y,t)$ *on* $(-\infty,\infty)\times[0,\infty)$ *with the properties:*

i) u_n *is a viscosity solution for the problem*

$$u_t + \Phi_n(u_y) = 0 \quad \textit{for } -\infty < y < \infty,\ 0 < t < \infty \tag{3.37}$$

$$u(y,0) = u_0^{(n)}(y) \quad \textit{for } -\infty < y < \infty \quad . \tag{3.38}$$

ii) There exists a Lipschitz continuous function $s_n(t)$ *such that*

$$s_n'(t) \in \left[-2C_*, -\frac{\Phi(-q_0)}{1+q_0}\right] \quad \textit{for a.e. } t \in [0,\infty) \tag{3.39}$$

and

$$u_n(y,t) \equiv y \quad \textit{for } y \le s_n(t), 0 \le t < \infty \tag{3.40}$$

$$-\frac{2}{c} \le \frac{\partial u_n}{\partial y}(y,t) \le -q_0 \quad \textit{for a.e. } y > s_n(t), 0 \le t < \infty \quad . \tag{3.41}$$

Proof: Suppose that $T \ge 0$ is the largest number such that there exists a (unique) u_n for $0 \le t < T$ with the properties:

i) u_n is Lipschitz continuous and satisfies (3.39)–(3.41) in the region $\{(y,t) \mid -\infty < y < \infty, 0 \le t < T\}$.

ii) u_n is a viscosity solution of (3.37) (3.38) in the region $\{(y,t) \mid -\infty < y < \infty, 0 \le t < T\}$.

iii) $q_n(y,t) = (u_n)_y(y,t)$ is an entropy solution with values in J in the region $\{(y,t) \mid -\infty < y < \infty, 0 \le t < T\}$, and

$$\operatorname*{Var}_{(-\infty,\infty)} q_n(\cdot,t) \le \operatorname*{Var}_{(-\infty,\infty)} q_n(\cdot,0) \quad . \tag{3.42}$$

From the previous argument we know that i)–iii) hold for $T=\tau$ if τ is sufficiently small. We claim: $T=\infty$. We shall assume $T<\infty$, and derive a contradiction. By the proof of Lemma 3.2 of [4], there exists a piecewise constant function $q_n(y,T)$ with values in J which has finite number of discontinuities such that

$$(u_n)_y(\cdot,t) \to q_n(\cdot,T) \quad \text{in } L^1_{loc}(-\infty,\infty) \text{ as } t \to T-0 \quad . \tag{3.43}$$

Since u_n is uniformly Lipschitz continuous on $(-\infty,\infty)\times[0,T)$, we can define $u_n(y,t)$ for $t=T$ so that it is Lipschitz continuous on $(-\infty,\infty)\times[0,T]$. From (3.43) we get

$$q_n(y,T) = \frac{d}{dy} u_n(y,T) \tag{3.44}$$

in the distribution sense. Since $u_n(y,T)$ is Lipschitz continuous and $q_n(y,T)$ is piecewise constant, we deduce that (3.44) is satisfied in the classical sense except for finitely many points.

Since $s_n(t)$ is uniformly Lipschitz continuous, we can define $s_n(T) = \lim_{t\to T-0} s_n(t)$ and we have

$$u_n(y,T) \equiv y \quad \text{for } y \le s_n(T) \tag{3.45}$$

$$-\frac{2}{c} \le (u_n)_y(y,T) \le -q_0 \quad \text{for a.e. } y > s_n(T) \quad . \tag{3.46}$$

Thus we can use $u_n(y,T)$ as the new initial data and obtain a solution on $(-\infty,\infty)\times[T,T')$ $(T' > T)$. Now we have a function $u_n(y,t)$ defined on a larger region $(-\infty,\infty)\times[0,T')$ and satisfying

i) u_n is Lipschitz continuous and satisfies (3.39)–(3.41) in the region $\{(y,t) \mid -\infty < y < \infty, 0 \le t < T'\}$.

ii) $(u_n)_y(y,t)$ is an entropy solution for $t \in [0,T)$ and $t \in [T,T')$. From the proof of Lemma 3.2 of [4] it is also an entropy solution for $t \in [0,T')$.

iii) u_n is a viscosity solution of (3.37) for $(x,t) \in (-\infty,\infty)\times(0,T)$ and $(x,t) \in (-\infty,\infty)\times(T,T')$ respetively. Thus, by Lemma 4.1 of [3], u_n is a viscosity solution for $(x,t) \in (-\infty,\infty)\times[0,T')$.

This is a contradiction to the maximality of T. □

LEMMA 3.2. *Suppose that (3.9)–(3.13) hold. Then there exists a Lipschitz continuous viscosity solution u of the problem*

$$u_t + \tilde{\Phi}(u_y) = 0 \quad \textit{for } -\infty < y < \infty,\ 0 < t < \infty \tag{3.47}$$

$$u(y,0) = u_0(y) \quad \textit{for } -\infty < y < \infty \quad . \tag{3.48}$$

such that

$$u(y,t) \equiv y \quad \textit{for } y \le s(t), 0 \le t < \infty \tag{3.49}$$

$$-\frac{2}{c} \le u_y(y,t) \le -q_0 \quad \textit{for a.e. } y > s(t), 0 < t < \infty \quad , \tag{3.50}$$

where $s(t)$ is Lipschitz continuous, $s(0) = 0$, and

$$s'(t) \in \left[-2C_*, -\frac{\Phi(-q_0)}{1+q_0}\right] \quad \textit{for a.e. } t \in [0,\infty) \tag{3.51}$$

Proof: Take sequences Φ_n and $u_0^{(n)}$ such that (3.14)–(3.19) hold. Denote by u_n the viscosity solution (obtained in Lemma 3.1) corresponding to Φ_n and $u_0^{(n)}$. We can then choose a subsequence of n's such that as $n \to \infty$

$$s_n(t) \to s(t) \text{ uniformly in any compact set of } [0,\infty) \tag{3.52}$$

$$u_n(y,t) \to u(y,t) \text{ uniformly in any compact set of } (-\infty,\infty)\times[0,\infty) \quad . \tag{3.53}$$

By [3, Theorem 1.4], u is a viscosity solution of (3.47) (3.48). Note that (3.50) is equivalent to

$$-\frac{2}{c}\sigma \le u(y+\sigma,t) - u(y,t) \le -q_0\sigma \quad \text{for } \sigma > 0, y > s(t), 0 < t < \infty \quad , \tag{3.54}$$

and (3.51) is equivalent to

$$-2C_*\sigma \le s(t+\sigma) - s(t) \le -\frac{\Phi(-q_0)}{1+q_0}\sigma \quad \text{for } \sigma > 0, 0 < t < \infty \quad , \tag{3.55}$$

from which the assertions (3.50) (3.51) follow. □

LEMMA 3.3. *The viscosity solution obtained in Lemma 3.2 is also a viscosity solution for the problem*

$$u_t + \Phi(u_y) = 0 \quad \textit{for } -\infty < y < \infty,\ 0 < t < \infty \tag{3.56}$$

$$u(y,0) = u_0(y) \quad \textit{for } -\infty < y < \infty \quad . \tag{3.57}$$

Proof: If $y_0 < s(t_0)$, then

$$p = 1 \quad \text{for any } (p,\tau) \in D^+u(y_0,t_0) \cup D^-u(y_0,t_0) \quad , \tag{3.58}$$

and hence $\Phi(p) = \tilde{\Phi}(p) = 0$.

If $y_0 > s(t_0)$, it follows from (3.50) that

$$-\frac{2}{c} \le p \le -q_0 \quad \text{for } (p,\tau) \in D^+u(y_0,t_0) \cup D^-u(y_0,t_0) \quad , \tag{3.59}$$

and therefore by (3.13) we have $\Phi(p) = \tilde{\Phi}(p)$.

Finally, if $y_0 = s(t_0)$, it follows from (3.49) (3.50) that $D^-u(y_0,t_0) = \emptyset$, whereas for $(p,\tau) \in D^+u(y_0,t_0)$, we have

$$\tau + \Phi(p) \le \tau + \tilde{\Phi}(p) \le 0 \quad , \tag{3.60}$$

since $\Phi \le \tilde{\Phi}$ everywhere. □

Lemma 3.3 provides a solution for the transformed problem. We now return to the original problem and prove:

THEOREM 3.4. *Suppose that*

$$f_1 \in W^{1,\infty}(R), \quad f_1(p) > 0 \quad \textit{for } -\infty < p < \infty \ , \quad \|f_1\|_{W^{1,\infty}(R)} < \infty \quad , \tag{3.61}$$

$$h \in W^{1,\infty}_{loc}, \quad h(0) = 0, \quad -\frac{1}{c_0} \le h'(x) \le -c_0 \quad \textit{for } x \in (-\infty, 0] \ \textit{for some } c_0 > 0 \quad , \tag{3.62}$$

and

$$g(x) = x \quad \textit{for } -\infty < x \le 0 \quad . \tag{3.63}$$

Then there exist Lipschitz continuous functions $s(t)$ $(t \ge 0)$ and $y(x,t)$ $(t \ge 0, x \le s(t))$ such that the equation (2.30) is satisfied in the viscosity sense, $y(x,0) = h(x)$ for $x \in (-\infty, 0]$ and (2.33) is satisfied; furthermore, there exist $\alpha, \sigma_0 > 0$ such that

$$-\frac{1}{\alpha} \le s'(t) \le -\alpha \quad \textit{for a.e. } t \in [0,\infty) \tag{3.64}$$

$$-\frac{1}{\sigma_0} \le y_x(x,t) \le -\sigma_0 \quad \textit{for a.e. } x < s(t), 0 < t < \infty \quad . \tag{3.65}$$

Proof: Define Φ by (3.4) and u_0 by (3.5). Then the assumptions of Lemmas 3.2 and 3.3 are satisfied. Hence there exists a viscosity solution $u(y,t)$ of (3.56) (3.57) satisfying (3.49)–(3.51).

In view of (3.50), the inverse function of $y \to u(y,t)$ exists in the region $y \geq s(t)$, for each fixed t. The inverse function $y = y(x,t)$ is defined for $-\infty < x \leq s(t)$ and

$$u(y(x,t),t) = x \quad \text{for } x \leq s(t), t > 0 \tag{3.66}$$

$$y(u(y,t),t) = y \quad \text{for } y \geq s(t), t > 0 \quad . \tag{3.67}$$

By virtue of (3.50), we get

$$-\frac{1}{q_0} \leq y_x(x,t) \leq -\frac{c}{2} \quad \text{for a.e. } x < s(t), t > 0 \quad , \tag{3.68}$$

and (3.64) (3.65) follow immediately from (3.51) and (3.68).

Since u is Lipschitz continuous, (3.56) is satisfied almost everywhere by [3, Theorem 1.2 (ii)]. Hence

$$-\max_{-2/c \leq q \leq 1} \Phi(q) \leq u_t(y,t) \leq 0 \quad \text{almost everywhere.} \tag{3.69}$$

From (3.50) (3.68) (3.69) it follows that $y(x,t)$ is also Lipschitz continuous in t. It is clear that the initial and boundary conditions are satisfied.

We next show that y is a viscosity solution for (2.30). Suppose that $\psi \in C^1$ and $y(x,t) - \psi(x,t)$ takes maximum M at (x_0,t_0), $x_0 < s(t_0)$. By (3.68) we get that $\psi_x(x_0,t_0) \leq -c/2 < 0$.

For any fixed value of t near t_0, the inverse function $y \to \psi^{-1}(y,t)$ exists and is C^1 in a neighborhood V of $(\psi(x_0,t_0),t_0)$.

Assume for simplicity that $M = 0$, i.e. $\psi(x_0,t_0) = y(x_0,t_0)$ and that $(\psi^{-1}(y,t),\ t) \in U$ for $(y,t) \in V$, where U is a neighborhood of (x_0,t_0) in which $\psi_x < 0$.

Since $y - \psi(u(y,t),t) = y(u(y,t),t) - \psi(u(y,t),t)$ takes maximum at $(y_0,t_0) \in V$, (here $y_0 = y(x_0,t_0) = \psi(x_0,t_0)$), we deduce that $\psi^{-1}(y,t) - \psi^{-1}(\psi(u(y,t),t),t) = \psi^{-1}(y,t) - u(y,t)$ takes the local minimum at (y_0,t_0), i.e., $u(y,t) - \psi^{-1}(y,t)$ takes the local maximum at (y_0,t_0), and this implies that

$$\frac{\partial \psi^{-1}}{\partial t}(y_0,t_0) + \Phi\left(\frac{\partial \psi^{-1}}{\partial y}(y_0,t_0)\right) \leq 0 \quad . \tag{3.70}$$

Noting that

$$\frac{\partial \psi^{-1}}{\partial y} = \frac{1}{\psi_x} < 0, \quad \frac{\partial \psi^{-1}}{\partial t} = -\frac{\psi_t}{\psi_x} \quad , \tag{3.71}$$

it follows that

$$\psi_t(x_0,t_0) - \psi_x(x_0,t_0)\Phi\left(\frac{1}{\psi_x(x_0,t_0)}\right) \leq 0 \tag{3.72}$$

i.e.,

$$\psi_t(x_0,t_0) + f_1(\psi_x(x_0,t_0)) \le 0 \quad . \tag{3.73}$$

This shows that u is a viscosity subsolution in the region $\{(x,t) \mid x < s(t), t > 0\}$. The proof that u is a viscosity supersolution in the same region is similar.

To prove (2.33), let $(p,\tau) \in D^+_{\Gamma\_}y(s(t_0),t_0)$. Using (3.68) and the definition of $D^+_{\Gamma\_}y(s(t_0),t_0)$, we get that $p \le -c/2$. Define

$$v(x,t) = \begin{cases} y(x,t) & \text{for } x \le s(t) \\ p(x-s(t)) + s(t) & \text{for } x > s(t) \end{cases} \quad . \tag{3.74}$$

Then v is Lipschitz continuous.

Letting $x = s(t)$ and $t \to t_0$ in the limit (2.34), we get

$$\limsup_{t\to t_0} \frac{s(t)-s(t_0)-p(s(t)-s(t_0))-\tau(t-t_0)}{\sqrt{(s(t)-s(t_0))^2+(t-t_0)^2}} \le 0 \quad ; \tag{3.75}$$

this is equivalent to

$$\limsup_{t\to t_0} \frac{s(t)-s(t_0)-p(s(t)-s(t_0))-\tau(t-t_0)}{|t-t_0|} \le 0 \tag{3.76}$$

since $s(t)$ is Lipschitz continuous. By (3.76), if $x > s(t)$ then

$$\begin{aligned} & v(x,t) - v(x_0,t_0) - p(x-x_0) - \tau(t-t_0) \qquad\qquad (x_0 = s(t_0)) \\ = \; & p(x-s(t)) + s(t) - s(t_0) - p(x-x_0) - \tau(t-t_0) \\ = \; & p(x_0-s(t)) + s(t) - s(t_0) - \tau(t-t_0) \\ = \; & s(t)-s(t_0)-p(s(t_0)-s(t))-\tau(t-t_0) \\ \le \; & o(|t-t_0|) \;\le\; o\left(\sqrt{(t-t_0)^2+(x-x_0)^2}\right) \quad . \end{aligned} \tag{3.77}$$

It now follows from (3.77), (2.34) and (3.74) that $(p,\tau) \in D^+v(x_0,t_0)$. By [3, Proposition 1.1], there exists a function $\psi \in C^1$ such that $v - \psi$ takes maximum at (x_0,t_0), and $(\psi_x(x_0,t_0), \psi_t(x_0,t_0)) = (p,\tau)$. Take ψ so that $\psi(x_0,t_0) = v(x_0,t_0) = s(t_0) \equiv y_0$. It then follows from the previous argument that

$$u(y,t) - \psi^{-1}(y,t) = v^{-1}(y,t) - \psi^{-1}(y,t) \le 0 \quad \text{for } (y,t) \in V, y \ge s(t) \quad , \tag{3.78}$$

where $v^{-1}(v(x,t),t) \equiv x$, $v(v^{-1}(y,t),t) \equiv y$. Since ψ^{-1} is decreasing in y,

$$u(y,t) - \psi^{-1}(y,t) = y - \psi^{-1}(y,t) \le s(t) - s(t) = 0 \quad \text{for } (y,t) \in V, y < s(t) \quad . \tag{3.79}$$

This shows that $u - \psi^{-1}$ takes local maximum at (y_0,t_0). Since $\psi_x(x_0,t_0) = p < 0$, it follows from the previous argument that

$$\tau + f_1(p) \le 0 \quad ,$$

which establishes (2.33). □

§4 Lower material problem

In §3, we obtained a viscosity solution for the upper material. At the same time we obtained a free boundary $x = s(t)$ which is Lipschitz continuous and, for some $\alpha > 0$,

$$-\frac{1}{\alpha} \le s'(t) \le -\alpha < 0 \quad \text{for } t > 0 \quad . \tag{4.1}$$

Now consider the lower material problem. Assume that

$$f_2 \in W^{1,\infty}(R), \qquad f_2(p) > 0 \quad \text{for } -\infty < p < \infty, \qquad \|f_2\|_{W^{1,\infty}(R)} \le M \quad , \tag{4.2}$$

and

$$h \in W^{1,\infty}_{loc}(R), \qquad h(0) = 0, \qquad -M \le h'(x) \le 1 - \sigma \quad \text{for } x \in [0, \infty) \tag{4.3}$$

where $0 < \sigma < 1$, $M > 0$ are constants.

It will be convenient to first assume that h, s are C^∞, and that, in addition to (4.1)-(4.3),

$$\|h\|_{W^{2,\infty}[0,\infty)} < \infty \quad . \tag{4.4}$$

For any small $\epsilon > 0$, we can choose $s_\epsilon \in C^\infty$ such that

$$s_\epsilon(0) = 0, \quad s_\epsilon(t) \to s(t) \text{ uniformly in } [0, T] \text{ as } \epsilon \to 0 \tag{4.5}$$

$$s'_\epsilon(0)(1 - h'(0)) + f_2(h'(0)) - \epsilon h''(0) = 0 \quad \text{(consistancy condition)} \tag{4.6}$$

$$-L \le s'_\epsilon(t) \le -l \quad \text{for } t \in [0, T] \quad , \tag{4.7}$$

where $0 < l < 1$, $L > 0$ depend only on α, σ, M and $f_2(h'(0))$ (notice that $s'_\epsilon(0) \approx -f_2(h'(0))/(1 - h'(0)) < 0$), l and L are independent of ϵ.

Consider the equations

$$\begin{aligned} y_t + f_2(y_x) - \epsilon y_{xx} &= 0 \quad \text{for } t > 0, s_\epsilon(t) < x < \infty & (4.8)\\ y(x, 0) &= h(x) \quad \text{for } x \ge 0 & (4.9)\\ y(s_\epsilon(t), t) &= s_\epsilon(t) \quad \text{for } t \ge 0 \quad . & (4.10) \end{aligned}$$

Under the assumptions (4.1)–(4.7), the parabolic equations (4.8)–(4.10) has a classical solution y_ϵ such that it is C^∞ in the interior of the domain and $(y_\epsilon)_t$, $(y_\epsilon)_{xx}$ are continuous up to the boundary; furthermore

$$\|(y_\epsilon)_t\|_{L^\infty(\Omega)} + \|(y_\epsilon)_x\|_{L^\infty(\Omega)} < \infty \quad ,$$

where $\Omega = \{(x, t) \in R \times (0, T] \mid x > s_\epsilon(t)\}$.

Set $\xi = x - s_\epsilon(t)$ and $w(\xi, t) = y_\epsilon(x, t) - s_\epsilon(t)$. Then w satisfies

$$\begin{aligned} L[w] &= 0 \quad \text{for } \xi > 0, t > 0 & (4.11)\\ w(\xi, 0) &= h(\xi) \quad \text{for } \xi \ge 0 & (4.12)\\ w(0, t) &= 0 \quad \text{for } t > 0 & (4.13) \end{aligned}$$

where

$$L[w] = w_t - s_\epsilon(t) w_\xi - \epsilon w_{\xi\xi} + s'_\epsilon(t) + f_2(w_\xi) \quad . \tag{4.14}$$

LEMMA 4.1. *Under the assumptions (4.1)–(4.7), we have*

$$-\frac{M}{l} \leq w_\xi(\xi,t) \leq 1 \quad \textit{for } \xi > 0, t > 0 \quad . \tag{4.15}$$

Proof: Clearly

$$L[\xi] = -s'_\epsilon(t) + s'_\epsilon(t) + f_2(1) > 0 \quad \text{for } \xi > 0, t > 0 \quad . \tag{4.16}$$

By (4.3),

$$h(\xi) < \xi \quad \text{for } \xi \geq 0 \quad . \tag{4.17}$$

Obviously also

$$w(0,t) = 0 = \xi \quad \text{for } \xi = 0, t \geq 0 \quad . \tag{4.18}$$

Thus by the comparison principle for parabolic equations in an unbounded domain (see Chapter 2 of [5]), we get

$$w(\xi,t) \leq \xi \quad \text{for } \xi \geq 0, t \geq 0 \quad . \tag{4.19}$$

Similarly

$$\begin{aligned} L\left[-\frac{M}{l}\xi\right] &= \frac{M}{l}s'_\epsilon(t) + s'_\epsilon(t) + f_2(\frac{M}{l}) \\ &\leq \left(\frac{M}{l}+1\right)(-l) - M < 0 \quad . \end{aligned} \tag{4.20}$$

Since $l < 1$, we get from (4.3) that

$$h(\xi) \geq -M\xi \geq -\frac{M}{l}\xi \quad \text{for } \xi \geq 0 \quad . \tag{4.21}$$

Using comparison again we deduce that

$$w(\xi,t) \geq -\frac{M}{l}\xi \quad \text{for } \xi \geq 0, t \geq 0 \quad . \tag{4.22}$$

Now it follows from (4.19) and (4.22) that

$$-\frac{M}{l} \leq w_\xi(0,t) \leq 1 \quad \text{for } t \geq 0 \quad . \tag{4.23}$$

This inequality clearly holds also on $t = 0$ by (4.3). Since w_ξ satisfies the parabolic equation

$$w_{\xi t} - s'_\epsilon(t)w_{\xi\xi} - \epsilon w_{\xi\xi\xi} + f'_2(w_\xi)w_{\xi\xi} = 0 \quad \text{for } \xi > 0, t > 0 \quad , \tag{4.24}$$

it follows by comparison that (4.15) follows. □

LEMMA 4.2. *Under the assumptions (4.1)–(4.7), we have*

$$\left|\frac{\partial y_\epsilon}{\partial t}(x,t)\right| < C \quad \textit{for } x > s_\epsilon(t), 0 < t < T \tag{4.25}$$

where C is independent of ϵ, but may depend on $\|h\|_{W^{2,\infty}[0,\infty)}$.

Proof: By Lemma 4.1,

$$-\frac{M}{l} \leq \frac{\partial y_\epsilon}{\partial x}(x,t) \leq 1 \quad \text{for } x > s_\epsilon(t), t > 0 \quad . \tag{4.26}$$

Diffrerntiating (4.8) with respect to t, we obtain

$$\left(\frac{\partial y_\epsilon}{\partial t}\right)_t - \epsilon\left(\frac{\partial y_\epsilon}{\partial t}\right)_{xx} + f_2'(\ (y_\epsilon)_x)\left(\frac{\partial y_\epsilon}{\partial t}\right)_x = 0 \quad \text{for } x > s_\epsilon(t), t > 0 \quad . \tag{4.27}$$

Since $(y_\epsilon)_{xx}$ and $(y_\epsilon)_t$ are continuous up to the boundary,

$$\frac{\partial y_\epsilon}{\partial t}(x,0) = \epsilon h''(x) - f_2(h'(x)) \quad \text{for } x > 0 \tag{4.28}$$

$$\frac{\partial y_\epsilon}{\partial t}(s_\epsilon(t),t) = s_\epsilon'(t) - (y_\epsilon)_x(s_\epsilon(t),t)s_\epsilon'(t) \quad \text{for } t > 0 \quad . \tag{4.29}$$

By assumptions (4.2) (4.4) (4.7) and by (4.26), it then follows that y_t is uniformly bounded on the parabolic boundary of the domain. Using $\pm C(t+1)$ as the comparison functions, we get (4.25) immediately. □

THEOREM 4.3. *Suppose that (4.1)–(4.3) hold. Then there exists a Lipschitz continuous viscosity solution $y(x,t)$ for the problem*

$$y_t + f_2(y_x) = 0 \qquad \textit{for } x > s(t), 0 < t < T \tag{4.30}$$

$$y(x,0) = h(x) \quad \textit{for } x \geq 0 \tag{4.31}$$

with the properties:

$$-\frac{M}{l} \leq y_x(x,t) \leq 1 \quad \textit{for a.e.} \quad x > s(t), t > 0 \quad , \tag{4.32}$$

$$-M \leq y_t(x,t) \leq -m < 0 \quad \textit{for a.e. } x > s(t), t > 0 \quad , \tag{4.33}$$

where

$$m = \min_{-M/l \leq p \leq 1} f_2(p) \quad . \tag{4.34}$$

Proof:

Step 1. Suppose in addition to the assumptions (4.1)–(4.3) that (4.4) holds.

Take $s_\epsilon \in C^\infty$ satisfying (4.5)–(4.7). By virtue of Lemmas 4.1 and 4.2, there exists a subsequence of ϵ's such that as $\epsilon \to 0$

$$y_\epsilon(x,t) \to y(x,t) \quad \text{uniformly in any compact set of } \{(x,t) \mid x > s(t), t \geq 0\}. \tag{4.35}$$

Using (4.5) (4.35), we deduce that y is a viscosity solution of (4.30) (4.31) by [3, Theorem 3.1].

Next, (4.32) follows from (4.26). Since y is Lipschitz, equation (4.30) is satisfied almost everywhere ([3, Theorem 1.2 (ii)]), and (4.33) follows.

Step 2. The general case.

Take $h_\delta \in C^\infty$ such that

$$\|h_\delta\|_{W^{2,\infty}[0,\infty)} < \infty \quad , \tag{4.36}$$

$$h_\delta \to h \quad \text{uniformly in any compact set.} \tag{4.37}$$

Then by Step 1, there exists Lipschitz continuous viscosity solutions of the problem (4.30) with the initial data $h_\delta(x)$.

Since the estimates in (4.32) (4.33) are independent of δ, the theorem follows by a compactness argument and [3, Theorem 1.4]. □

Combining Theorem 3.4 and Theorem 4.3, we obtain:

COROLLARY 4.4: *Under the assumptions of Theorems 3.4 and 4.3, there exists a viscosity soluiton (in the sense of Definition 2.2).* □

§5 Uniqueness

We denote by $\mathcal{E}_M$ the collection of function pairs (y, s) with the properties:

i) $y \in C(R \times [0,T]), s \in C([0,T])$, and $s(0) = 0$,

ii) (y, s) is a viscosity solution satisfying Definition 2.2,

iii) $y(x,t)$ is strictly decreasing in x for $x < s(t)$, y is Lipschitz continuous and

$$\|y_t\|_{L^\infty(R\times[0,T])} \leq M \ , \quad \|y_x\|_{L^\infty(R\times[0,T])} \leq M \tag{5.1}$$

and

iv) s is Lipschitz continuous and

$$\|s'\|_{L^\infty[0,T]} \leq M \quad . \tag{5.2}$$

Set

$$\mathcal{E}_\infty = \bigcup_{M>0} \mathcal{E}_M \quad .$$

We had already proved the existence of a viscosity solution in $\mathcal{E}_\infty$. Uniqueness for the viscosity solution is equivalent to the assertion that $\mathcal{E}_\infty$ consists of a single point.

We first prove uniqueness for the upper material problem.

LEMMA 5.1. *Assume that (3.61) (3.62) hold and that* $(y_j, s_j) \in \mathcal{E}_\infty$ *for* $j = 1$ *and* $j = 2$. *Then*

$$s_1(t) = s_2(t) \quad \textit{for}\, 0 \leq t \leq T \tag{5.3}$$

$$y_1(x,t) = y_2(x,t) \quad \textit{for}\, x \leq s_1(t), 0 \leq t \leq T \quad . \tag{5.4}$$

Proof: We may assume that $y_1(x,t)$ is the solution obtained in §3. Then (3.65) is valid for y_1. Take M such that

$$(y_j, s_j) \in \mathcal{E}_M \quad (j=1,2) \quad . \tag{5.5}$$

By the cone dependence ([7, Theorem 2.4]), we obtain

$$y_1(x,t) = y_2(x,t) \quad \text{for } x \leq -x^* - Lt, 0 \leq t \leq T \tag{5.6}$$

where $L = \|f_1'\|_{L^\infty(R)}$, and x^* is choosen such that

$$-x^* - Lt < s_j(t) \quad \text{for } 0 \leq t \leq T \ \ (j=1,2) \ . \tag{5.7}$$

By virtue of iii), for any fixed $t \in [0,T]$, the inverse function $y \to w_j(y,t)$ $(j=1,2)$ of $x \to y_j(x,t)$ $(j=1,2)$ for $x \leq s_j(t)$ exists. By (5.6) and iii) we know that $w_j(y,t)$ is well defined for $s_j(t) \leq y \leq \infty, 0 \leq t \leq T$, and

$$w_j(s_j(t),t) = s_j(t) \quad \text{for } 0 \leq t \leq T \quad . \tag{5.8}$$

Define

$$u_j(y,t) = \begin{cases} y & \text{for } y \leq s_j(t), 0 \leq t \leq T \\ w_j(y,t) & \text{for } y > s_j(t), 0 \leq t \leq T \end{cases} \quad . \tag{5.9}$$

Then u_j is continuous. Further, by §3, u_1 is uniformly continuous in all of $R \times [0,T]$. Using (5.6), we deduce that u_2 is also uniformly continuous in $R \times [0,T]$.

Next, we prove (similar to the proof of Lemma 3.4) that u_j is a viscosity solution for the transformed problem (3.56) (3.57).

For $y_0 > s_j(t_0)$, if $u_j(y,t) - \psi(y,t)$ takes strict maximum (minimum) at (y_0,t_0) for some $\psi \in C^1$, then $\psi_y(y_0,t_0) \leq 0$ since u_j is decreasing in y near (y_0,t_0). If $\psi_y(y_0,t_0) < 0$, then we can proceed, as in Lemma 3.4, to prove that

$$\psi_t(y_0,t_0) + \Phi(\psi_y(y_0,t_0)) \leq 0 \quad (\geq 0) \quad . \tag{5.10}$$

If $\psi_y(y_0,t_0) = 0$ we can work with $\psi(y,t) - \epsilon(y-y_0)$ instead of $\psi(y,t)$, where $\epsilon > 0$ is sufficiently small. Obviously, $u_j(y,t) - (\psi(y,t) - \epsilon(y-y_0))$ takes a local maximum (minimum) at (y_ϵ, t_ϵ), where $y_\epsilon < s_j(t_\epsilon)$ and $(y_\epsilon,t_\epsilon) \to (t_0,t_0)$ as $\epsilon \to 0^+$. Thus we can still obtain (5.10) by letting $\epsilon \to 0^+$.

For $y_0 < s_j(t_0)$, u_j satisfies the equation (3.56) in the classical sense near (y_0,t_0).

Finally for $y_0 = s_j(t_0)$, it is obvious that $D^- u_j(y_0,t_0) = \emptyset$, whereas for $(p,\tau) \in D^+ u_j(y_0,t_0)$, we have $\tau \leq 0$. Thus

$$\tau + \Phi(p) = \tau + 0 \leq 0 \tag{5.11}$$

provided $p \geq 0$. On the other hand if $p < 0$, we can use (2.33) and proceed as in Lemma 3.4 to prove that (5.11) is still valid.

Thus u_j is a viscosity solution of the problem (3.56) (3.57).

By (3.61), we get that the function Φ defined in (3.4) is uniformly continuous in R. Thus by comparison theorem ([1, Theorem 1]), we conclude that $u_1(y,t) \equiv u_2(y,t)$. It follows immediately that $y_1(x,t) \equiv y_2(x,t)$ for $x \leq s_j(t), 0 \leq t \leq T$ and $s_1(t) \equiv s_2(t)$ for $0 \leq t \leq T$. □

LEMMA 5.2. *Suppose that $y(x,t)$ is a viscosity solution for the problem*

$$y_t + f(y_x) = 0 \quad \text{for } x > s(t),\ 0 < t < T \tag{5.12}$$

where $s(t)$ is C^1. Then $w(\xi,t) = y(x,t)$ $(\xi = x - s(t))$ is a viscosity solution of the problem

$$w_t - s'(t)w_\xi + f(w_\xi) = 0 \quad \text{for } \xi > 0,\ 0 < t < T \quad . \tag{5.13}$$

Proof: Suppose that $w(\xi,t) - \psi(\xi,t)$ takes a local maximum (minimum) at (ξ_0,t_0), $\xi_0 > 0,\ 0 < t_0 < T$; then $u(x,t) - \psi(x - s(t),t)$ takes a local maximum (minimum) at (x_0,t_0), where $x_0 = \xi_0 + s(t_0)$. Hence we get

$$[\psi_t(\xi_0,t_0) + \psi_\xi(\xi_0,t_0)(-s'(t_0))] + f(\psi_\xi(\xi_0,t_0)) \leq 0 \quad (\geq 0) \quad . \quad \square \tag{5.14}$$

We now state the main result of this section.

THEOREM 5.3. *Assume that (3.61) (3.62) and (4.2) (4.3) hold. Then $\mathcal{E}_\infty$ consists of a single point, i.e., there exists a unique solution in $\mathcal{E}_\infty$.*

Proof: Suppose there exist two solutions (y_1,s_1) (y_2,s_2). By Lemma 5.1, we have

$$s_1(t) = s_2(t) \equiv s(t) \quad \text{for } 0 \leq t \leq T \tag{5.15}$$

and

$$y_1(x,t) = y_2(x,t) \quad \text{for } -\infty \leq x \leq s(t), 0 \leq t \leq T \quad . \tag{5.16}$$

Take $s_\delta \in C^1[0,T]$ such that

$$s(t) \leq s_\delta(t) \leq s(t) + \delta \quad \text{for } 0 \leq t \leq T \tag{5.17}$$

and set $w_j(\xi,t) = y_j(x,t)$, $(\xi = x - s_\delta(t))$. Then by Lemma 5.2, $w_j(\xi,t)$ satisfy the equation

$$w_t - s'_\delta(t)w_\xi + f_2(w_\xi) = 0 \quad \text{for } \xi > 0,\ 0 < t < T \tag{5.18}$$

in the viscosity sense.

By assumption, $w_1(\xi,0) = w_2(\xi,0)$ for $\xi \geq 0$. In view of (5.1) and (5.17), $|w_1(0,t) - w_2(0,t)| \leq 2M\delta$ for $0 \leq t \leq T$, where M is choosen as in Lemma 5.1.

Applying comparison theorem ([1, Theorem 1]) to functions w_1 and $w_2 + 2M\delta t$, we get $w_1(\xi,t) \leq w_2(\xi,t) + 2M\delta T$ for $0 < t < T$. Switching w_1 and w_2 we deduce that

$$|w_1(\xi,t) - w_2(\xi,t)| \leq 2M\delta T \quad \text{for } \xi \geq 0,\ 0 \leq t \leq T \quad , \tag{5.19}$$

which implies that

$$|y_1(x,t) - y_2(x,t)| \leq 2M\delta T \quad \text{for } x \geq s_\delta(t),\ 0 \leq t \leq T \quad . \tag{5.20}$$

Taking $\delta \to 0$, it follows that $y_1 \equiv y_2$. □

Remark: All the results of the paper (existence, uniqueness, regularity) are still valid if the assumption $g(x) = x$ is replaced by

$$0 < c \leq g'(x) \leq C \quad \text{for } -\infty < x \leq 0.$$

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References

[1] M.G.Crandall and P.L.Lions, On existence and uniqueness of solutions of Hamilton-Jacobi equations. *Nonlinear Anal., Vol* **10**, *No* 4, (1986), 353-370

[2] M.G.Crandall and P.L.Lions, Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.,* **277**(1983), 1-42.

[3] M.G.Crandall, L.C.Evans and P.L.Lions, Some properties of viscosity solutions of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.,* **282**(1984), 487-502.

[4] C.M.Dafermos, Polygonal approximations of solutions of the initial value problems for a conservation law, *J. Math. Anal. Appl.,* **38**(1972), 33-41.

[5] A.Friedman, "Partial differential equations of parabolic type," Prentice-Hall, Englewood Cliffs, N.J., 1964.

[6] A.Friedman, "Mathematics in industrial problems," Springer-Verlag, New York, 1988.

[7] H.Ishii, Uniqueness of unbounded solutions of Hamilton-Jacobi equaitons, *Indiana Univ. Math. J.* **33**(1984), 721-748.

[8] D.S.Ross, Evolution of material boundaries under ion bombardment, *J. Electrochemical Society, 10 South Main street, Pennigton, N. J.* to appear.

[9] J.Smoller, "Shock waves and reaction diffusion equations, " Springer-Verlag, New York, 1980.