SEMILINEAR EQUATIONS IN $\mathbb{R}^N$
WITHOUT CONDITIONS AT INFINITY

BY
HAIM BREZIS

IMA Preprint Series #48
December 1983

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA
514 Vincent Hall
206 Church Street SE.
Minneapolis, Minnesota 55455
<table>
<thead>
<tr>
<th>Preprint #</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Workshop Summaries from the September 1982 workshop on Statistical Mechanics, Dynamical Systems and Turbulence</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Raphael De la Llave</td>
<td>A Simple Proof of C. Siegel's Center Theorem</td>
</tr>
<tr>
<td>3</td>
<td>H. Simpson, S. Spector</td>
<td>On Copositive Matrices and Strong Ellipticity for Isotropic Elastic Materials</td>
</tr>
<tr>
<td>4</td>
<td>George R. Sell</td>
<td>Vector Fields in the Vicinity of a Compact Invariant Manifold</td>
</tr>
<tr>
<td>5</td>
<td>Milan Miklavcic</td>
<td>Non-linear Stability of Asymptotic Suction</td>
</tr>
<tr>
<td>6</td>
<td>Hans Weinberger</td>
<td>A Simple System with a Continuum of Stable Inhomogeneous Steady States</td>
</tr>
<tr>
<td>7</td>
<td>Bau-Sen Du</td>
<td>Period 3 Bifurcation for the Logistic Mapping</td>
</tr>
<tr>
<td>8</td>
<td>Hans Weinberger</td>
<td>Optimal Numerical Approximation of a Linear Operator</td>
</tr>
<tr>
<td>9</td>
<td>L.R. Angel, D.F. Evans,</td>
<td>Three Component Ionic Microemulsions</td>
</tr>
<tr>
<td></td>
<td>B. Ninham</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>D.F. Evans, D. Mitchell,</td>
<td>Surfactant Diffusion; New Results and Interpretations</td>
</tr>
<tr>
<td></td>
<td>S. Mukherjee, B. Ninham</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>Leif Arkeryd</td>
<td>A Remark about the Final Aperiodic Regime for Maps on the Interval</td>
</tr>
<tr>
<td>12</td>
<td>Luis Magalhaes</td>
<td>Manifolds of Global Solutions of Functional Differential Equations</td>
</tr>
<tr>
<td>13</td>
<td>Kenneth Meyer</td>
<td>Tori in Resonance</td>
</tr>
<tr>
<td>14</td>
<td>C. Eugene Wayne</td>
<td>Surface Models with Nonlocal Potentials: Upper Bounds</td>
</tr>
<tr>
<td>16</td>
<td>George R. Sell</td>
<td>Smooth Linearization Near a Fixed Point</td>
</tr>
<tr>
<td>17</td>
<td>David Wollkind</td>
<td>A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification</td>
</tr>
<tr>
<td>18</td>
<td>Pierre Collet</td>
<td>Local $C^m$ Conjugacy on the Julia Set for some Holomorphic Perturbations of $z + z^2$</td>
</tr>
<tr>
<td>19</td>
<td>Henry C. Simpson, Scott J. Spector</td>
<td>On the Modified Bessel Functions of the First Kind and BARRELLING for a Material in Finite Elasticity</td>
</tr>
<tr>
<td>20</td>
<td>George R. Sell</td>
<td>Linearization and Global Dynamics</td>
</tr>
<tr>
<td>21</td>
<td>P. Constantin, C. Foias</td>
<td>Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 2D Navier-Stokes Equations</td>
</tr>
<tr>
<td>22</td>
<td>Milan Miklavcic</td>
<td>Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator</td>
</tr>
<tr>
<td>23</td>
<td>P. Collet, H. Epstein, G. Gallavotti</td>
<td>Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties</td>
</tr>
<tr>
<td>24</td>
<td>J.E. Dunn, J. Serrin</td>
<td>On the Thermomechanics of Interstitial working</td>
</tr>
<tr>
<td>25</td>
<td>Scott J. Spector</td>
<td>On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension</td>
</tr>
<tr>
<td>26</td>
<td>W.A. Coppel</td>
<td>Maps on an Interval</td>
</tr>
<tr>
<td>27</td>
<td>James Kirkwood</td>
<td>Phase Transitions in the Ising Model with Traverse Field</td>
</tr>
<tr>
<td>28</td>
<td>Luis Magalhaes</td>
<td>The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different</td>
</tr>
<tr>
<td>29</td>
<td>Charles Tresser</td>
<td>Homoclinic Orbits for Flow in $\mathbb{R}^3$</td>
</tr>
<tr>
<td>30</td>
<td>Charles Tresser</td>
<td>About some Theorems by L.P. Sil'nikov</td>
</tr>
<tr>
<td>31</td>
<td>Michael Alzernmann</td>
<td>On the Renormalized Coupling Constant and the Susceptibility in $\phi_4^4$ Field Theory and the Ising Model in Four Dimensions</td>
</tr>
<tr>
<td>32</td>
<td>C. Eugene Wayne</td>
<td>The KAM Theory of Systems with Short Range Interactions I</td>
</tr>
</tbody>
</table>

(continued on back cover)
SEMILINEAR EQUATIONS IN $\mathbb{R}^N$
WITHOUT CONDITIONS AT INFINITY

Haim Brezis
Université Paris VI
1. Introduction

The purpose of this paper is to point out that some nonlinear elliptic (and parabolic) problems are well-posed in all of $\mathbb{R}^N$ without conditions at infinity. A typical example is the following:

**Theorem 1.** Let $1 < p < \infty$. For every $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ there exists a unique $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$-\Delta u + |u|^{p-1}u = f(x) \text{ in } \mathcal{D}'(\mathbb{R}^N).$$

Moreover, if $f \geq 0$ a.e. then $u \geq 0$ a.e.

**Remark 1.** It was previously known that for every $f \in L^1(\mathbb{R}^N)$ there exists a unique $u \in L^p(\mathbb{R}^N)$ satisfying (1) (see [3], Theorem 5.11). However, we emphasize that in Theorem 1 there is no limitation on the growth at infinity of the data $f$ and the solution $u$ is unique without prescribing its behavior at infinity.

2. Proof of Theorem 1

Existence

Let $B_R = \{x \in \mathbb{R}^N; |x| < R\}$. We start with some local estimate:

**Lemma 1.** Let $R < R'$ and assume $u \in L^p_{\text{loc}}(B_{R'})$ satisfies

$$-\Delta u + |u|^{p-1}u = f(x) \text{ in } \mathcal{D}'(B_{R'}).$$

with $f \in L^1(B_{R})$. Then

$$\int_{B_{R}} |u|^p \leq C \left(1 + \int_{B_{R'}} |f| \right)$$

where $C$ depends only on $p$, $R$ and $R'$. 

1
Remark 2. The conclusion of Lemma 1 is a rather unusual localization property. Indeed, let $\Omega$ and $\Omega'$ be bounded open sets in $\mathbb{R}^N$ such that $\overline{\Omega} \cap \overline{\Omega'} = \emptyset$ and let $u$ be the solution of (1). On the one hand the values of $f$ in $\Omega'$ "affect" the solution $u$ in $\Omega$: for example, if $f > 0$ in $\Omega'$ and $f \equiv 0$ outside $\Omega'$ it follows from the strong maximum principle that $u > 0$ in $\Omega$. On the other hand the values of $f$ in $\Omega'$ affect only "mildly" $u$ in $\Omega$: in view of (3) $u \mid_{\Omega}$ may be estimated independently of $f \mid_{\Omega'}$; even if $f \to \infty$ on $\Omega'$, $\int_{\Omega} |u|^p$ still remains bounded.

Proof of Lemma 1. We use a device introduced by P. Baras and M. Pierre [2]. By Kato's inequality (see [10]) and (2) we have

$$-\Delta |u| + |u|^p \leq |f| \text{ in } \mathcal{D}'(B_{R'})$$  \hspace{1cm} (4)

Let $\xi \in \mathcal{D}(B_{R'})$ be such that $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $B_{R}$. Multiplying (4) through by $\xi^\alpha$ where $\alpha$ is an integer, and integrating we find

$$\int |u|^p \xi^\alpha \leq \int |f| + C \int |u| \xi^{\alpha - 2} \leq \int |f| + C \int |u| \xi^{\alpha/p} ,$$  \hspace{1cm} (5)

provided $\alpha - 2 \geq \alpha/p$, i.e., $\alpha \geq 2p/(p-1)$ and we fix any such $\alpha$. The condition of Lemma 1 follows easily from (5).

Proof of Theorem 1 - Existence

Let

$$f_n(x) = \begin{cases} f(x) & \text{if } |x| < n \\ 0 & \text{if } |x| \geq n \end{cases}$$

Let $u_n \in L^p(\mathbb{R}^N)$ be the unique solution of

$$-\Delta u_n + |u_n|^{p-1} u_n = f_n \text{ in } \mathcal{D}'(\mathbb{R}^N)$$  \hspace{1cm} (6)

(see [3], Theorem 5.11).
We deduce from Lemma 1 that there is a constant $C$ such that

$$\|u_n\|_{L^p(B_R)} \leq C$$

where $C$ depends only on $p$, $R$, and $f$, and thus we also have

$$\|\Delta u_n\|_{L^1(B_R)} \leq C.$$ 

It follows that (for some subsequence still denoted by $u_n$) we have

$$u_n \rightarrow u \quad \text{in} \quad L^1_{loc}(\mathbb{R}^N)$$

$$u_n \rightarrow u \quad \text{a.e. on} \quad \mathbb{R}^N.$$ 

We claim that

$$|u_n|^{p-1}u_n \rightarrow u^{p-1}u \quad \text{in} \quad L^1_{loc}(\mathbb{R}^N).$$

It suffices to verify that $|u_n|^{p-1}u_n$ is a Cauchy sequence in $L^1(B_R)$ for any $R$. By Kato's inequality and (6) we have

$$-\Delta |u_n - u_m| + |u_n|^{p-1}u_n - |u_m|^{p-1}u_m \leq \|f_n - f_m\|.$$ 

Let $\xi \in \mathcal{D}(\mathbb{R}^N)$ be such that $0 \leq \xi \leq 1$ and $\xi = 1$ on $B_R$. We have

$$\int \|u_n|^{p-1}u_n - |u_m|^{p-1}u_m\| \xi \leq \int \|f_n - f_m\| \xi + \int \|u_n - u_m\| \Delta \xi$$

and the RHS tend to zero as $m, n \rightarrow \infty$. Passing to the limit in (6) we obtain (1).

**Uniqueness**

We shall need the following:

**Lemma 2.** Assume $u \in L^p_{loc}(\mathbb{R}^N)$ satisfies
-Δu + |u|^{p-1}u ≤ 0 \text{ in } \mathcal{O}'(\mathbb{R}^N). \tag{7}

Then \( u \leq 0 \) a.e. on \( \mathbb{R}^N \).

**Remark 3.** Lemma 2 is closely related to the results of J. Keller [13] and R. Osserman [14] (see also the earlier works quoted in these papers).

**Proof.** We use a comparison function of the same type as in Osserman [14] (see also C. Loewner and L. Nirenberg [12]). Set

\[
U(x) = \frac{CR^\alpha}{(R^2 - |x|^2)^\alpha} \text{ in } \mathcal{B}_R
\]

where \( R > 0 \), \( \alpha = 2/(p-1) \) and \( C^{p-1} = 2\alpha \text{Max}\{N, \alpha+1\} \). A direct computation shows that

\[
-\Delta U + U^p > 0 \text{ in } \mathcal{B}_R \tag{8}
\]

and thus

\[
-\Delta(u-U) + |u|^{p-1}u - U^p \leq 0 \text{ in } \mathcal{O}'(\mathcal{B}_R). \tag{9}
\]

Using a variant of Kato's inequality (see Lemma A.1 in the Appendix) we deduce from (9) that

\[
-\Delta(u-U)^+ + (|u|^{p-1}u - U^p) \text{sign}^+(u-U) \leq 0 \text{ in } \mathcal{O}'(\mathcal{B}_R) \tag{10}
\]

and therefore

\[
-\Delta(u-U)^+ \leq 0 \text{ in } \mathcal{O}'(\mathcal{B}_R). \tag{11}
\]

From Lemma A.1 and (7) we deduce that

\[
-\Delta u^+ + (u^+)^p \leq 0 \text{ in } \mathcal{O}'(\mathbb{R}^N)
\]

and therefore

\[
-\Delta u^+ \leq 0 \text{ in } \mathcal{O}'(\mathbb{R}^N),
\]

i.e., \( u^+ \) is subharmonic and in particular \( u^+ \in L^\infty_{\text{loc}}(\mathbb{R}^N) \). It follows that for some \( \delta > 0 \) we have
\( (u-U)^+ = 0 \quad \text{for} \quad R - \delta < |x| < R \quad (12) \)

(since \( U(x) \to +\infty \) as \( |x| \to R, \ x \in B_R \)). Combining (11) and (12) we obtain that
\( (u-U)^+ = 0 \) a.e. on \( B_R \), i.e.,

\[ u \leq U \quad \text{a.e. on} \quad B_R . \]

Keeping \( x \) fixed and letting \( R \to \infty \) we see that \( u \leq 0 \) a.e. on \( \mathbb{R}^N \).

**Proof of Theorem 1 - Uniqueness**

Let \( u_1 \) and \( u_2 \) be two solutions of (1) and let \( u = u_1 - u_2 \). By Kato's inequality we have

\[ -\Delta |u| + \left| |u_1|^{p-1} u_1 - |u_2|^{p-1} u_2 \right| \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N) . \quad (13) \]

On the other hand, there is a constant \( \delta > 0 \) —depending only on \( p \)— such that

\[ \left| |a|^{p-1} a - |b|^{p-1} b \right| \geq \delta |a-b|^p \quad \forall a, b \in \mathbb{R} . \quad (14) \]

From (13) and (14) we deduce that

\[ -\Delta |u| + \delta |u|^p \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N) . \]

Using Lemma 2 we conclude that \( u = 0 \).

3. **Miscellaneous Remarks and Generalizations**

A) **Monotone nonlinearities**

The proof of Theorem 1 extends easily to the case where \( |u|^{p-1} u \) is replaced by a more general function \( g(u) \). Assume \( g : \mathbb{R} \to \mathbb{R} \) is a \( C^1 \) function such that

\[ g'(u) \geq a |u|^{p-1} \quad \forall u \in \mathbb{R} , \]

for some constants \( a > 0 \) and \( 1 < p < \infty \) (for example, \( g(u) = \sinh u \), etc. ...).
Theorem 1'. For every \( f \in L^1_{\text{loc}}(\mathbb{R}^N) \) there exists a unique \( u \in L^p_{\text{loc}}(\mathbb{R}^N) \) with \( g(u) \in L^1_{\text{loc}}(\mathbb{R}^N) \) satisfying

\[
-\Delta u + g(u) = f(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{16}
\]

B) Nonmonotone g's

Let \( g(x, u) : \mathbb{R}^N \times \mathbb{R} \) be measurable in \( x \) and continuous in \( u \). We assume that:

\[
g(x, u) \text{sign } u \geq a |u|^p - \omega(x) \quad \text{for a.e. } x \in \mathbb{R}^N, \text{ for all } u \in \mathbb{R} \tag{17}
\]

where \( \omega \in L^1_{\text{loc}}(\mathbb{R}^N) \) and \( a > 0, \ 1 < p < \infty \) and also

\[
h_M(x) = \sup_{|u| \leq M} |g(x, u)| \in L^1_{\text{loc}}(\mathbb{R}^N) \quad \text{for all } M > 0. \tag{18}
\]

Theorem 2. There exists \( u \in L^1_{\text{loc}}(\mathbb{R}^N) \) such that \( g(\cdot, u) \in L^1_{\text{loc}}(\mathbb{R}^N) \) satisfying

\[
-\Delta u + g(x, u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{19}
\]

Sketch of the Proof

First we consider the case of a smooth bounded domain \( \Omega \subset \mathbb{R}^N \).

Claim: There exists \( u \in W^{1, 1}_0(\Omega) \) such that \( g(\cdot, u) \in L^1(\Omega) \), satisfying

\[
-\Delta u + g(x, u) = 0 \quad \text{on } \Omega. \tag{20}
\]

This type of result is closely related—but not quite contained in \([6]\). Here it suffices to assume (17) with \( a = 0 \).

For \( r \in \mathbb{R} \) and \( n \in \mathbb{N} \) we set

\[
\tau_{n, r} = \begin{cases} r & \text{if } |r| \leq n \\ n & \text{if } r > n \\ -n & \text{if } r < -n \end{cases}
\]
By the Schauder fixed point theorem there exists \( u_n \in W^{1,1}_0(\Omega) \) satisfying

\[
- \Delta u_n + g(x, \tau u_n) = 0 \quad \text{on} \quad \Omega .
\]  

(21)

Using the fact that \( -\int_\Omega \Delta u_n \text{sign } u_n \geq 0 \) we find

\[
\int_\Omega |g(x, \tau u_n)| \leq 2 \int_\Omega |\omega|. 
\]

Therefore

\[
\int_\Omega |\Delta u_n| \leq 2 \int_\Omega |\omega|. 
\]  

(22)

After extracting a subsequence we may assume that

\[
u_n \rightarrow u \quad \text{in} \quad W^{1,1}(\Omega)
\]

\[
u_n \rightarrow u \quad \text{a.e.}
\]

\[g(x, \tau u_n) \rightarrow g(x, u) \quad \text{a.e.}
\]

In order to show that \( g(x, \tau u_n) \rightarrow g(x, u) \) in \( L^1(\Omega) \) we use a new device introduced in [8] by Th. Gallouet and J. M. Morel (with an observation of L. Boccardo).

Set

\[
p_M(r) = \begin{cases} 
1 & \text{if } r > M \\
0 & \text{if } -M \leq r \leq M \\
-1 & \text{if } r < -M 
\end{cases}
\]

where \( r \in \mathbb{R} \) and \( M > 0 \). It is well known that
\[
- \int_{\Omega} \Delta u \cdot p_M(u) \geq 0 \quad \forall u \in W^{1,1}_0(\Omega), \; \Delta u \in L^1(\Omega).
\]

Therefore we have

\[
\int_{\Omega} g(x, \tau u_n) p_M(u_n) \leq 0,
\]

That is,

\[
\int_{|u_n| \geq M} g(x, \tau u_n) \text{sign}(u_n) \leq 0
\]

and hence

\[
\int_{|u_n| > M} |g(x, \tau u_n)| \leq 2 \int_{|u_n| > M} |\omega|.
\] (23)

From (22) we see that \( \|u_n\|_1^1 \leq C \) and thus \( \text{meas} \{ |u_n| > M \} \leq C \).

Given \( \epsilon > 0 \) we may fix \( M \) large enough so that \( 2 \int_{|u_n| > M} |\omega| < \epsilon \). Next, for any measurable \( A \subseteq \Omega \), we have

\[
\int_{|u_n| \leq M} |g(x, \tau u_n)| \leq \int_{|u_n| \leq M} |g(x, \tau u_n)| + \int_{|u_n| > M} |g(x, \tau u_n)|
\]

\[
\leq \int_A h_M(x) + \epsilon \leq 2\epsilon
\]

provided \( \text{meas} A < \delta \) and \( \delta \) is small enough. In other words, we have established that
\[ \forall \epsilon > 0 \; \exists \delta > 0 \; \text{ s.t. } \int_A |g(x, \tau_n u_n)| < 2\epsilon \text{ when } \text{meas}A < \delta. \]

We conclude that \( g(x, \tau_n u_n) \to g(x, u) \) in \( L^1(\Omega) \).

We turn now to problem (19). For each \( n \) let \( \Omega_n = \left\{ x \in \mathbb{R}^N; \; |x| < n \right\} \).

By the previous step there exists \( u_n \in W^{1,1}_0(\Omega_n) \) such that \( g(\cdot, u_n) \in L^1(\Omega_n) \) satisfying

\[ -\Delta u_n + g(x, u_n) = 0 \quad \text{on} \quad \Omega_n. \tag{24} \]

From Kato's inequality and (24) we obtain

\[ -\Delta |u_n| + g(x, u_n) \text{ sign } u_n \leq 0 \quad \text{in } \mathcal{D}'(\Omega_n). \]

And therefore we also have

\[ -\Delta |u_n| + a|u_n|^p \leq \omega \quad \text{in } \mathcal{D}'(\Omega_n) \tag{25} \]

\[ -\Delta |u_n| + |g(x, u_n)| \leq 2|\omega| \quad \text{in } \mathcal{D}'(\Omega_n). \tag{26} \]

Using the same device as in the proof of Theorem 1 we deduce from (25) that \( u_n \) is bounded in \( L^p_{\text{loc}}(\mathbb{R}^N) \).

It follows from (26) that \( g(\cdot, u_n) \) is bounded in \( L^1_{\text{loc}}(\mathbb{R}^N) \) and thus \( \Delta u_n \) is bounded in \( L^1_{\text{loc}}(\mathbb{R}^N) \).

Hence we may assume that
\[ u_n \to u \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^N) \]
\[ u_n \to u \quad \text{a.e. on} \quad \mathbb{R}^N \]
\[ g(x, u_n) \to g(x, u) \quad \text{a.e. on} \quad \mathbb{R}^N. \]

Finally we prove that \( g(x, u_n) \to g(x, u) \) in \( L^1_{\text{loc}}(\mathbb{R}^N) \). By a variant of Kato's inequality (see Lemma A.2) we have

\[-\Delta p_{M_n}(u) + g(x, u_n) p_{M_n}(u_n) \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega) \]

where

\[ p_M(t) = \int_0^t p_M(s) \text{d}s. \]

Therefore we have

\[ \int_{|u_n| > M} |g(x, u_n)| \xi \leq 2 \int_{|u_n| > M} |\omega| \xi + \int_{|u_n| > M} |u_n| \Delta \xi \quad \forall \xi \in \mathcal{D}'(\Omega). \]

It follows easily that \( g(x, u_n) \) is equi-integrable on bounded sets of \( \mathbb{R}^N \) and thus \( g(x, u_n) \to g(x, u) \) in \( L^1_{\text{loc}}(\mathbb{R}^N) \).

C. Measures or more general distributions as right hand side data

Let \( T \) be a distribution of the form \( T = f + \Delta \phi \) where \( f \in L^1_{\text{loc}}(\mathbb{R}^N) \) and \( \phi \in L^p_{\text{loc}}(\mathbb{R}^N) \). Then the problem

\[-\Delta u + |u|^{p-1} u = T \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N) \quad \text{(27)}\]

has a unique solution \( u \in L^p_{\text{loc}}(\mathbb{R}^N) \).

Indeed it suffices to consider the new unknown \( v = u + \phi \) and to apply the result of Section B to \( v \) (see also [8] for similar questions on bounded domains).
Suppose now that $T$ is a measure on $\mathbb{R}^N$ (not necessarily a bounded measure). Suppose $1 < p < N/(N-2)$ (no restriction when $N = 1, 2$). Then there exists a unique solution $u \in L^p_{\text{loc}}(\mathbb{R}^N)$ for (27). Related questions for bounded measures are considered in [5] and [8].

D. Nonlinearities with growth close to linear

Suppose $g: \mathbb{R} \to \mathbb{R}$ is continuous and $g(u)$ behaves like $|u|\log u|^k$ as $|u| \to \infty$ with $k > 2$. Then for every $f \in L^1(\mathbb{R}^N)$ there exists $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $g(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$-\Delta u + g(u) = f \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N).$$

As before, we use Kato's inequality to find

$$-\Delta |u| + g(u) \text{ sign } u \leq |f|.$$

We multiply (28) through by $\eta = e^{-1/\xi^\beta}$ where $\beta > 2/(k-2)$. Then we estimate

$$\int |u||\Delta \eta|$$

with the help of Young's inequality.

E. Unbounded domains

Let $\Omega \subset \mathbb{R}^N$ be any domain (bounded or unbounded) with smooth boundary. Using the same principles as in the proof of Theorem 1 one can show that for every $f \in L^1_{\text{loc}}(\overline{\Omega})$ and $\phi \in L^1_{\text{loc}}(\partial \Omega)$ there exists a unique $u \in L^p_{\text{loc}}(\overline{\Omega})$ satisfying

$$\begin{cases}
-\Delta u + |u|^{p-1} u = f & \text{in } \Omega \\
u = \phi & \text{on } \partial \Omega,
\end{cases}$$

where $1 < p < \infty$ and the boundary condition is understood in some appropriate sense.

F. Local regularity

Let $\Omega \subset \mathbb{R}^N$ be any domain. Let $g: \mathbb{R} \to \mathbb{R}$ be a continuous and nondecreasing function.
Theorem 3. Suppose \( u \in L^1_{\text{loc}}(\Omega) \) is such that \( g(u) \in L^1_{\text{loc}}(\Omega) \) and satisfies

\[
-\Delta u + g(u) = f(x) \quad \text{in } \mathcal{D}'(\Omega)
\]

(29)

where \( f \in L^q_{\text{loc}}(\Omega) \) and \( 1 < q < \infty \).

Then \( u \in W^{2,q}_{\text{loc}}(\Omega) \).

Proof. We may assume that \( g(0) = 0 \). We have

\[
-\Delta |u| + g(u) \text{ sign } u \leq |f| \quad \text{in } \mathcal{D}'(\Omega)
\]

and thus

\[
-\Delta |u| \leq |f| \quad \text{in } \mathcal{D}'(\Omega).
\]

It follows that \( u \in L^q_{\text{loc}}(\Omega) \).

Set

\[
g_n(x) = g(\tau_n r) \quad \text{and} \quad P_n(r) = \text{sign } r \int_0^r |g_n(s)|^{q-1} \, ds
\]

so that

\[
|P_n(r)| \leq |r| |g_n(r)|^{q-1} \quad \forall r \in \mathbb{R}.
\]

By Lemma A.2 and (29) we have

\[
\Delta P_n(u) \geq |g_n(u)|^{q-1} \text{ sign } u (g(u) - f) \geq |g_n(u)|^{q-1} |f| |g_n(u)|^{q-1}.
\]

(30)

Let \( \xi \in \mathcal{D}(\Omega) \) with \( 0 \leq \xi \leq 1 \); from (30) we see that

\[
\int |g_n(u)|^q \xi^\alpha \leq C \int |P_n(u)| \xi^{\alpha-2} + \int |f||g_n(u)|^{q-1} \xi^\alpha
\]

\[
\leq C \int |u||g_n(u)|^{q-1} \xi^{\alpha-2} + \int |f||g_n(u)|^{q-1} \xi^\alpha
\]

where \( C \) is independent of \( u \).
Fix any integer $\alpha \geq 2q$; by Hölder’s inequality we have

$$\int g_n(u)|q|^q \xi^\alpha \leq C \int_{\text{supp } \xi} (|u|^q + |f|^q) .$$

As $n \to \infty$ we see that $g(u) \in L^q_{\text{loc}}(\Omega)$.

G. Parabolic equations

Consider the problem

$$\begin{cases}
  u_t - \Delta u + |u|^{p-1} u = 0 & \text{on } \mathbb{R}^N \times (0, +\infty) \text{ with } 1 < p < \infty \\
  u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N .
\end{cases} \tag{31}$$

Using the same principles as in the proof of Theorem 1 one can show that for every $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$ there is a unique function $u \in C^2(\mathbb{R}^N \times (0, +\infty)) \cap C(\{0, +\infty\}; L^1_{\text{loc}}(\mathbb{R}^N))$ satisfying (31).

Results of the same nature for the problem

$$\begin{cases}
  u_t - \Delta (|u|^{m-1} u) = 0 & \text{on } \mathbb{R}^N \times (0, \infty) \\
  u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N
\end{cases}$$

have been obtained by M. Herrero – M. Pierre \cite{9} when $0 < m < 1$. When $m > 1$ the situation is totally different; see \cite{1}, \cite{4}, \cite{7}.

Appendix: Some variants of Kato’s inequality

Let $\Omega \subset \mathbb{R}^N$ be any open set.

**Lemma A.1.** Let $u \in L^1_{\text{loc}}(\Omega)$ and $f \in L^1_{\text{loc}}(\Omega)$ be such that

$$\Delta u \geq f \text{ in } \mathcal{D}'(\Omega) .$$

Then

$$\Delta u^+ \geq f \text{ sign } u^+ \text{ in } \mathcal{D}'(\Omega) .$$
Lemma A.2. Let \( p: \mathbb{R} \rightarrow \mathbb{R} \) be a monotone, nondecreasing function such that \( p \) is continuous except at a finite number of jumps and \( p(\mathbb{R}) \) is bounded.

Let \( P(r) = \int_0^r p(s)ds \) and let \( u \in L^1_{\text{loc}}(\Omega) \) with \( \Delta u \in L^1_{\text{loc}}(\Omega) \). Then

\[
\Delta P(u) \geq (\Delta u)p(u) \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

The proofs are easy modifications of Kato's original argument in [10], and we shall omit them.

Acknowledgments: This paper was written during a visit at the University of Minnesota. I thank the Mathematics Department and the Institute for Mathematics and its Applications for their invitation and hospitality. I thank D. Aronson, C. Kenig and H. Weinberger for valuable conversations.
References


<table>
<thead>
<tr>
<th>Page</th>
<th>Author(s)</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>M. Slemrod, J. E. Marsden</td>
<td>Temporal and Spatial Chaos in a Van der Waals Fluid due to Periodic Thermal Fluctuations</td>
</tr>
<tr>
<td>34</td>
<td>J. Kirkwood, C. E. Wayne</td>
<td>Percolation in Continuous Systems</td>
</tr>
<tr>
<td>35</td>
<td>Luis Magalhaes</td>
<td>Invariant Manifolds for Functional Differential Equations Close to Ordinary Differential Equations</td>
</tr>
<tr>
<td>36</td>
<td>C. Eugene Wayne</td>
<td>The KAM Theory of Systems with Short Range Interactions II</td>
</tr>
<tr>
<td>37</td>
<td>Jean De Canniere</td>
<td>Passive Quasi-Free States of the Noninteracting Fermi Gas</td>
</tr>
<tr>
<td>38</td>
<td>Elias C. Aifantis</td>
<td>Maxwell and van der Waals Revisited</td>
</tr>
<tr>
<td>39</td>
<td>Elias C. Aifantis</td>
<td>On the Mechanics of Modulated Structures</td>
</tr>
<tr>
<td>40</td>
<td>William Ruckle</td>
<td>The Strong $\phi$ Topology on Symmetric Sequence Spaces</td>
</tr>
<tr>
<td>41</td>
<td>Charles R. Johnson</td>
<td>A Characterization of Borda's Rule Via Optimization</td>
</tr>
<tr>
<td>42</td>
<td>Hans Weinberger, Kazuo Kishimoto</td>
<td>The Spatial Homogeneity of Stable Equilibria of Some Reaction-Diffusion Systems on Convex Domains</td>
</tr>
<tr>
<td>43</td>
<td>K.A. Pericak-Spector, W.O. Williams</td>
<td>On Work and Constraints in Mixtures</td>
</tr>
<tr>
<td>44</td>
<td>H. Rosenberg, E. Toubiana</td>
<td>Some Remarks on Deformations of Minimal Surfaces</td>
</tr>
<tr>
<td>45</td>
<td>Stephan Pelikan</td>
<td>The Duration of Transients</td>
</tr>
<tr>
<td>46</td>
<td>V. Capasso, K.L. Cooke, M. Witten</td>
<td>Random Fluctuations of the Duration of Harvest</td>
</tr>
<tr>
<td>47</td>
<td>E. Fabes, D. Stroock</td>
<td>The $L^p$-intergrability of Green's functions and fundamental solutions for elliptic and parabolic equations</td>
</tr>
</tbody>
</table>