MAXWELL'S EQUATIONS IN A PERIODIC STRUCTURE

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MAXWELL'S EQUATIONS IN A PERIODIC STRUCTURE*

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Abstract. Consider a diffraction of a beam of particles in $\mathbf{R}^3$ when the dielectric coefficient is a constant ϵ_1 above a surface S and a constant ϵ_2 below a surface S, and the magnetic permeability is constant throughout $\mathbf{R}^3$. S is assumed to be periodic in the x_1 direction and of the form $x_1 = f_1(s), x_3 = f_3(s), x_2$ arbitrary. We prove that there exists a unique solution to the time–harmonic Maxwell equations in $\mathbf{R}^3$ having the form of refracted waves for $x_3 \gg 1$ and of transmitted waves for $-x_3 \gg 1$ if and only if there exist a unique solution to a certain system of two coupled Fredholm equations. Thus, in particular, for all the ϵ's, except for a discrete number, there exists a unique solution to the Maxwell equations.

Key words. Maxwell's equations, transmission, reflection, Fredholm equations

AMS(MOS) Mathematics subject classification 1980 (1985 Revised). Primary: 78A10, 78A45. Secondary: 35p25, 47A40

Introduction. In this paper we consider the Maxwell equations for time harmonic solutions in the entire space $\mathbf{R}^3$, with piecewise constant dielectric coefficient having jump across a periodic surface. The magnetic permeability μ is assumed to be constant whereas the dielectric coefficient ϵ is given by: $\epsilon = \epsilon_1$ above a surface $S : x_3 = f(x_1)$ and $\epsilon = \epsilon_2$ below the surface S; ϵ_1 and ϵ_2 are different constants. If S is a half space $\{x_3 = 0\}$ then the solution $\vec{E}_0, \vec{H}_0$ can be computed explicitly. We assume in this paper that S is periodic, i.e.,

$$f(x_1 + L) = f(x_1) \qquad \forall x_1 \in \mathbf{R} \qquad (L > 0).$$

We wish to find a solution $\vec{E}, \vec{H}$ such that

$$\begin{aligned} &\vec{E} - \vec{E}_0 \quad \text{and } \vec{H} - \vec{H}_0 \text{ are superpositions of "transmitted" waves} \\ &\text{in } \{x_3 < -A\} \text{ and of "reflected" waves in } \{x_3 > A\} \text{ where } A > \max|f|. \end{aligned} \tag{0.1}$$

In §§1–7 we assume that $f \in C^2$ and we reduce the solution of the Maxwell equations to a Fredholm system of four integral equations; in §8 we reduce it further to a Fredholm system of two integral equations. Thus for all but a discrete sequence of values of the physical parameters there exists a unique solution to the integral equations, yielding a solution of the Maxwell equations; the solution satisfies (0.1). In §9 we prove that any solution of the Maxwell equations which satisfies (0.1) is uniquely determined.

In §10 we generalize the previous results to the case where the curve $\widehat{S} \equiv S \cap \{x_2 = 0\}$ is not necessarily of the form $x_3 = f(x_1)$ with $f \in C^2$; in fact $\widehat{S}$ is assumed to be a piecewise

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C^2 curve, which is not necessarily an x_3-graph. In particular, the case where f is a step–function is included; this case arises in the design of digital lenses (oral communication from Dr. Allen Cox at Honeywell).

Integral equations have been used by Benaldi [2] to solve the Maxwell equations; Benaldi [2] uses finite–elements schemes for computing the solutions of the integral equation. The Maxwell equations in periodic structure corresponding to arrays of antennas were studied by Nedelec and Starling [8]; Bellout and Friedman [1] studied the Schrödinger equation for a periodic potential, corresponding to quantum scattering by a slab with periodically varying potential energy.

For the specific problem dealt with in this paper there is a numerical approach due to Gaylord and Moharam [5], which is based on approximating $f(x_1)$ by step–functions and using the "separation of variables" method for solving the approximating problems.

A good background on diffraction optics in grating material, especially from engineering and numerical points of view, can be found in a collection of articles edited by R. Petit [9].

§**1. The Maxwell equations.** We denote points in $\mathbf{R}^3$ by $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$. Let S be a surface in $\mathbf{R}^3$ given by

$$S : x_3 = f(x_1)$$

where $f(x_1)$ is periodic of period L:

$$f(x_1 + L) = f(x_1) \qquad \forall\, x_1 \ ;$$

we also assume that

$$f \in C^2 \ .$$

Introduce the domains

$$\begin{aligned} \Omega_1 &= \{X = (x_1, x_2, x_3); \quad x_1 > f(x_3)\}, \\ \Omega_2 &= \{X = (x_1, x_2, x_3); \quad x_1 < f(x_3)\} \end{aligned}$$

we assume that the magnetic permeability μ is constant throughout space whereas the dialectric coefficient ϵ satisfies:

$$\epsilon = \begin{cases} \epsilon_1 & \text{in} \quad \Omega_1 \\ \epsilon_2 & \text{in} \quad \Omega_2 \end{cases}$$

where ϵ_1, ϵ_2 are complex constants and $\epsilon_1 \neq \epsilon_2$. Writing for $j = 1, 2$

$$\epsilon = \epsilon' + i\epsilon'' \ , \ \epsilon_j = \epsilon'_j + i\, \epsilon''_j \ ,$$

we further assume that

$$\epsilon_j' > 0, \quad \epsilon_j'' \geq 0 \,;$$

the case $\epsilon_j'' > 0$ accounts for absorption; see, for instance, [4; §82].

Suppose a beam of particles is incident to the periodic surface S from above and it is time periodic of period $2\pi/\omega$. We wish to find the corresponding time harmonic solution

$$\vec{E}(\vec{r})e^{-i\omega t} \,,\ \vec{H}(\vec{r})e^{-i\omega t} \quad (\vec{r} = X = (x_1, x_2, x_3))$$

of the Maxwell equations and, in particular, to investigate the asymptotic behavior of the solution as $|x_3| \to \infty$. Setting

$$\vec{E}^j = \vec{E}\big|_{\Omega_j} \,,\ \ \vec{H}^j = \vec{H}\big|_{\Omega_j} \,,$$

Maxwell's equations in each Ω_j are:

$$\nabla \times \vec{E}^j - \frac{i\omega\mu}{c}\,\vec{H}^j = 0 \quad \text{in} \quad \Omega_j, \tag{1.1}$$

$$\nabla \times \vec{H}^j + \frac{i\omega\epsilon}{c}\,\vec{E}^j = 0 \quad \text{in} \quad \Omega_j, \tag{1.2}$$

where c is the speed of light. The weak form of Maxwell's equations in a neighborhood of S reduce to the following jump relations:

$$\vec{n} \times (\vec{E}^1 - \vec{E}^2) = 0 \qquad \text{on} \quad S, \tag{1.3}$$

$$\vec{n} \times (\vec{H}^1 - \vec{H}^2) = 0 \qquad \text{on} \quad S, \tag{1.4}$$

$$\vec{n} \cdot (\epsilon_1 \vec{E}^1 - \epsilon_2 \vec{E}^2) = 0 \qquad \text{on} \quad S, \tag{1.5}$$

$$\vec{n} \cdot (\vec{H}^1 - \vec{H}^2) = 0 \qquad \text{on} \quad S, \tag{1.6}$$

where $\vec{n}$ is the downward pointing unit normal to S.

The incident beam of particles coming from Ω_1 can be represented by the solution

$$\vec{H}_*(\vec{r}) = (0, h_0, 0), \quad \vec{E}_*(\vec{r}) = \frac{c}{i\omega\epsilon_1}\left(\frac{\partial h_0}{\partial x_3}, 0, -\frac{\partial h_0}{\partial x_1}\right), \qquad h_0 = e^{i(\alpha x_1 - \beta x_3)} \tag{1.7}$$

where

$$\alpha^2 + \beta^2 = \frac{\omega^2 \mu \epsilon_1}{c^2}, \ \alpha \text{ real}, \ Im\beta \geq 0. \tag{1.8}$$

In the special case

$$S = \{x_3 = 0\}$$

the corresponding solution of the Maxwell equations (1.1)–(1.6) is

(1.9)
$$\begin{aligned} &\vec{H}_0 = (0, h, 0),\ \vec{E}_0 = \frac{c}{i\omega\epsilon}\left(\frac{\partial h}{\partial x_3}, 0, -\frac{\partial h}{\partial x_1}\right),\\ &h = h(x_1, x_3) = \begin{cases} e^{i(\alpha x_1 - \beta x_3)} + r(\alpha,\beta)e^{i(\alpha x_1 + \beta x_3)} & \text{if } x_3 > 0 \\ t(\alpha,\beta)e^{i(\alpha x_1 - \widehat{\beta} x_3)} & \text{if } x_3 < 0 \end{cases} \end{aligned}$$

where

(1.10)
$$\widehat{\beta} = \{(\alpha^2+\beta^2)n_2^2 - \alpha^2\}^{1/2},\ \ Im\widehat{\beta} \geq 0,\ \ n_2 = \left(\frac{\epsilon_2}{\epsilon_1}\right)^{1/2} \qquad \text{(Snell's law)},$$

(1.11)
$$t(\alpha,\beta) = \frac{2\beta n_2^2}{\beta n_2^2 + \widehat{\beta}} \qquad \text{(transmission coefficient)},$$

(1.12)
$$r(\alpha,\beta) = \frac{\beta n_2^2 - \widehat{\beta}}{\beta n_2^2 + \widehat{\beta}} \qquad \text{(reflection coefficient)}.$$

Notice that $Im\beta > 0$ and $Im\widehat{\beta} > 0$ imply exponential decay away from S (due to absorption).

We wish to find a solution to (1.1)–(1.6) for general periodic surface S such that (0.1) holds; this last condition is precisely the condition (9.2) under which uniqueness is proved in §9.

We look for a solution satisfying:

(1.13)
$$e^{-i\alpha x_1}\vec{E}(\vec{r}) \text{ and } e^{-i\alpha x_1}\vec{H}(\vec{r}) \text{ are periodic in } x_1 \text{ of period } L, \text{ and are independent of } x_2.$$

In the sequel we use the notation

$$\vec{e}_1 = (1,0,0),\quad \vec{e}_2 = (0,1,0),\quad \vec{e}_3 = (0,0,1).$$

Note that along S,

(1.14)
$$\vec{n}(X) \equiv \vec{n}(x_1) = \frac{f'(x_1)\vec{e}_1 - \vec{e}_3}{\{1+f'(x_1)^2\}^{1/2}}.$$

Throughout this paper we write

(1.15)
$$\sigma(x_1) = \{1 + f'(x_1)^2\}^{1/2}\ .$$

We conclude this section by proving that equations (1.5), (1.6) follow from (1.1)–(1.4).

Lemma 1.1. *If $(\vec{E}, \vec{H})$ is a solution of (1.1)-(1.4) then $(\vec{E}, \vec{H})$ satisfies also the equations (1.5), (1.6).*

Proof. Take any bounded subdomain F of S with smooth boundary. By (1.1) (which is assumed to hold up to the boundary of Ω_j)

$$\int_F \vec{n} \cdot \frac{i\omega\mu}{c} \vec{H}^j dS = \int_F \vec{n} \cdot (\nabla \times \vec{E}^j)\, dS = \int_{\partial F} \vec{E}^j \cdot (\vec{n} \times \vec{n}_0)\, dl$$
$$= -\int_{\partial F} \vec{n}_0 \cdot (\vec{n} \times \vec{E}^j)\, dl$$

where $\vec{n}_0$ is the outward unit normal to ∂F. Hence

$$\int_F \frac{i\omega\mu}{c} \vec{n} \cdot (\vec{H}^1 - \vec{H}^2)\, dS = -\int_{\partial F} \vec{n}_0 \cdot \left(\vec{n} \times (\vec{E}^1 - \vec{E}^2)\right)\, dl = 0$$

by (1.3). Since F is arbitrary, $\vec{n} \cdot (\vec{H}^1 - \vec{H}^2) = 0$ on S. The proof of (1.5) follows similarly from (1.2), (1.4).

§2. Integral representation of solutions. It is easily seen that $\vec{E}$ and $\vec{H}$ both satisfy, outside S, the equation

$$\Delta v + k^2 v = 0, \quad \text{where } k = \frac{\omega}{c}\sqrt{\mu\epsilon}; \tag{2.1}$$

we shall always choose k such that

$$Re k > 0, \quad Im\, k \geq 0.$$

It is natural to expect an integral representation of $\vec{E}, \vec{H}$ by means of the fundamental solution Φ_k of (2.1):

$$\Phi_k(X - Y) = \frac{e^{ik|X-Y|}}{4\pi|X - Y|}\ ; \tag{2.2}$$

$k = \frac{\omega}{c}\sqrt{\mu\epsilon_j} \equiv k_j$ in Ω_j.

Motivated by [7] we shall try to find a solution of the form

$$\vec{E}(X) = \int_S \left\{ \frac{i\omega\mu}{c} \vec{I}(Y)\Phi_k(X - Y) - \vec{J}(Y) \times \nabla_Y \Phi_k(X - Y) \right.$$
$$\left. + \frac{c}{i\omega\epsilon} \nabla_X \left[\vec{I}(Y) \cdot \nabla_Y \Phi_k(X - Y) \right] \right\} dS_Y, \tag{2.3}$$

$$\vec{H}(X) = \int_S \Big\{ \frac{i\omega\epsilon}{c} \vec{J}(Y)\Phi_k(X-Y) + \vec{I}(Y) \times \nabla_Y \Phi_k(X-Y) \tag{2.4}$$
$$+ \frac{c}{i\omega\mu} \nabla_X \Big[\vec{J}(Y) \cdot \nabla_Y \Phi_k(X-Y) \Big] \Big\} \, dS_Y.$$

LEMMA 2.1. *Suppose $\vec{I}(Y)$ and $\vec{J}(Y)$ are bounded by $O(|Y|^{-1-\delta})$ as $|Y| \to \infty$, $\delta > 0$ (so that the integrals in (2.3), (2.4) and their derivatives are all well defined). Then*

$$\nabla \times \vec{E} - \frac{i\omega\mu}{c}\vec{H} = 0 \qquad \text{on} \quad \mathbb{R}^3 \setminus S, \tag{2.5}$$

$$\nabla \times \vec{H} + \frac{i\omega\epsilon}{c}\vec{E} = 0 \qquad \text{on} \quad \mathbb{R}^3 \setminus S. \tag{2.6}$$

Proof. Since $\nabla_X \Phi_k = -\nabla_Y \Phi_k$, we have

$$\vec{J} \times \nabla_Y \Phi_k = -\vec{J} \times \nabla_X \Phi_k = \nabla_X \times (\vec{J}\Phi_k).$$

It follows that

$$\vec{E} = \frac{i\omega\mu}{c} \int \vec{I}\Phi_k - \nabla_X \times \int \vec{J}\Phi_k + \nabla_X \int \frac{c}{i\omega\epsilon} \vec{I} \cdot \nabla_Y \Phi_k.$$

Using the relations $\nabla \times \nabla = 0$ and

$$\nabla \times (\nabla \times \vec{g}) = \nabla(\nabla \cdot \vec{g}) - \Delta \vec{g}$$

we deduce that

$$\begin{aligned}
\nabla \times \vec{E} &= \frac{i\omega\mu}{c} \nabla_X \times \int \vec{I}\Phi_k - \nabla_X \times (\nabla_X \times \int \vec{J}\Phi_k) + \nabla_X \times \Big[\nabla_X \int \frac{c}{i\omega\epsilon} \vec{I} \cdot \nabla_Y \Phi_k \Big] \\
&= \frac{i\omega\mu}{c} \int \vec{I} \times \nabla_Y \Phi_k - \nabla_X (\nabla_X \cdot \int \vec{J}\Phi_k) + \Delta_X \int \vec{J}\Phi_k \\
&= \frac{i\omega\mu}{c} \int \Big\{ \vec{I} \times \nabla_Y \Phi_k + \frac{c}{i\omega\mu} \nabla_X (\vec{J} \cdot \nabla_Y \Phi_k) + \frac{i\omega\epsilon}{c} \vec{J}\Phi_k \Big\} \\
&= \frac{i\omega\mu}{c} \vec{H},
\end{aligned}$$

i.e., (2.5) holds. The proof of (2.6) is similar.

In the sequel we shall be interested only in $\vec{I}$ and $\vec{J}$ which are independent of y_2 and are such that

$$e^{-i\alpha y_1} \vec{I}(y_1) \quad \text{and } e^{-i\alpha y_1} \vec{J}(y_1) \text{ are periodic in } y_1 \text{ of period } L. \tag{2.7}$$

We then must show that the integrals in (2.3), (2.4) make sense. This is done in the next section.

§3. Periodic fundamental solution. In the sequel we assume that

$$k_j^2 \neq \left(\frac{2\pi n}{L} - \alpha\right)^2 \qquad \text{for all } \ n = 0, \pm 1, \pm 2, \dots \ . \tag{3.1}$$

Notice that if $Im \epsilon_j > 0$ then (3.1) is certainly satisfied.

LEMMA 3.1. *The following formula holds:*

$$\int_{-\infty}^{\infty} \Phi_k(X - Y)\, dy_2 = \frac{i}{4} H_0^{(1)}(k|x - y|) \tag{3.2}$$

where $x = (x_1, x_3),\ y = (y_1, y_3)$ *and* $H_0^{(1)}$ *is a Hankel function.*

Proof. Substituting $\zeta = (a^2 + y^2)^{1/2} a^{-1/2}$, we get

$$\int_0^{\infty} \frac{e^{ik(a^2+y^2)^{1/2}}}{(a^2 + y^2)^{1/2}} dy = \int_1^{\infty} \frac{e^{ika\zeta}}{(\zeta^2 - 1)^{1/2}}\, d\zeta = \frac{\pi i}{2} H^{(1)}(ka);$$

in the last equality we used [6; p.322, 387 # 4] for $Imk \geq 0$, where $H_0^{(1)}$ is a Hankel function.

In the sequel we use the notation

$$x = (x_1, x_3), \quad y = (y_1, y_2).$$

and set

$$\Psi_k(x - y) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\alpha(x_1 - y_1 + nL)} \Phi_k(X - Y + nLe_1)\, dy_2. \tag{3.3}$$

For any complex number k in $\mathbf{C}^+ = \{Imk > 0\}$, set

$$\begin{aligned} \beta_n(k) &= \sqrt{|k^2 - \left(\frac{2\pi n}{L} - \alpha\right)^2|}\; e^{i\theta/2} \\ &\text{if } \ k^2 - \left(\frac{2\pi n}{L} - \alpha\right)^2 = |k^2 - \left(\frac{2\pi n}{L} - \alpha\right)^2|\; e^{i\theta}, 0 \leq \theta < 2\pi. \end{aligned} \tag{3.4}$$

Since $k^2 \neq \left(\frac{2\pi n}{L} - \alpha\right)^2$ for $k \in \mathbf{C}^+$, $\beta_n(k)$ is a complex analytic function in $\mathbf{C}^+$; furthermore, it is continuous up to $\{Imk = 0\}$ and

$$\beta_n(k) = \begin{cases} \sqrt{k^2 - \left(\frac{2\pi n}{L} - \alpha\right)^2} & \text{for } k \text{ real, } k^2 > \left(\frac{2\pi n}{L} - \alpha\right)^2 \\ i\sqrt{\left(\frac{2\pi n}{L} - \alpha\right)^2 - k^2} & \text{for } k \text{ real, } k^2 < \left(\frac{2\pi n}{L} - \alpha\right)^2. \end{cases} \tag{3.5}$$

THEOREM 3.2. *For any $k \in \mathbf{C}^+$ there holds:*

$$\Psi_k(x-y) = \frac{i}{2L} \sum_{n=-\infty}^{\infty} \frac{e^{i\beta_n(k)|x_3-y_3|} e^{-i\frac{2\pi n}{L}(x_1-y_1)}}{\beta_n(k)}. \tag{3.6}$$

Notice that for $k > 0$ the infinite series contains a finite number of oscillating terms, namely those with

$$\left(\frac{2\pi n}{L} - \alpha\right)^2 < k^2;$$

the remaining terms are exponentially decaying as $|x_3 - y_3| \to \infty$, and so is their sum. On the other hand, if $k \in \mathbf{C}^+$ then all the terms in (3.6) are exponentially decaying as $|x_3 - y_3| \to \infty$.

Proof. It suffices to establish (3.6) for $y = 0$. Set

$$f(s, y, x_3) = e^{-i\alpha s} \frac{e^{ik\sqrt{s^2+y^2+x_3^2}}}{\sqrt{s^2+y^2+x_3^2}},$$

$$F(t, y, x_3) = \int_{-\infty}^{\infty} e^{its} f(s, y, x_3)\, ds.$$

By the Poisson summation formula [3; p. 52]

$$\sum_{n=-\infty}^{\infty} f(x_1 + nL, y, x_3) = \frac{1}{L} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi n}{L}, y, x_3\right) e^{-i\frac{2\pi n x_1}{L}}.$$

Consequently

$$\Psi_k(x) = \sum_{n=-\infty}^{\infty} \frac{1}{4\pi L} \left(\int_{-\infty}^{\infty} F\left(\frac{2\pi n}{L}, y, x_3\right) dy \right) e^{-i\frac{2\pi n x_1}{L}}. \tag{3.7}$$

By Bellout–Friedman [1], each integral in the last sum is equal to

$$4 \int_0^\infty \int_0^\infty \cos\left[\left(\frac{2\pi n}{L} - \alpha\right) s\right] \frac{e^{ik\sqrt{s^2+y^2+x_3^2}}}{\sqrt{s^2+y^2+x_3^2}}\, ds\, dy$$

$$= 2\pi \int_0^\infty \frac{r e^{ik\sqrt{r^2+x_3^2}}}{\sqrt{r^2+x_3^2}} J_0\left(\left|\frac{2\pi n}{L} - \alpha\right| r\right) dr \equiv g_n(k).$$

Since

$$J_0(z) \sim \sqrt{\frac{2}{\pi z}}\, \sin z \qquad \text{as} \qquad z \to \infty$$

we easily deduce that the function $g_n(k)$ is complex analytic in k for $k \in \mathbf{C}^+$, continuous up to $Imk = 0$. It was proved by Bellout and Friedman [1] that the boundary values of $g_n(k)$ on $Imk = 0$ are given by

$$g_n(k) = 2\pi i \frac{e^{i\beta_n(k)|x_3|}}{\beta_n(k)} \qquad \text{for} \quad k > 0\ . \tag{3.8}$$

Since both functions

$$g_n(k) \quad \text{and} \quad 2\pi i \frac{e^{i\beta_n(k)|x_3|}}{\beta_n(k)}$$

are complex analytic in $\mathbf{C}^+$ with the same boundary values on $Imk = 0,\ k > 0$, it follows, by applying unique continuous to their difference, that they must coincide in $\mathbf{C}^+$. Thus (3.8) holds for all complex k $Imk \geq 0$, and using this in (3.7) the assertion (3.6) follows.

REMARK 3.1. Recall, by [6; p. 951, 8.405 #1] that

$$H_0^{(1)}(ka) = J_0(ka) + iN_0(ka) \tag{3.9}$$

and, as $z \to 0$,

$$\begin{aligned} &J_0(z) = 1 + O(z) \qquad ([6;\ \text{p. } 959,\ 8.440]),\\ &\pi N_0(z) = 2J_0(z)\log z + O(z) \qquad ([6;\ \text{p. } 960,\ 8.444]). \end{aligned}$$

It follows that

$$H_0^{(1)}(z) = \frac{2i}{\pi}\log z + O(1) \qquad \text{as} \quad z \to 0.$$

Setting

$$\Phi_{k,0}(z) = \int_{-\infty}^{\infty} \Phi_k(X)\ dx_2 \tag{3.10}$$

we conclude from Lemma 3.1 that

$$\Phi_{k,0}(x) = \frac{1}{2\pi}\log\frac{1}{|x|} + O(1) \qquad (|x| \to 0)\ . \tag{3.11}$$

REMARK 3.2. Set

$$\Phi_{k,n}(x) = \int_{-\infty}^{\infty} \Phi_k(x_1 + n, x_2, x_3)\ dx_2,$$

$$g_N = \sum_{\substack{n=-N\\ n\neq 0}}^{N} \Phi_{k,n}.$$

Then,

$$\Delta g_N + k^2 g_N = 0 \qquad \text{for all } x\ ,\ -\frac{L}{2} \le x_1 \le \frac{L}{2}\ ,$$

and g_N is uniformly convergent for $|x| = R$ $(\forall\ 0 < R < \frac{L}{2})$ since the same is true of $g_N + \Phi_{k,0}$ (by Theorem 3.2). Hence, by elliptic estimates, g_N is uniformly convergent also in $\{|x| < R\}$. It follows that

$$\Psi_k(x-y) - e^{-i\alpha(x_1-y_1)}\Phi_{k,0}(x-y)$$

is a smooth function also as $x - y \to 0$ and then, by Remark 3.1,

$$\Psi_k(x-y) - \frac{1}{2\pi}\log\frac{1}{|x-y|} \qquad \text{is smooth } \ \forall\ x, y. \tag{3.12}$$

The above argument is due to Nedelec and Starling [8].

REMARK 3.3. If we use Lemma 3.1 and apply the Poisson summation formula to $f(s) = e^{-\alpha s}H_0^{(1)}\left(k\sqrt{s^2+x_3^2}\right)$, and then compare the resulting expression for $\Psi_k(x)$ with that derived in Theorem 3.2, we obtain a formula for the Fourier transform of $H_0^{(1)}\left(k\sqrt{s^2+a^2}\right)$ for $Imk \ge 0$; this formula is known in case $Imk = 0$ (see [6; p. 736]).

§4. Integral representation for the periodic case.

Set

$$\vec{E}_\alpha(X) = e^{-i\alpha x_1}\vec{E}(X), \quad \vec{H}_\alpha(X) = e^{-i\alpha x_1}\vec{H}(X)\ . \tag{4.1}$$

We can assume that $\vec{I}$ and $\vec{J}$ depend only on y_1, and set

$$\vec{I}_\alpha(y_1) = e^{-i\alpha y_1}\vec{I}(y_1), \quad \vec{J}_\alpha(y_1) = e^{-i\alpha y_1}\vec{J}(y_1). \tag{4.2}$$

Multiplying both sides of (2.3) by $e^{-i\alpha x_1}$ and setting

$$\Phi_{k,\alpha}(X-Y) = e^{-i\alpha(x_1-y_1)}\Phi_k(X-Y) \tag{4.3}$$

we get, after some easy manipulations,

$$\begin{aligned}
\vec{E}_\alpha(X) = \int\limits_S \Big\{ \frac{i\omega\mu}{c}\vec{I}_\alpha\Phi_{k,\alpha}(X-Y) - \vec{J}_\alpha \times \nabla_Y\Phi_{k,\alpha}(X-Y) + \frac{c}{i\omega\epsilon}\nabla_X\left[\vec{I}_\alpha(y_1)\cdot\nabla_Y\Phi_{k,\alpha}(X-Y)\right]\Big\} \\
+ i\alpha\int\limits_S(\vec{J}_\alpha \times \vec{e}_1)\Phi_{k,\alpha}(X-Y) + \frac{ic\alpha}{i\omega\epsilon}\int\left[\vec{I}_\alpha\cdot\nabla_Y\Phi_{k,\alpha}(X-Y)\vec{e}_1 + \left(\vec{I}_\alpha\cdot\vec{e}_1\right)\nabla_Y\Phi_{k,\alpha}(X-Y)\right] \\
- \frac{c(i\alpha)^2}{i\omega\epsilon}\vec{e}_1\int_S\vec{I}_\alpha\cdot\vec{e}_1\Phi_{k,\alpha}(X-Y)\ .
\end{aligned} \tag{4.4}$$

If we integrate with respect to $y_2, -\infty < y_2 < \infty$, we obtain on the right–hand side of (4.4), the same expressions but with $\Phi_\alpha(X-Y)$ replaced by

$$\widetilde{\Phi}_{k,\alpha}(x-y) \equiv \int_{-\infty}^{\infty} \Phi_{k,\alpha}(x_1-y_1, x_2-y_2, x_3-y_3)\ dy_2.$$

Indeed, we first replace each $\nabla_Y \Phi_\alpha$ by $-\nabla_X \Phi_\alpha$, then perform the integration with respect to y_2, and finally replace back the $-\nabla_X \widetilde{\Phi}_\alpha$ by $\nabla_Y \widetilde{\Phi}_\alpha$. Assuming that

$$\vec{I}_\alpha(y_1),\ \vec{J}_\alpha(y_1) \quad \text{are} \quad L\text{-periodic in } y_1, \tag{4.5}$$

we can then rewrite (4.4) in the form
(4.6)

$$\begin{aligned}\vec{E}_\alpha(X) = \int_0^L \left\{ \frac{i\omega\mu}{c} \vec{I}_\alpha \Psi_k(x-\tilde{y}) - \vec{J}_\alpha \times \nabla_Y \Psi_k(x-\tilde{y}) + \frac{c}{i\omega\epsilon} \nabla_X \left[\vec{I}_\alpha(y_1) \cdot \nabla_Y \Psi_k(x-\tilde{y}) \right] \right\} \sigma \\ + i\alpha \int_0^L (\vec{J}_\alpha \times \vec{e}_1) \Psi_k(x-\tilde{y}) \sigma + \frac{c\alpha}{\omega\epsilon} \int_0^L \left[\vec{I}_\alpha \cdot \nabla_Y \Psi_k(x-\tilde{y}) \vec{e}_1 + \left(\vec{I}_\alpha \cdot \vec{e}_1 \right) \nabla_Y \Psi_k(x-\tilde{y}) \right] \sigma \\ - \frac{ic\alpha^2}{\omega\epsilon} \vec{e}_1 \int_0^L (\vec{I}_\alpha \cdot \vec{c}_1) \Psi_k(x-\tilde{y}) \sigma dy_1,\end{aligned}$$

where $\sigma = \sigma(y_1)$ is defined as in (1.16), and $\tilde{y} = (y_1, f(y_1))$.

Using the notation

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, 0, \frac{\partial}{\partial x_3} \right),\ \nabla_y = \left(\frac{\partial}{\partial y_1}, 0, \frac{\partial}{\partial y_3} \right),$$

omitting the index α in $\vec{I}_\alpha$, $\vec{J}_\alpha$, and noting that the right–hand side of (4.6) is independent of x_2, we can rewrite (4.6) in the form
(4.7)

$$\begin{aligned}\vec{E}_\alpha(x) = \int_0^L \left\{ \frac{i\omega\mu}{c} \vec{I} \Psi_k(x-\tilde{y}) - \vec{J} \times \nabla_y \Psi_k(x-\tilde{y}) + \frac{c}{i\omega\epsilon} \nabla_x \left[\vec{I}(y_1) \cdot \nabla_y \Psi_k(x-\tilde{y}) \right] \right\} \sigma(y_1)\ dy_1 \\ + i\alpha \int_0^L (\vec{J} \times \vec{e}_1) \Psi_k(x-\tilde{y}) \sigma(y_1)\ dy_1 + \frac{c\alpha}{\omega\epsilon} \int_0^L \left[\vec{I} \cdot \nabla_y \Psi_k(x-\tilde{y}) \vec{e}_1 + I_1 \nabla_y \Psi_k(x-\tilde{y}) \right] \sigma(y_1)\ dy_1 \\ - \frac{ic\alpha^2}{\omega\epsilon} \vec{c}_1 \int_0^L I_1 \Psi_k(x-\tilde{y}) \sigma(y_1)\ dy,\end{aligned}$$

where $\vec{I} = \vec{I}(y_1) = (I_1, I_2, I_3)$, $\vec{J} = \vec{J}(y) = (J_1, J_2, J_3)$; here we wrote $\vec{E}_\alpha(x) = \vec{E}_\alpha(X)$.

Similarly, from (2.4) we obtain

(4.8)

$$\vec{H}_\alpha(x) = \int_0^L \left\{ \frac{i\omega\epsilon}{c} \vec{J} \Psi_k(x-\tilde{y}) + \vec{I} \times \nabla_y \Psi_k(x-\tilde{y}) + \frac{c}{i\omega\mu} \nabla_x \left[\vec{J}(y_1) \cdot \nabla_y \Psi_k(x-\tilde{y}) \right] \right\} \sigma(y_1)\, dy_1$$

$$-i\alpha \int_0^L (\vec{I} \times \vec{e}_1) \Psi_k(x-\tilde{y}) \sigma(y_1)\, dy_1 + \frac{c\alpha}{\omega\mu} \int_0^L \left[\vec{J} \cdot \nabla_y \Psi_k(x-\tilde{y}) \vec{e}_1 + J_1 \nabla_y \Psi_k(x-\tilde{y}) \right] \sigma(y_1) dy_1$$

$$-\frac{ic\alpha^2}{\omega\mu} \vec{e}_1 \int_0^L J_1 \Psi_k(x-\tilde{y}) \sigma(y_1)\, dy_1 .$$

LEMMA 4.1. *For any functions $\vec{I}(y_1), \vec{J}(y_1)$ $(0 < y_1 < L)$ the functions $\vec{E}, \vec{H}$ defined by (4.1), (4.7) and (4.8) satisfy the Maxwell equations (2.5), (2.6); further*

$$\begin{aligned} \vec{E}_\alpha(x_1 + L, x_3) &= \vec{E}_\alpha(x_1, x_3), \\ \vec{H}_\alpha(x_1 + L, x_3) &= \vec{H}_\alpha(x_1, x_3). \end{aligned} \tag{4.9}$$

Proof. To prove (2.5), (2.6) we can proceed by appealing to Lemma 2.1 and rigorously establishing the passage from (2.3), (2.4) to (4.7), (4.8). Alternately (and more simply) we can establish (2.5), (2.6) directly, using the method of proof of Lemma 2.1. Finally, the validity of (4.9) is obvious.

We shall henceforth refer to S as either the surface $\{x_3 = f(x_1)\}$ in $\mathbf{R}^3$ or the curve $\{x_3 = f(x_1)\}$ in $\mathbf{R}^2$. We denote points on the curve S by

$$\tilde{x} = (x_1, f(x_1)), \qquad \tilde{y} = (y_1, f(y_1)).$$

LEMMA 4.2. *If $\vec{I}(y_1)$ is L-periodic in y_1 and if $\vec{I} \cdot \vec{n} = 0$ on S, then, for any $z \neq 0$,*

$$\int_0^L \vec{I}(y_1) \cdot \big(\nabla_y \Psi_k(\tilde{x}_z - \tilde{y})\big) \sigma(y_1)\, dy_1 = -\int_0^L \rho(\vec{I}) \Psi_k(\tilde{x}_z - \tilde{y})\, dy_1 \tag{4.10}$$

where $\tilde{x}_z = \big(x_1 + z, f(x_1)\big)$, and

$$\rho(\vec{I}) = \frac{d}{dy_1}\big(I_1(y_1)\sigma(y_1)\big)\ , \qquad \vec{I} = (I_1, I_2, I_3). \tag{4.11}$$

Proof. Since $\vec{I} \cdot \vec{n} = 0$,

$$I_3 = I_1 f'.$$

Hence

$$\int_0^L \sigma \vec{I} \cdot \nabla_y \Psi_k(\tilde{x}_z - \tilde{y}) = \int_0^L \sigma \vec{I} \cdot \left(\frac{\partial \Psi_k}{\partial y_1} \vec{e}_1 + \frac{\partial \Psi_k}{\partial y_3} \vec{e}_3 \right)$$

$$= \int_0^L \left(\sigma I_1 \frac{\partial \Psi_k}{\partial y_1} + \sigma I_3 \frac{\partial \Psi_k}{\partial y_3} \right) = \int_0^L \sigma I_1 \frac{d}{dy_1} \Psi_k = - \int_0^L \frac{d}{dy_1} (\sigma I_1)\ \Psi_k;$$

the boundary terms disappear since $\sigma I_1 \Psi_k$ is periodic in y_1. This completes the proof.

Using Lemma 4.2 we can rewrite the formulas (4.7), (4.8) in a more convenient way, by getting rid of some gradients ∇_y. The new representation is:
(4.12)

$$\vec{E}_\alpha(x) = \int_0^L \left\{ \frac{i\omega\mu}{c} \sigma \vec{I} \Psi_k(x - \tilde{y}) - \sigma \vec{J} \times \nabla_y \Psi_k(x - \tilde{y}) + \frac{c}{i\omega\epsilon} \rho(\vec{I}) \nabla_y \Psi_k(x - \tilde{y}) \right\}\ dy_1$$

$$+ i\alpha \int_0^L (\vec{J} \times \vec{e}_1) \Psi_k(x - \tilde{y}) \sigma(y_1) dy_1 - \frac{c\alpha}{\omega\epsilon} \int_0^L \rho(\vec{I}) \Psi_k(x - \tilde{y}) \vec{e}_1 dy_1$$

$$+ \frac{c\alpha}{\omega\epsilon} \int_0^L \sigma(y_1) I_1 \nabla_y \Psi_k(x - \tilde{y})\ dy_1 - \frac{ic\alpha^2}{\omega\epsilon}\ \vec{c}_1 \int_0^L \vec{I}_1 \Psi_k(x - \tilde{y}) \sigma(y_1)\ dy_1,$$

(4.13)

$$\vec{H}_\alpha(x) = \int_0^L \left\{ \frac{i\omega\epsilon}{c} \sigma \vec{J} \Psi_k(x - \tilde{y}) + \sigma \vec{I} \times \nabla_y \Psi_k(x - \tilde{y}) + \frac{c}{i\omega\mu} \rho(\vec{J}) \nabla_y \Psi_k(x - \tilde{y}) \right\}\ dy_1$$

$$- i\alpha \int_0^L (\vec{I} \times \vec{e}_1) \Psi_k(x - \tilde{y}) \sigma(y_1)\ dy_1 - \frac{c\alpha}{\omega\mu} \int_0^L \rho(\vec{J}) \Psi_k(x - \tilde{y}) e_1 dy_1$$

$$+ \frac{c\alpha}{\omega\mu} \int_0^L \sigma(y_1) J_1 \nabla_y \Psi_k(x - \tilde{y})\ dy_1 - \frac{ic\alpha^2}{\omega\mu} \vec{e}_1 \int_0^L \vec{J}_1 \Psi_k(x - \tilde{y}) \sigma(y_1)\ dy_1.$$

§5. Auxiliary estimates. Let

$$\Psi^*(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \log \frac{nL}{|x + nL\vec{e}_1|}. \tag{5.1}$$

The series is convergent in the sense of principal value, i.e.,

$$\Psi^*(x) = \frac{1}{2\pi} \lim_{N \to \infty} \sum_{n=-N}^{N} \log \frac{nL}{|x + nL\vec{e}_1|};$$

further

$$\nabla\Psi^*(x) = -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(x_1+nL, x_3)}{(x_1+nL)^2+x_3^2}$$

where the convergence is again in the sense of principal value, i.e.,

$$\nabla\Psi^*(x) = -\frac{1}{2\pi}\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{(x_1+nL, x_3)}{(x_1+nL)^2+x_3^2}\;; \tag{5.2}$$

the convergence is uniform in x in every compact set which does not contain integers.

Set

$$\Psi_0^*(x) = \frac{1}{2\pi}\log\frac{1}{|x|}. \tag{5.3}$$

Lemma 5.1. *For any $z \neq 0$, write $\tilde{x}_z = \tilde{x} + z\vec{e}_3$ and let*

$$\vec{R}_{k,z}(\tilde{x},\tilde{y}) \equiv \nabla_y\Psi_k(\tilde{x}_z-\tilde{y}) - \nabla_y\Psi^*\big(x_1-y_1, z+f'(x_1)(x_1-y_1)\big). \tag{5.4}$$

where $\nabla_y\Psi_k = -\nabla\Psi_k$. Then

$$|\vec{R}_{k,z}(\tilde{x},\tilde{y})| \leq C \tag{5.5}$$

where C is a constant independent of $\tilde{x},\tilde{y},z$.

Proof. By (3.9) and (5.2) it suffices to show that

$$|\nabla_y\Psi_0^*(\tilde{x}_z-\tilde{y}) - \nabla_y\Psi_0^*\big(x_1-y_1, z+f'(x_1)(x_1-y_1)\big)| \leq C\;,$$

i.e.,

$$K \equiv \left|\frac{\big(x_1-y_1, z+f(x_1)-f(y_1)\big)}{(x_1-y_1)^2+\big(z+f(x_1)-f(y_1)\big)^2} - \frac{\big(x_1-y_1, z+f'(x_1)(x_1-y_1)\big)}{(x_1-y_1)^2+\big(z+f'(x_1)(x_1-y_1)\big)^2}\right| \leq C.$$

The difference of the numerators is $(0, f(x_1)-f(y_1)-f'(x_1)(x_1-y_1))$, which is $O\big((x_1-y_1)^2\big)$. Hence

$$K \leq C_0 + \big(|x_1-y_1|+|z|\big)\Big\{2\big|z[f'(x_1)(x_1-y_1)-\big(f(x_1)-f(y_1)\big)]\big|$$
$$+\big|f'(x_1)^2(x_1-y_1)^2-\big(f(x_1)-f(y_1)\big)^2\big|\Big\}/N$$

where N is the product of the denominators, or

$$K \leq C_0 + C_1(|x_1-y_1|+|z|)[|z|(x_1-y_1)^2+|x_1-y_1|^3]/N.$$

If $|z| > (1+C_2)|x_1-y_1|$ where $C_2 = \max|f'|$, then $N \geq c_0|z|^4$ ($c_0 > 0$) and consequently $K \leq C$. On the other hand if $|z| < (1+C_2)|x_1-y_1|$ then

$$K \leq C_0 + C_3|x_1-y_1|^4/N \leq C_0 + C_4.$$

Thus K is bounded in both cases, independently of $z, \tilde{x}, \tilde{y}$.

Lemma 5.2. *For all $z \neq 0$,*

$$\int_0^L \nabla\Psi^*(x_1, z + ax_1)\, dx_1 = -\frac{1}{2}\, \frac{\vec{e}_3 - a\vec{e}_1}{1+a^2} \operatorname{sgn} z. \tag{5.6}$$

Proof. Notice that $\nabla\Psi^*$ is defined as in (5.2). Therefore, the left–hand side of (5.6) is equal to the integral

$$\int_{-\infty}^{\infty} \nabla\Psi_0^*(x_1, z + ax_1)\, dx_1 \tag{5.7}$$

taken in the p.v. (principal value) sense:

$$\lim_{N\to\infty} \int_{-N}^{N} \nabla\Psi_0^*(x_1, z + ax_1)\, dx_1.$$

To compute the integral in (5.7) we write it in the form

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{x_1\vec{e}_1 + (z+ax_1)\vec{e}_3}{x_1^2 + (z+ax_1)^2}\, dx_1 = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(x_1 + \frac{az}{1+a^2})(\vec{e}_1 + a\vec{e}_3)}{(1+a^2)(x_1 + \frac{az}{1+a^2})^2 + \frac{z^2}{1+a^2}}$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\frac{z}{1+a^2}(\vec{e}_3 - a\vec{e}_1)}{(1+a^2)\left(x_1 + \frac{az}{1+a^2}\right)^2 + \frac{z^2}{1+a^2}}\, dx_1\ .$$

The first integral on the right–hand side (taken in the p.v. sense) is equal to zero since the integrand is an odd function in $x_1 + az/(1+a^2)$. Since further

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\epsilon}{(1+a^2)(\xi_1^2 + \epsilon^2)} d\xi_1 = \frac{1}{2}\frac{\operatorname{sgn}\epsilon}{1+a^2},$$

the assertion of the lemma follows.

Lemma 5.3. *Let $u(x_1)$ be an L-periodic function, locally in L^2. Then*

$$\begin{aligned} &\lim_{\substack{z\to 0\\ z\neq 0}} \int_0^L (u(y_1) - u(x_1))\nabla_y\Psi^*(x_1 - y_1, z + a(x_1 - y_1))dy_1 \\ &= \int_0^L (u(y_1) - u(x_1))\nabla_y\Psi^*(x_1 - y_1, a(x_1 - y_1))\, dy_1 = \frac{1}{2}\frac{\vec{e}_1 + a\vec{e}_3}{1+a^2} H(u) \end{aligned} \tag{5.8}$$

where $\nabla_y \Psi^*$ *means* $-\operatorname{grad}\Psi^*$, *and*

$$
\begin{aligned}
Hu = (Hu)(x_1) &= p.v.\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{u(y_1)}{x_1 - y_1}\, dy_1 \\
&= \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \left\{ \frac{1}{\pi}\int\limits_{-N}^{x_1-\epsilon} \frac{u(y_1)}{x_1-y_1}\, dy_1 + \frac{1}{\pi}\int\limits_{x_1+\epsilon}^{N} \frac{u(y_1)}{x_1-y_1}\, dy_1 \right\}
\end{aligned} \tag{5.9}
$$

is the Hilbert transform.

Proof. From (5.2) we see that (5.8) is equivalent to

$$
\begin{aligned}
&\lim_{\substack{z\to 0\\ z\neq 0}} \int\limits_{-\infty}^{\infty} \big(u(y_1) - u(x_1)\big)\nabla_y \Psi_0^*(x_1 - y_1, z + a(x_1 - y_1))\, dy_1 \\
&= \int\limits_{-\infty}^{\infty} \big(u(y_1) - u(x_1)\big)\nabla_y \Psi_0^*(x_1 - y_1, a(x_1 - y_1))\, dy_1. \\
&= \frac{1}{2}\frac{\vec{e}_1 + a\vec{e}_3}{1+a^2} Hu;
\end{aligned} \tag{5.10}
$$

the integrals are taken in the p.v. sense, $\lim\limits_{N\to\infty} \int\limits_{-N}^{N}$.

Consider first the integral on the left–hand side of (5.10); it is equal to

$$
\begin{aligned}
&\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \big(u(y_1) - u(x_1)\big)\frac{(x_1 - y_1)(\vec{e}_1 + a\vec{e}_3)}{(1+a^2)(x_1 - y_1)^2}\, dy_1 \\
&= -\frac{\vec{e}_1 + a\vec{e}_3}{2(1+a^2)}\,\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{u(y_1) - u(x_1)}{|x_1 - y_1|}\operatorname{sgn}(x_1 - y_1)\, dy_1 \\
&= -\frac{\vec{e}_1 + a\vec{e}_3}{2(1+a^2)}\,\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{u(y_1)}{x_1 - y_1}\, dy_1 = \frac{\vec{e}_1 + a\vec{e}_3}{2(1+a^2)} H(u),
\end{aligned}
$$

since

$$
p.v.\int\limits_{-\infty}^{\infty} \frac{u(x_1)}{x_1 - y_1}\, dy_1 = 0.
$$

Thus the second equality in (5.10) follows. The first equality in (5.10) follows by similar considerations.

§6. The jump relations. Fix a point $\tilde{x}$. For any $z > 0$, the point $x = \tilde{x} + z\vec{e}_3$ belongs to Ω_1. We shall evaluate $\lim\limits_{z\to 0} \vec{E}_\alpha(\tilde{x} + ze_3)$, where $\vec{E}_\alpha$ is defined by (4.12).

Notice that all the integrals which do not involve $\nabla_y\Psi_k$ are well defined also at $z = 0$ and are continuous up to $z \geq 0$. Thus the sum of the terms in $\vec{E}_\alpha$ which may produce some discontinuity is

$$\begin{aligned} -\int_0^L \sigma(y_1)\vec{J}(y_1) \times \nabla_y\Psi_k(x-\tilde{y})\,dy_1 + \int_0^L \frac{c}{i\omega\epsilon}\rho(\vec{I})\nabla_y\Psi_k(x-\tilde{y})\,dy_1 \\ +\frac{c\alpha}{\omega\epsilon}\int_0^L \sigma(y_1)I_1\nabla_y\Psi_k(x-\tilde{y})\,dy_1. \end{aligned} \tag{6.1}$$

Write the first term in (6.1) in the form

$$\begin{aligned} &-\int_0^L \sigma(y_1)\vec{J}(y_1) \times \vec{R}_{k,z}(\tilde{x},\tilde{y})\,dy_1 \\ &-\int_0^L \Big[\sigma(y_1)\vec{J}(y_1) - \sigma(x_1)\vec{J}(x_1)\Big] \times \nabla_y\Psi^*(x_1-y_1, z + a(x_1-y_1))\,dy_1 \\ &-\sigma(x_1)\vec{J}(x_1) \times \int_0^L \nabla_y\Psi^*\big(x_1-y_1,\ z+a(x_1-y_1)\big)\,dy_1 \qquad \big(a = f'(x_1)\big). \end{aligned}$$

Then, as $z \downarrow 0$ we get, by Lemmas 5.1–5.3,

$$\begin{aligned} -\int_0^L \sigma(y_1)\vec{J}(y_1) \times \vec{R}_k(x_1,y_1)dy_1 - \frac{1}{2}H(\sigma\vec{J}) \times \frac{\vec{e}_1 + a\vec{e}_3}{1+a^2} \\ -\frac{1}{2}\,\sigma\vec{J} \times \frac{\vec{e}_3 - a\vec{e}_1}{1+a^2}, \end{aligned} \tag{6.2}$$

where

$$\vec{R}_k(x_1,y_1) = \nabla_y\Psi_k(\tilde{x}-\tilde{y}) - \nabla_y\Psi^*\big(x_1-y_1, f'(x_1)(x_1-y_1)\big). \tag{6.3}$$

Recalling (1.15) and introducing the unit tangent

$$\vec{\tau} = \frac{\vec{e}_1 + f'(x_1)\vec{e}_3}{\{1+f'(x_1)^2\}^{1/2}} \tag{6.4}$$

to S, we can write with (6.2) in the form

$$-\int_0^L \sigma(y_1)\vec{J}(y_1)\times\vec{R}_k(x_1,y_1)\,dy_1 - \frac{H(\sigma\vec{J})}{2\sigma}\times\vec{\tau} + \frac{1}{2}\vec{J}\times\vec{n}. \tag{6.5}$$

If $z<0, z\to 0$ then we obtain the same limit except that $\frac{1}{2}\vec{J}\times\vec{n}$ is replaced by $-\frac{1}{2}\vec{J}\times\vec{n}$.

Similarly, for the second term in (6.1) we get, as $z \downarrow 0$,

$$\frac{c}{i\omega\epsilon}\int_0^L \rho(\vec{I})\vec{R}_k(x_1,y_1)\,dy_1 + \frac{c}{i\omega\epsilon}\frac{H(\rho(\vec{I}))}{2\sigma}\vec{\tau} - \frac{1}{2}\,\frac{c}{i\omega\epsilon}\frac{\rho(\vec{I})}{\sigma}\vec{n} \tag{6.6}$$

and for the third term we get the limit

$$\frac{c\alpha}{\omega\epsilon}\int_0^L \sigma(y_1)I_1(y_1)\vec{R}_k(x_1,y_1)\,dy_1 + \frac{c\alpha}{\omega\epsilon}\frac{H(\sigma I_1)}{2\sigma}\vec{\tau} - \frac{1}{2}\frac{c\alpha}{\omega\epsilon}I_1\vec{n}. \tag{6.7}$$

If $z<0, z\to 0$ we get the same results, but the last term in (6.6) and in (6.7) have the reverse sign.

Setting $\Psi_j = \Psi_{k_j}, \vec{R}_j = \vec{R}_{k_j}$ and

$$\vec{E}^j_\alpha(\tilde{x}) = \lim_{\substack{x\to\tilde{x}\\ x\in\Omega_j}} \vec{E}_\alpha(X),$$

we conclude that

$$\begin{aligned}\vec{E}^j_\alpha(\tilde{x}) = &\int_0^L \left\{\frac{i\omega\mu}{c}\sigma\vec{I}\Psi_j - \sigma\vec{J}\times\vec{R}_j + \frac{c}{i\omega\epsilon_j}\rho(\vec{I})\vec{R}_j\right\}\\ &+i\alpha\int_0^L(\vec{J}\times\vec{e}_1)\sigma\Psi_j - \frac{c\alpha}{\omega\epsilon_j}\int_0^L\rho(\vec{I})\Psi_j\vec{e}_1 + \frac{c\alpha}{\omega\epsilon_j}\int_0^L\sigma I_1\vec{R}_j\\ &-\frac{ic\alpha^2}{\omega\epsilon_j}\,\vec{e}_1\int_0^L I_1\sigma\Psi_j - \frac{1}{2\sigma}H(\sigma\vec{J})\times\vec{\tau} + \frac{c}{i\omega\epsilon_j}\frac{H(\rho(\vec{I}))}{2\sigma}\vec{\tau}\\ &+\frac{c\alpha}{\omega\epsilon_j}\frac{H(\sigma I_1)}{2\sigma}\vec{\tau} \pm \frac{1}{2}\vec{J}\times\vec{n} \mp \frac{c}{i\omega\epsilon_j}\frac{\rho(\vec{I})}{2\sigma}\vec{n} \mp \frac{1}{2}\,\frac{c\alpha}{\omega\epsilon_j}I_1\vec{n}\end{aligned} \tag{6.8}$$

where "+" is for $j=1$ and "–" for $j=2$; here $\Psi_j = \Psi_j(\tilde{x}-\tilde{y})$ and $\vec{R}_j = \vec{R}_j(x_1,y_1)$.

Similarly, setting

$$\vec{H}^j_\alpha(\tilde{x}) = \lim_{\substack{x \to \tilde{x} \\ x \in \Omega_j}} \vec{H}_\alpha(x)$$

we get:

$$\begin{aligned} \vec{H}^j_\alpha(\tilde{x}) = & \int_0^L \left\{ \frac{i\omega\epsilon_j\sigma}{c} \vec{J}\Psi_j + \sigma\vec{I} \times \vec{R}_j + \frac{c}{i\omega\mu}\rho(\vec{J})\vec{R}_j \right\} \\ & -i\alpha \int_0^L (\vec{I} \times \vec{e}_1)\sigma\Psi_j - \frac{c\alpha}{\omega\mu}\int_0^L \rho(\vec{J})\Psi_j\vec{e}_1 + \frac{c\alpha}{\omega\mu}\int_0^L \sigma J_1\vec{R}_j \\ & -\frac{ic\alpha^2}{\omega\mu}\,\vec{e}_1 \int_0^L J_1\sigma\Psi_j + \frac{1}{2\sigma}H(\sigma\vec{I}) \times \vec{\tau} + \frac{c}{i\omega\mu}\frac{H(\rho(\vec{J}))}{2\sigma}\vec{\tau} \\ & +\frac{c\alpha}{\omega\mu}\,\frac{H(\sigma J_1)}{2\sigma}\vec{\tau} \mp \frac{1}{2}\,\vec{I} \times \vec{n} \mp \frac{c}{i\omega\mu}\,\frac{\rho(\vec{J})}{2\sigma}\vec{n} \mp \frac{1}{2}\,\frac{c\alpha}{\omega\mu}J_1\vec{n}. \end{aligned} \tag{6.9}$$

We summarize:

Lemma 6.1. *If $\vec{I} \cdot \vec{n} = 0$, $\vec{J} \cdot \vec{n} = 0$ along S, then the $\vec{E}_\alpha(x), \vec{H}_\alpha(x)$ defined by (4.7), (4.8) are uniformly continuous in each $\overline{\Omega}_j$ and their limits on $\partial\Omega_j$ are given by (6.8), (6.9) respectively.*

§7. Reduction to integral equations.

Definition 7.1. We denote by $(\vec{E}_f, \vec{H}_f)$ the vector field defined by (1.9)–(1.12) where $x_3 > 0$ and $x_3 < 0$, in (1.9), are replaced by $x_3 > f(x_1)$ and $x_3 < f(x_1)$ respectively.

We shall try to solve (1.1)–(1.4) in the form

$$(\vec{E}, \vec{H}) = (\vec{E}_f, \vec{H}_f) + (\vec{E}_\alpha, \vec{H}_\alpha)e^{i\alpha x_1} \tag{7.1}$$

where $(\vec{E}_\alpha, \vec{H}_\alpha)$ are defined by (4.7), (4.8) for some L-periodic surface fields $\vec{I}, \vec{J}$ (i.e., $\vec{I} \cdot \vec{n} = 0, \vec{J} \cdot \vec{n} = 0$). A solution of the form (7.1) will incorporate the condition (1.13), and as we shall see later on, the condition (0.1) will also be satisfied.

Set

$$\begin{aligned} T_1(\vec{I}, \vec{J}) = \int_0^L \Bigg\{ & \frac{i\omega\mu}{c}\sigma\vec{I}(\Psi_1 - \Psi_2) - \sigma\vec{J} \times (\vec{R}_1 - \vec{R}_2) + \frac{c}{i\omega}\rho(\vec{I})\left(\frac{\vec{R}_1}{\epsilon_1} - \frac{\vec{R}_2}{\epsilon_2}\right) \\ & + i\alpha(\vec{J} \times \vec{e}_1)\sigma(\Psi_1 - \Psi_2) - \frac{c\alpha}{\omega}\rho(\vec{I})\vec{e}_1\left(\frac{\Psi_1}{\epsilon_1} - \frac{\Psi_2}{\epsilon_2}\right) + \frac{c\alpha}{\omega}\sigma I_1\left(\frac{\vec{R}_1}{\epsilon_1} - \frac{\vec{R}_2}{\epsilon_2}\right) \\ & - \frac{ic\alpha^2}{\omega}I_1\vec{e}_1\sigma\left(\frac{\Psi_1}{\epsilon_1} - \frac{\Psi_2}{\epsilon_2}\right)\Bigg\}, \end{aligned} \tag{7.2}$$

$$
(7.3)\qquad
\begin{aligned}
T_2(\vec{I},\vec{J}) = \int_0^L \Big\{ & \frac{i\omega}{c}\sigma \vec{J}(\epsilon_1\Psi_1 - \epsilon_2\Psi_2) + \sigma\vec{I}\times(\vec{R}_1 - \vec{R}_2) \\
& + \frac{c}{i\omega\mu}\rho(\vec{J})(\vec{R}_1 - \vec{R}_2) - i\alpha(\vec{I}\times\vec{e}_1)\sigma(\Psi_1 - \Psi_2) - \frac{c\alpha}{\omega\mu}\rho(\vec{J})(\Psi_1 - \Psi_2)\vec{e}_1 \\
& + \frac{c\alpha}{\omega\mu}\sigma J_1(\vec{R}_1 - \vec{R}_2) - \frac{ic\alpha^2}{\omega\mu}\sigma J_1\vec{e}_1(\Psi_1 - \Psi_2)\Big\}.
\end{aligned}
$$

Lemma 7.1. *$(\vec{E},\vec{H})$ is a solution of (1.1)–(1.4) if and only if $\vec{I},\vec{J}$ satisfy the following conditions:*

$$
(7.4)\qquad \vec{n}\times(\vec{H}_0^1 - \vec{H}_0^2)e^{-i\alpha x_1} - \vec{I} + \vec{n}\times T_2(\vec{I},\vec{J}) = 0\ ,
$$

$$
(7.5)\qquad
\begin{aligned}
\vec{n}\times(\vec{E}_0^1 - \vec{E}_0^2)e^{-\alpha x_1} + \vec{J} + \frac{\epsilon_1-\epsilon_2}{\epsilon_1\epsilon_2}\frac{c\vec{e}_2}{2i\sigma\omega}\Big[H\big(\rho(\vec{I})\big) + i\alpha H(\sigma I_1)\Big] \\
+ \vec{n}\times T_1(\vec{I},\vec{J}) = 0.
\end{aligned}
$$

Here and in the sequel, $\vec{H}_0^j$ and $\vec{E}_0^j$ stand for $\vec{E}_f$ and $\vec{H}_f$ in Ω_j.

Proof. If $\vec{E},\vec{H}$ satisfy (1.3), (1.4) then (7.4), (7.5) follow using (6.8), (6.9) and noting that $\vec{n}\times\vec{\tau} = -\vec{e}_2, \vec{n}\times(\vec{J}\times\vec{n}) = \vec{J}, \vec{n}\times(\vec{I}\times\vec{n}) = \vec{I}$. Conversely, if $\vec{I},\vec{J}$ satisfy (7.4), (7.5) then clearly $\vec{I}\cdot\vec{n} = 0, \vec{J}\cdot\vec{n} = 0$ and therefore if $\vec{E}_\alpha,\vec{H}_\alpha$ are defined by (4.7), (4.8) then they satisfy (6.8), (6.9). It follows that (7.4), (7.5) imply (1.3), (1.4).

Definition 7.2. Introduce the space

(7.6)
$X = \{g(x_1);\ g(x_1)$ is L-periodic continuous function on $\mathbf{R}^1$ with $\|g\| = \max_{-\frac{L}{2}\le x\le\frac{L}{2}} |g(x_1)|\}$.

Consider the operator

$$
(7.7)\qquad T : g \to \int_0^L G(x_1,y_1)g(y_1)\,dy_1
$$

where $G(x_1,y_1)$ is continuous for all $(x_1,y_1), x_1\neq y_1$, and $|G(x_1,y_1)|\le C|x_1-y_1|^{-1+\epsilon}$ $(\epsilon > 0)$. Then T is a compact operator from X into X.

We need to look more carefully at T_1 and T_2. By integration by parts

$$
\int_0^L \rho(\vec{J})(\vec{R}_1 - \vec{R}_2) = -\int_0^L \sigma J_1\frac{d}{dy_1}(\vec{R}_1 - \vec{R}_2),
$$

$$
\int_0^L \rho(\vec{J})(\Psi_1 - \Psi_2)\vec{e}_1 = -\int_0^L \sigma J_1\vec{e}_1\frac{d}{dy_1}(\Psi_1 - \Psi_2).
$$

Using this in T_2 we find that

$$T_2 = \int_0^L \frac{i\omega}{c}\sigma(\epsilon_1 - \epsilon_2)\vec{J}\Psi_1 + \widehat{T}_2 \tag{7.8}$$

where

$$\begin{aligned} \widehat{T}_2(\vec{I}, \vec{J}) = \int_0^L \Big\{ & \frac{i\omega}{c}\sigma\epsilon_2\vec{J}(\Psi_1 - \Psi_2) + \sigma\vec{I} \times (\vec{R}_1 - \vec{R}_2) - \frac{c}{i\omega\mu}\sigma\ J_1\frac{d}{dy_1}(\vec{R}_1 - \vec{R}_2) \\ & - i\alpha(\vec{I} \times \vec{e}_1)\sigma(\Psi_1 - \Psi_2) + \frac{c\alpha}{\omega\mu}\sigma J_1\frac{d}{dy_1}(\Psi_1 - \Psi_2)\vec{e}_1 \\ & + \frac{c\alpha}{\omega\mu}\alpha J_1(\vec{R}_1 - \vec{R}_2) - \frac{ic\alpha^2}{\omega\mu}\sigma J_1\vec{e}_1(\Psi_1 - \Psi_2)\Big\} \\ \equiv & \int_0^L G_2(x_1, y_1)(\vec{I}, \vec{J}) \ ; \end{aligned} \tag{7.9}$$

G_2 is a 3×6 matrix and $(\vec{I}, \vec{J})$ is a column vector with components I_j, J_j. Since $\Psi_1 - \Psi_2$ and $\vec{R}_1 - \vec{R}_2$ are smooth functions,

$$|\nabla G_2| \le C \qquad \left(\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right)\right) \tag{7.10.}$$

In particular, $\widehat{T}_2$ is a compact operator from X into X (more precisely, from X^6 into X^3).

Similarly we can write

$$T_1 = \left(\frac{1}{\epsilon_1} - \frac{1}{\epsilon_2}\right)\frac{c}{i\omega}\int_0^L \rho(\vec{I})(\vec{R}_1 - i\alpha\Psi_1\vec{e}_1) + \widehat{T}_1 \tag{7.11}$$

where

$$\begin{aligned} \widehat{T}_1(\vec{I}, \vec{J}) = \int_0^L \Big\{ & \frac{i\omega\mu}{c}\sigma\vec{I}(\Psi_1 - \Psi_2) - \sigma\vec{J} \times (\vec{R}_1 - \vec{R}_2) - \frac{c}{i\omega\epsilon_2}\sigma I_1\frac{d}{dy_1}(\vec{R}_1 - \vec{R}_2) \\ & + i\alpha(\vec{J} \times \vec{e}_1)\sigma(\Psi_1 - \Psi_2) + \frac{c\alpha}{\omega\epsilon_2}\sigma I_1(\Psi_1 - \Psi_2)\vec{e}_1 \\ & + \frac{c\alpha}{\omega}\sigma I_1\left(\frac{\vec{R}_1}{\epsilon_1} - \frac{\vec{R}_2}{\epsilon_2}\right) - \frac{ic\alpha^2}{\omega}I_1\vec{c}_1\sigma\left(\frac{\Psi_1}{\epsilon_1} - \frac{\Psi_2}{\epsilon_2}\right)\Big\} \\ \equiv & \int_0^L G_1(x_1, y_1)(\vec{I}, \vec{J}) \end{aligned}$$

and G_1 is a 3×6 matrix,

$$|G_1(x_1, y_1)| \le C \log \frac{1}{|x_1 - y_1|} + C_1 \; ; \tag{7.13}$$

$G_1(x_1, y_1)$ is continuous in x_1, y_1 for all $x_1 \neq y_1$. It follows that $\widehat{T}_1$ is a compact operator from X into X.

If we substitute (7.8), (7.11) into (7.4), (7.5), we get

$$\begin{aligned} &\vec{n} \times (\vec{H}_0^1 - \vec{H}_0^2) e^{-i\alpha x_1} - \vec{I} \\ &\qquad + \frac{i\omega}{\epsilon}(\epsilon_1 - \epsilon_2) \vec{n} \times \int_0^L \sigma \vec{J} \Psi_1 + \vec{n} \times \widehat{T}_2(\vec{I}, \vec{J}) = 0, \end{aligned} \tag{7.14}$$

$$\begin{aligned} &n \times (\vec{E}_0^1 - \vec{E}_0^2) e^{-i\alpha x_1} + \vec{J} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 \epsilon_2} \, \frac{c \vec{e}_2}{2 i \sigma \omega} \left[H(\rho(\vec{I})) + i\alpha H(\sigma I_1) \right] \\ &\qquad + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 \epsilon_2} \frac{c}{i\omega} \vec{n} \times \int_0^L \rho(\vec{I})(\vec{R}_1 - i\alpha \Psi_1 \epsilon_1) + \vec{n} \times \widehat{T}_1 = 0. \end{aligned} \tag{7.15}$$

LEMMA 7.2. *If* $\vec{K} = \vec{n} \times \int_0^L \sigma \vec{J} \Psi_j$ *and* $\vec{J} \cdot \vec{n} = 0$ *then*

$$\rho(\vec{K}) = \frac{d}{dx_1}(K_1 \sigma) = -\frac{1}{2} \, H(\sigma J_2) - \int_0^L \sigma J_2 \left(R_j^1 + f'(x_1) R_j^3 \right) dy_1 \tag{7.16}$$

where $\vec{R}_j = (R_j^1, R_j^2, R_j^3)$.

Proof. Notice that

$$\vec{J} = J_1(\vec{e}_1 + f'(x_1)\vec{e}_3) + J_2 \vec{e}_2.$$

We have

$$K_1 = \vec{K} \cdot \vec{e}_1 = -(\vec{n} \times \vec{e}_1) \cdot \int_0^L \sigma \vec{J} \Psi_j = \frac{1}{\sigma} \int_0^L \sigma J_2 \Psi_j.$$

Let $x_z = \tilde{x} + z \vec{e}_3 = \big(x_1, z + f(x_1)\ \ (z \neq 0)$ and consider

$$K_1(z) = \frac{1}{\sigma(x_1)} \int_0^L \sigma(y_1) J_2(y_1) \Psi_j(x_z - \tilde{y}) \, dy_1 \; .$$

Then

$$\frac{\partial}{\partial x_1}(K_1(z)\sigma) = \int_0^L \sigma(y_1)J_2(y_1)\left(\frac{\partial\Psi_j}{\partial x_1} + f'(x_1)\frac{\partial\Psi_j}{\partial x_3}\right)(x_z - \tilde{y})\;dy_1$$

$$= \sigma(x_1)J_2(x_1)\int_0^L \left(\frac{\partial\Psi^*}{\partial x_1} + f'(x_1)\frac{\partial\Psi^*}{\partial x_3}\right)(x_1 - y_1, z + f'(x_1)(x_1 - y_1))\;dy_1$$

$$+ \int_0^L [\sigma(y_1)J_2(y_1) - \sigma(x_1)J_2(x_1)]\left(\frac{\partial\Psi^*}{\partial x_1} + f'(x_1)\frac{\partial\Psi^*}{\partial x_3}\right)(x_1 - y_1, z + f'(x_1)(x_1 - y_1))\;dy_1$$

$$- \int_0^L \sigma(y_1)J_2(y_1)\left[R^1_{k_j,z}(\tilde{x},\tilde{y}) + f'(x_1)R^3_{k_j,z}(\tilde{x},\tilde{y})\right]\;dy_1$$

where $R_{k_j,z}$ is defined as in (5.4).

Using calculations as in Lemma 5.2 we find that the first integral on the right–hand side is equal to zero. Similarly, by the calculations in Lemma 5.3, the second integral on the right–hand side is continuous in z (up to $z = 0$) and is equal to $-\frac{1}{2}\;H(\sigma J_2)$ when $z = 0$.

Since the last integral in (7.17) is also continuous in z, up to $z = 0$, we conclude that (7.16) holds.

Let us now substitute $\overrightarrow{I}$ from (7.14) into those expressions in (7.15) which involve $\rho(\overrightarrow{I})$, noting that by Lemma 7.2,

$$\begin{aligned}\rho(\overrightarrow{I}) =& \rho(\overrightarrow{n} \times (\overrightarrow{H}^1_0 - \overrightarrow{H}^2_0)e^{i\alpha x_1}) + \rho(\overrightarrow{n} \times \widehat{T}_2(\overrightarrow{I}, \overrightarrow{J})) \\ & - \frac{i\omega}{c}(\epsilon_1 - \epsilon_2)\left\{\frac{1}{2}H(\sigma J_2) + \int_0^L \sigma J_2(R^1_1 + f'(x_1)R^1_3)\right\}.\end{aligned} \tag{7.18}$$

Since $H^2u = H^2(u\chi_{[-L,L]}) + H^2(u\chi_{[-L,L]^c})$ and $H^2v = -v$ for any $v \in L^2(\mathbf{R}^1)$, we obtain

$$\overrightarrow{n} \times (\overrightarrow{E}^1_0 - \overrightarrow{E}^2_0)e^{-\alpha x_1} + \overrightarrow{J} + \frac{(\epsilon_1 - \epsilon_2)^2}{4\epsilon_1\epsilon_2}(J_2\overrightarrow{e}_2 + \overrightarrow{e}_2H^2(\chi_{[-L,L]^c}J_2))$$

$$+ \frac{\epsilon_1 - \epsilon_2}{\epsilon_1\epsilon_2}\frac{c\overrightarrow{e}_2}{2i\sigma\omega}\left\{H\left[\rho(\overrightarrow{n} \times (\overrightarrow{H}^1_0 - \overrightarrow{H}^2_0)e^{-i\alpha x_1}) + \rho(\hat{n} \times \widehat{T}_2(\overrightarrow{I} \times \overrightarrow{J}))\right.\right.$$

$$\left.\left.- \frac{i\omega}{c}(\epsilon_1 - \epsilon_2)\int_0^L \sigma J_2(R^1_1 + f'(x_1)R^1_3)\right] + i\alpha H(\sigma I_1)\right\} +$$

$$
(7.19) \quad + \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 \epsilon_2} \frac{c}{i\omega} \vec{n} \times \int_0^L \Big\{ \rho(\vec{n} \times (\vec{H}_0^1 - \vec{H}_0^2) e^{-i\alpha x_1}) + \rho(\vec{n} \times \widehat{T}_2(\vec{I}, \vec{J})) - \frac{i\omega}{c}(\epsilon_1 - \epsilon_2) \left[\frac{1}{2} H(\sigma J_2) + \int_0^L \sigma J_2 (R_1^1 + f'(x_1) R_3^1) \right] \Big\} \cdot (\vec{R}_1 - i\alpha \Psi_1 \vec{e}_1) + \vec{n} \times \widehat{T}_1 = 0.
$$

In view of (7.10),

$$
(7.20) \qquad \widehat{T}_3 \equiv \rho(\vec{n} \times \widehat{T}_2) \qquad \text{is a compact integral operator.}
$$

Also, from (7.14)

$$
(7.21) \qquad \vec{I} = \vec{n} \times (\vec{H}_0^1 - \vec{H}_0^2) e^{-i\alpha x_1} + \widehat{T}_4(\vec{I}, \vec{J}), \quad \widehat{T}_4 \ \text{ a compact integral operator.}
$$

Using (7.21) in evaluating the term $i\alpha H(\sigma I_1)$ in (7.19), using also (7.20), and noting that

$$
\int H(\sigma J_2)(\vec{R}_1 - \alpha \Psi_1 \vec{e}_1)
$$

is a compact operator, we can rewrite (7.19) in the form

$$
(7.22) \qquad \vec{J} + \frac{(\epsilon_1 - \epsilon_2)^2}{4\epsilon_1 \epsilon_2} J_2 \vec{e}_2 = \vec{J}_0 + \widehat{T}_5(\vec{I}, \vec{J})
$$

where $\vec{J}_0$ depends only on the vectors $\vec{E}_0^1 - \vec{E}_0^2, \vec{H}_0^1 - \vec{H}_0^2$, and it vanishes if both vectors vanish, and $\widehat{T}_5(\vec{I}, \vec{J})$ is a compact integral operator.

If we finally denote

$$
J_2 + \frac{(\epsilon_1 - \epsilon_2)^2}{4\epsilon_1 \epsilon_2} \quad \text{by} \quad J_2 \ ,
$$

we see that (7.21), (7.22) reduce to:

$$
(7.23) \qquad \begin{aligned} \vec{I} &= \vec{I}_0 + \widehat{T}_4(\vec{I}, \vec{J}), \\ \vec{J} &= \vec{J}_0 + \widehat{T}_5(\vec{I}, \vec{J}) \end{aligned}
$$

with $\vec{J}_0$ as above $\vec{I}_0 = n \times (\vec{H}_0^1 - \vec{H}_0^2) e^{-i\alpha x_1}$ and slightly modified $\widehat{T}_4, \widehat{T}_5$.

We have proved:

THEOREM 7.3. *The system (7.4), (7.5) is equivalent to the system (7.23) where $\widehat{T}_4, \widehat{T}_5$ are compact linear integral operator in X; consequently the Fredholm alternative holds.*

It follows that the system has a unique solution except for a countable number of values of the parameters $\epsilon_1, \epsilon_2, \mu$.

§8 A Simplified integral system. Recall that

$$\vec{\tau}(x_1) = \frac{1}{\sigma(x_1)}(\vec{e}_1 + f'(x_1)\vec{e}_3),$$
$$\vec{n}(x_1) = \frac{1}{\sigma(x_1)}(f'(x_1)\vec{e}_1 - \vec{e}_3)$$

and $(\vec{e}_2, \vec{\tau}(x_1), \vec{n}(x_1))$ is a moving orthonormal frame along S, with $\vec{\tau} \times \vec{n} = \vec{e}_2$.

Let us write the possible solution $\vec{I}, \vec{J}$ of (7.23) in the form

$$\begin{aligned} \vec{I}(x_1) &= I_\tau(x_1)\vec{\tau}(x_1) + I_2(x_1)\vec{e}_2, \\ \vec{J}(x_1) &= J_\tau(x_1)\vec{\tau}(x_1) + J_2(x_1)\vec{e}_2. \end{aligned} \tag{8.1}$$

Notice that along S

$$\begin{aligned} \vec{n} \times (\vec{H}_0^1 - \vec{H}_0^2)e^{-i\alpha x_1} &= a(x_1)\vec{\tau}(x_1), \\ \vec{n} \times (\vec{E}_0^1 - \vec{E}_0^2)e^{-i\alpha x_1} &= b(x_1)\vec{e}_2. \end{aligned} \tag{8.2}$$

If we take the scalar product of the equations (7.23) with $\vec{e}_2$ and $\vec{\tau}(x_1)$ respectively, we get a system of linear homogeneous equations for I_2, J_τ of the form

$$I_2 = W_1(I_2, J_\tau), \quad J_\tau = W_2(I_2, J_\tau). \tag{8.3}$$

Similarly, if we multiply the equations in (7.23) scalarly by $\vec{\tau}(x_1)$ and $\vec{e}_2$, respectively, we get a system of linear integral equations of the form

$$\begin{aligned} I_\tau &= a + W_3(I_\tau, J_2), \\ J_2 &= b + W_4(I_\tau, J_2). \end{aligned} \tag{8.4}$$

If the first Fredholm alternative holds for (7.23) then the first Fredholm alternative holds for both (8.3), (8.4). Therefore we must have

$$I_2 = 0, \quad J_\tau = 0. \tag{8.5}$$

Of course, the trivial solution (8.5) is always a solution of (8.3). Thus, in seeking a solution of (7.23), it seems reasonable to always choose $I_2 = 0, J_\tau = 0$ and concentrate just on finding a solution to (8.4).

THEOREM 8.1. *If the first Fredholm alternative holds for (7.23) then the unique solution has the form*

$$\vec{I} = I_\tau \vec{\tau}\ ,\quad \vec{J} = J_2 \vec{e}_2 \tag{8.6}$$

and the solution of (1.1)–(1.4) has the form

$$\vec{E} = E_1 \vec{e}_1 + E_3 \vec{e}_3\ ,\quad \vec{H} = H_2 \vec{e}_2 \quad \text{outside } S. \tag{8.7}$$

Proof. We have already proved (8.6). To prove (8.7), notice that $J_1 = 0$ and $\rho(\vec{J}) = 0$. It is now easy to check that all the terms on the right–hand side of (4.13) are vectors parallel to $\vec{e}_2$. It follows that $\vec{H} = H_2 \vec{e}_2$. Similarly one can check that all the terms on the right–hand side of (4.12) are orthogonal to $\vec{e}_2$, and the first assertion of (8.7) thus follows.

9. Uniqueness.

Set

$$\begin{aligned} \beta_n^1 &= \left\{ k_1^2 - \left(\alpha - \frac{2\pi n}{L}\right)^2 \right\}^{1/2}, Im\beta_n^1 \geq 0 \quad (n = 0, \pm 1, \pm 2, \dots), \\ \beta_n^2 &= \left\{ k_2^2 - \left(\alpha - \frac{2\pi n}{L}\right)^2 \right\}^{1/2}, Im\beta_n^2 \geq 0 \quad (n = 0, \pm 1, \pm 2, \dots). \end{aligned} \tag{9.1}$$

From (7.1), the representation (4.12), (4.13) and Theorem 3.2 we deduce that

$$(\vec{E}, \vec{H}) = \begin{cases} (\vec{E}_0, \vec{H}_0) + e^{i\alpha x_1} \displaystyle\sum_{n=-\infty}^{\infty} N_n^1 e^{-i\frac{2\pi n x_1}{L} + i\beta_n^1 x_3} & \text{if } x_3 > |f|_{L^\infty}, \\ e^{i\alpha x_1} \displaystyle\sum_{n=-\infty}^{\infty} N_n^2 e^{-i\frac{2\pi n x_1}{L} - i\beta_n^2 x_3} & \text{if } x_3 < -|f|_{L^\infty}. \end{cases} \tag{9.2}$$

and $|N_n^j| \leq \dfrac{C}{\sqrt{1+|n|}}$. The coefficients N_2^1 are called the reflection coefficients and the coefficients N_n^2 are called the transmission coefficients.

THEOREM 9.1. *suppose $(\vec{E}, \vec{H})$ is a solution of the Maxwell equations (1.1), (1.2) outside S, satisfying (1.3), (1.4) such that $e^{-i\alpha x_1}(\vec{E}, \vec{H})$ is independent of x_2 and is periodic in x_1 of period L. If $(\vec{E}, \vec{H})$ has the form (9.2) in $\{|x_3| > |f|_{L^\infty}\}$ with vectors N_n^j uniformly bounded, then $(\vec{E}, \vec{H})$ is unique provided the first Fredholm alternative holds the system (7.23).*

Proof. Suppose there are two solutions and denote by their difference (E, H) . Then

$$e^{-i\alpha x_1}(E,H) = \begin{cases} \sum\limits_{n=-\infty}^{\infty} M_n^1 e^{-i\frac{2\pi n x_1}{L} + i\beta_n^1 x_3} & \text{if } x_3 > |f|_{L^\infty}, \\ \sum\limits_{n=-\infty}^{\infty} M_n^2 e^{-i\frac{2\pi n x_1}{L} - i\beta_n^2 x_3} & \text{if } x_3 < -|f|_{L^\infty}. \end{cases} \tag{9.3}$$

Set $\vec{I} = -\vec{n} \times \vec{H}$, $\vec{J} = \vec{n} \times \vec{E}$, $\vec{I}_\alpha = \vec{I} e^{-i\alpha x_1}$, $\vec{J}_\alpha = \vec{J} e^{-i\alpha x_1}$ and define

$$\begin{aligned} F(\vec{I}(y), \vec{J}(y), \Psi_k(x-y)) = &\Big\{ \frac{i\omega\mu}{c} \vec{I}(y)\Psi_k(x-y) - \vec{J}(y) \times \nabla_y \Psi_k(x-y) \\ &+ \frac{c}{i\omega\epsilon}\, \nabla_x \left[I(y) \cdot \nabla_y \Psi_k(x-y)\right] \\ &+ i\alpha \vec{J}(y) \times \vec{e}_1 \Psi_k(x-y) + \frac{c\alpha}{\omega\epsilon}\left[\vec{I}(y) \cdot \nabla_y \Psi_k(x-y)\vec{e}_1 + I_1 \nabla_y \Psi_k(x-y)\right] \\ &- \frac{ic\alpha^2}{\omega\epsilon}\vec{e}_1\ I_1(y)\Psi_k(x-y)\Big\} \end{aligned}$$

If we use the integral representation [7; p. 130] for $\vec{E}, \vec{H}$ in

$$D_m = \{f(x_1) < x_3 < Y,\ -mL < x_1 < mL\},$$

where $Y > |f|_{L^\infty}$, then we obtain a representation similar to (2.3), (2.4) with S replaced by ∂D_m. Integrating with respect to $y_2, -\infty < y_2 < \infty$, and the letting $m \to \infty$ as we find (since

$$H_0^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}}\ e^{i(z - \frac{\pi}{4})} \quad \text{as} \quad z \to \infty;$$

e.g. [6; p. 962, §8451, #3]) that the boundary integrals over $x_1 = \pm mL$ converge to zero. To the remaining integrals (on S and on $y_3 = Y$) we apply the process which led from (2.3), (2.4) to (4.7), (4.8). We thus obtain the representation

$$e^{-i\alpha x_1}\vec{E}(x) = \int\limits_{\partial D_Y} F(\vec{I}(y), \vec{J}(y), \Psi_1(x-y))\ ds$$

where $D_Y = \{(x_1, x_3);\ 0 < x_1 < L,\ f(x_1) < x_3 < Y\}$. By periodicity, the integrals over $x_1 = 0$ and $x_1 = L$ cancel each other. Therefore

$$\begin{aligned} e^{-i\alpha x_1}\vec{E}(x) - \int\limits_0^L F(\vec{I}, \vec{J}, \Psi_1(x-y))(y_1, f(y_1))\sigma(y_1)\ dy_1 \\ = -\int\limits_0^L F(\vec{I}, \vec{J}, \Psi_1(x-y))(y_1, Y)\ dy_1. \end{aligned} \tag{9.4}$$

By Theorem 3.2 and (9.3), the left-hand side has the form

$$\sum K_n e^{i\alpha_n x_1 + i\beta_n^1 x_3} \qquad \left(\alpha_n = \frac{2\pi n}{L}\right)$$

whereas, by Theorem 3.2, the right-hand side of (9.4) has the form

$$\sum R_n e^{i\alpha_n x_1 - i\beta_n^1 x_3}$$

where $R_n = R_n(Y)$. Letting $Y \to \infty$ we get

$$\sum K_n e^{i\alpha_n x_1 + i\beta_n^1 x_3} = \sum R_n(\infty) e^{i\alpha_n x_1 - i\beta_n^1 x_3}$$

for all $x_3 > |f|_{L^\infty}$. It follows that $K_n e^{i\beta_n^1 x_3} = R_n(\infty) e^{-i\beta_n^1 x_3}$ for all $x_3 > |f|_{L^\infty}$, which implies that $K_n = R_n(\infty) = 0$. Thus, the right hand side of (9.4) converges to zero as $Y \to \infty$, and we obtain

$$e^{-i\alpha x_1} \vec{E}(X) = \int_0^L F(\vec{I}, \vec{J}, \Psi_1(x-y))(y_1, f(y_1))\sigma(y_1)\, dy_1.$$

This representation is the same as (4.7), and similarly we derive a representation for $\overline{H}$ as in (4.8). A similar representation holds in $\{x_3 < -f\}$. Letting $x_3 \to f(x_1)$ and using the jump relations derived in Section 5, we find that $\overline{I}, \overline{J}$ must satisfy the homogeneous version of the system of integral equations (7.23). Since for this system the first Fredholm alternative holds, $\vec{I} = \vec{J} = 0$ and therefore $\vec{E} = \vec{H} = 0$.

§10. The Piecewise smooth interface. In this section we extend the results of the previous sections to the case where S is piecewise smooth and is not necessarily a graph in the x_3-direction. We take S to have the form

$$(10.1) \quad S = \{(x_1, x_2, x_3);\ x_1 = f_1(s),\ x_3 = f_3(s),\ -\infty < s < \infty \ \text{ and } \ -\infty < x_2 < \infty\}$$

where s is the length parameter, and assume that for some $l_0 > 0,\ L > 0$,

$$(10.2) \qquad \begin{aligned} &f_1(s + nl_0) = nL + f_1(s), \\ &f_3(s + nl_0) = f_3(s) \ \ (-\infty < s < \infty,\ n = \pm 1, \pm 2, \dots), \end{aligned}$$

$$(10.3) \qquad f_j(s) \quad \text{are continuous for all} \quad s \in \mathbf{R}^1.$$

Denoting by $s_1 < s_2 < \dots, s_n$ the points of discontinuity of the derivatives of f_j in the interval $0 \le s \le l_0$ and setting $s_{n+1} = l_0 + s_1$, we further assume that

$$f_j(s) \quad \text{is in} \quad C^2[s_i, s_{i+1}] \text{ for } 1 \le i \le n. \tag{10.4}$$

Notice that our assumption on S include the case where $S \cap \{x_2 = 0\}$ is a step–function, i.e., $x_3 = f(x_1)$ where f is piecewise constant.

Set $l = S \cap \{x_2 = 0\}$, $\quad \vec{l}(s) = \big(f_1(s), 0, f_3(s)\big)$ and introduce the tangent

$$\vec{\tau} = \frac{d\vec{l}}{ds}$$

and normal $\vec{n} = \vec{e}_2 \times \vec{\tau}$. We shall also write $y(t) = \vec{l}(t)$.

We begin with the representation (2.3), (2.4), but take $\vec{I}, \vec{J}$ to have the form

$$\begin{aligned} \vec{I}\big(y(t)\big) &\equiv \vec{I}(t) = I(t)\vec{\tau}(t)\ , \qquad I(t) \equiv I\big(y(t)\big), \\ \vec{J}\big(y(t)\big) &\equiv \vec{J}(t) = J(t)\vec{e}_2\ , \qquad J(t) \equiv J\big(y(t)\big); \end{aligned} \tag{10.5}$$

the general case where

$$\begin{aligned} \vec{I} &= I(t)\vec{\tau}(t) + I_2(t)\vec{e}_2, \\ \vec{J} &= J(t)\vec{e}_2 \ \ + J_\tau(t)\vec{\tau}(t) \end{aligned} \tag{10.6}$$

can be handled similarly. However, as in §8, we can derive a system of equation for (I, J) and (separately) for (I_2, J_τ), and the system for I_2, J_τ has a solution $I_2 = 0,\ J_\tau = 0$.

Assuming (10.5) we have

$$J(t)\nabla_y \Psi_k\big(x - y(t)\big) = J(t)\vec{e}_2 \cdot (\partial_{y_1}\Psi_k \vec{e}_1 + \partial_{y_3}\Psi_k \vec{e}_3) = 0, \tag{10.7}$$

$$\rho\big(\vec{I}(t)\big) = \frac{d}{dt}\left[\sigma\big(y(t)\big)\vec{I}\big(y(t)\big) \cdot \vec{e}_1\right] = \frac{dI(t)}{dt}\ . \tag{10.8}$$

Proceeding with (2.4) and using (10.5) (10.7) we obtain analogously to (4.8):

$$\begin{aligned} H_\alpha(x) = \int_0^{l_0} \Big\{ &\frac{i\omega\epsilon}{c}\vec{J}\Psi_k\big(x - y(t)\big) + \vec{I}(t) \times \nabla_y \Psi_k\big(x - y(t)\big) \\ &-i\alpha\big(\vec{I}(t) \times \vec{e}_1\big)\Psi_k(x - y(t)\big) \Big\}\ dt. \end{aligned} \tag{10.9}$$

Similarly, proceeding with (2.4) and using (10.5), (10.8) we obtain analogously to (4.12):

$$
\begin{aligned}
E_\alpha(x) = \int_0^{l_0} \Big\{ & \frac{i\omega\mu}{c}\vec{I}(t)\Psi_k(x-y(t)) - \vec{J}(t)\times\nabla_y\Psi_k(x-y(t)) \\
& + \frac{c}{i\omega\epsilon}\frac{dI(t)}{dt}\nabla_y\Psi_k\big(x-y(t)\big)\Big\}\,dt \\
+ \int_0^{l_0} & \left\{ i\alpha\vec{J}(t)\times\vec{e}_1 - \frac{c\alpha}{\omega\epsilon}\frac{dI(t)}{dt}\vec{c}_1 - \frac{ic\alpha^2}{\omega\epsilon}I(t)\vec{c}_1(t)\right\}\Psi_k\big(x-y(t)\big)dt \\
& + \frac{c\alpha}{\omega\epsilon}\int_0^{l_0} I(t)\nabla_y\Psi_k\big(x-y(t)\big)\,dt.
\end{aligned}
\tag{10.10}
$$

We set

$$\Sigma = \{0, s_1, \dots, s_n\}. \tag{10.11}$$

Lemma 10.1. *If $s \notin \Sigma$ then, for any $u \in L^2(0,l_0)$,*

$$\lim_{\epsilon\to 0}\int_{\substack{0\\ \{|t-s|>\epsilon\}}}^{l_0} u(t)\frac{\vec{l}(s)-\vec{l}(t)}{|\vec{l}(s)-\vec{l}(t)|^2}\,dt \qquad \textit{exists.}$$

Proof. We have

$$\vec{l}(s)-\vec{l}(t) = (f_1'(s)\vec{e}_1 + f_3'(s)\vec{e}_3)(s-t) + (s-t)^2F(t) \tag{10.12}$$

where $F(t)$ is continuous in t. Hence

$$\frac{\vec{l}(s)-\vec{l}(t)}{|\vec{l}(s)-\vec{l}(t)|^2} = \frac{f_1'(s)\vec{e}_1 + f_3'(s)\vec{e}_3}{s-t} + F_1(t)$$

where $F_1(t)$ is continuous in t. Using the Hilbert transform properties, the assertion of the lemma follows.

Lemma 10.2. *Let $x = \vec{l}(s)$, $s \notin \Sigma$. Then for any L^2 function $u(t)$,*

$$
\begin{aligned}
\lim_{\substack{z\to\pm 0\\ z\neq 0}}\int_0^{l_0} u(t)\nabla_y\Psi_k\big(x+z\vec{n}(s)-y(t)\big)\,dt &= \pm\frac{1}{2}\vec{n}(s)u(s) \\
&+ \frac{1}{2\pi}p.v.\int_0^{l_0} u(t)\frac{\vec{l}(s)-\vec{l}(t)}{|\vec{l}(s)-\vec{l}(t)|^2}\,dt \\
+ \int_0^{l_0} u(t)\nabla_y & \left(\Psi_k\big(x-y(t)\big) - \frac{1}{2\pi}\log\frac{1}{|x-y(t)|}\right)\,dt
\end{aligned}
$$

holds for a.a. s.

Proof. Since

$$\widetilde{\Psi}_k(x-y) \equiv \Psi_k(x-y) - \frac{1}{2\pi}\log\frac{1}{|x-y|}$$

is a continuous function, it suffices to prove that

$$\begin{aligned}\lim_{\substack{z\to\pm 0\\ z\neq 0}} \frac{1}{2\pi}\int_0^{l_0} u(t)\frac{\vec{l}(s)-\vec{l}(t)+z\vec{n}(s)}{|\vec{l}(s)-\vec{l}(t)+z\vec{n}(s)|^2}\,dt &= \pm\frac{1}{2}\vec{n}(s)u(s)\\ &+\frac{1}{2\pi}\,p.v.\int_0^{l_0} u(t)\frac{\vec{l}(s)-\vec{l}(t)}{|\vec{l}(s)-\vec{l}(t)|^2}\,dt.\end{aligned} \tag{10.13}$$

Since

$$\begin{aligned}|\vec{l}(s)-\vec{l}(t)+z\vec{n}(s)|^2 &= |\vec{l}(s)-\vec{l}(t)|^2 + z^2 + 2z\vec{n}(s)\cdot\big(\vec{l}(s)-\vec{l}(t)\big)\\ &= (s-t)^2+z^2+zO\big((s-t)^2\big)+O\big((s-t)^3\big) \qquad \text{(by (10.12))},\end{aligned}$$

we have

$$\frac{\vec{l}(s)-\vec{l}(t)}{|l(s)-l(t)+z\vec{n}(s)|^2} = \frac{(s-t)\vec{\tau}(s)}{(s-t)^2+z^2} + \Delta(s,t,z)$$

where $\Delta(s,t,z)$ is continuous in all variables. Hence

$$\begin{aligned}\lim_{z\to\pm 0}\int_0^{l_0} u(t)\frac{\vec{l}(s)-\vec{l}(t)}{|\vec{l}(s)-\vec{l}(t)+z\vec{n}(s)|^2}\,dt &= \lim_{z\to\pm 0}\int_0^{l_0}\frac{(s-t)\vec{\tau}(s)u(s)}{(s-t)^2+z^2}\,dt\\ &+\lim_{z\to\pm 0}\int_0^{l_0} u(t)\Delta(s,t,z)\,dt + \lim_{z\to\pm 0}\int_0^{l_0}\frac{(s-t)\vec{\tau}(s)\big(u(t)-u(s)\big)}{(s-t)^2+z^2}\,dt\\ &= p.v.\left[\int_0^{l_0}\frac{(s-t)\big(u(t)-u(s)\big)}{(s-t)^2}\,dt\right]\vec{\tau}(s) + \int_0^{l_0} u(t)\Delta(s,t,0)\,dt\\ &= p.v.\int_0^{l_0} u(t)\left[\frac{1}{s-t}\vec{\tau}(s)+\Delta(s,t,0)\right]\,dt\\ &= p.v.\int_0^{l_0}\frac{\vec{l}(s)-\vec{l}(t)}{|\vec{l}(s)-l(t)|^2}u(t)\,dt,\end{aligned} \tag{10.14}$$

by reversing the previous steps in the case $z = 0$. Next

$$\frac{z\vec{n}(s)}{|\vec{l}(s) - \vec{l}(t) + z\vec{n}(s)|^2} = z\vec{n}(s)\left[\frac{1}{(s-t)^2 + z^2} + O\left(\frac{z + |s-t|}{(s-t)^2 + z^2}\right)\right]$$

and therefore

$$\begin{aligned}\lim_{z\to\pm 0}\int_0^{l_0}\frac{z\vec{n}(s)}{|\vec{l}(s) - \vec{l}(t) + zu(s)|^2}\,dt &= \vec{n}(s)\lim_{z\to\pm 0}\int_0^{l_0}\frac{u(t)z}{(s-t)^2 + z^2}\,dt \\ &= \pm\pi\vec{n}(s)u(s).\end{aligned}$$

Combining this with (10.14), the assertion (10.13) follows.

Using Lemma 10.2 we can now deduce from (10.9), (10.10) that

$$\vec{E}^{\pm}(x) = \lim_{z\to\pm 0}\vec{E}_\alpha\big(x + z\vec{n}(s)\big),\ \vec{H}^{\pm}(x) = \lim_{z\to\pm 0}\vec{H}_\alpha\big(x + z\vec{n}(s)\big)$$

exist for any $x = x(s), s \notin \Sigma$ and, setting $\vec{E}^{\pm}(s) = \vec{E}^{\pm}\big(x(s)\big), \vec{H}^{\pm}(s) = \vec{H}^{\pm}\big(x(s)\big)$, we easily get

$$\begin{aligned}\vec{n}(s)\times(H^+(s) - H^-(s)) = \Bigg\{ & I(s) \\ & + \frac{i\omega}{c}\int_0^{l_0}(\epsilon_1\Psi_1 - \epsilon_2\Psi_2)\big(x(s) - y(t)\big)J(t)\,dt \\ & + \int_0^{l_0} I(t)\left((f_3'(t)\frac{\partial}{\partial y_1} - f_1'(t)\frac{\partial}{\partial y_3}\right)(\Psi_1 - \Psi_2)\big(x(s) - y(t)\big)\,dt \\ & - \int_0^{l_0} i\alpha I(t)(\Psi_2 - \Psi_1)\big(x(s) - y(s)\big)f_3'(t)\,dt\Bigg\}\vec{\tau}(s),\end{aligned} \tag{10.15}$$

and (using also (10.13))
(10.16)

$$\begin{aligned}
\vec{n}(s) \times \left(\vec{E}^{+}(s) - \vec{E}^{-}(s)\right) = \Bigg\{ & - J(s) \\
& + \frac{c}{2\pi\omega i}\left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_1}\right) \int\limits_0^{l_0} \left[\frac{dI(t)}{dt} + i\alpha I(t)\right] \frac{f_1'(s)\big(f_1(s) - f_1(t)\big) + f_3'(s)\big(f_3(s) - f_3(t)\big)}{|\vec{l}(s) - \vec{l}(t)|^2}\, dt \\
& + \frac{c}{i\omega} \int\limits_0^{l_0} \left[\frac{dI(t)}{dt} + i\alpha I(t)\right] \left(f_1'(s)\frac{\partial}{\partial y_1} + f_3'(s)\frac{\partial}{\partial y_3}\right) \left(\frac{\tilde{\Psi}_2}{\epsilon_2} - \frac{\tilde{\Psi}_1}{\epsilon_1}\right) \big(x(s) - y(t)\big)\, dt \\
& + \int\limits_0^{l_0} J(t) \left[f_1'(s)\frac{\partial}{\partial y_3} - f_3'(s)\frac{\partial}{\partial y_1}\right] (\Psi_1 - \Psi_2)\big(x(s) - y(t)\big)\, dt \\
& + \frac{c\alpha}{\omega} \int\limits_0^{l_0} \left[\frac{dI(t)}{dt} + i\alpha I(t)\right] f_1'(s) \left(\frac{\Psi_2}{\epsilon_2} - \frac{\Psi_1}{\epsilon_1}\right) \big(x(s) - y(t)\big)\, dt \\
& + \int\limits_0^{l_0} \left[i\alpha f_3'(s) J(t) - \frac{i\omega\mu}{c}\left[f_1'(s)f_1'(t) + f_3'(s)f_3'(t)\right] I(t)\right] \\
& \qquad \cdot (\Psi_1 - \Psi_2)\big(x(s) - y(t)\big)\, dt \Bigg\} \vec{e}_2.
\end{aligned}$$

Following the procedure of §7 we now wish to substitute I from (10.15) into (10.16) in order to get rid of the derivatives dI/dt in (10.16). We require here a lemma analogous to Lemma 7.2 whose proof uses Lemma 10.2:

LEMMA 10.3. *If*

$$Lu = \int\limits_0^{l_0} u(t)(\epsilon_1\Psi_1 - \epsilon_2\Psi_2)\big(x(s) - y(t)\big)\, dt$$

then

$$\begin{aligned}
\left(\frac{d}{ds}Lu\right)(s) = {} & \epsilon_2 \frac{d}{ds} \int\limits_0^{l_0} u(t)(\Psi_1 - \Psi_2)\big(x(s) - y(t)\big)\, dt \\
& + \frac{\epsilon_2 - \epsilon_1}{2\pi}\ p.v. \int\limits_0^{l_0} u(t) \frac{\vec{\tau}(s) \cdot \big(\vec{l}(s) - \vec{l}(t)\big)}{|\vec{l}(s) - \vec{l}(t)|^2}\, dt \\
& + (\epsilon_2 - \epsilon_1) \int\limits_0^{l_0} u(t) \left(f_1'(s)\frac{\partial}{\partial y_1} + f_3'(s)\frac{\partial}{\partial y_3}\right) \tilde{\Psi}_1\big(x(s) - y(t)\big)\, dt.
\end{aligned}$$

The proof is omitted.

Introduce the operator $\widetilde{H}$:

$$\widetilde{H}u(s) = \frac{1}{\pi}\ p.v. \int_0^{l_0} u(t) \frac{\vec{\tau}(s) \cdot \left(\vec{l}(s) - \vec{l}(t)\right)}{|\vec{l}(s) - \vec{l}(t)|^2}\, dt. \tag{10.17}$$

If we substitute I from (10.15) into (10.16) and use Lemma 10.3 and the notation (10.17), we get a system

$$\begin{aligned} & I + \widetilde{T}_1(I, J) = -I_0\ , \\ & J - \frac{(\epsilon_1 - \epsilon_2)^2}{4\epsilon_1\epsilon_2} \widetilde{H}^2(J) + \widetilde{T}_2(I, J) = -\widetilde{J}_0 \end{aligned} \tag{10.18}$$

where $I_0\vec{\tau}(s) = -\vec{n}(s) \times \left(\vec{H}_0^+(s) - \vec{H}_0^-(s)\right)$ and $\widetilde{J}_0$ depends on both I_0 and $\vec{n} \times \Big(\vec{E}_0^+ - \vec{E}_0^-\Big)e^{-i\alpha x_1}$; for any $1 < p < \infty$ the operators $\widetilde{T}_1, \widetilde{T}_2$ are compact integral operators in the space X_p of L^p l_0-periodic functions defined on the curve $S \cap \{x_2 = 0\}$ with the $L^p(0, l_0)$ norm.

Thus the only essential difference between (10.18) and (7.21), (7.22) or (8.4) is that in the previous system $\widetilde{H}$ was the bounded operator H (the Hilbert transform) satisfying $H^2 = -1$ whereas now we have an operator $\widetilde{H}$ which is not as "nice" as the Hilbert transform. We nonetheless have:

LEMMA 10.4. *$\widetilde{H}$ is a bounded operator in X_p, that is,*

$$\|\widetilde{H}u\|_{L^p(0,l_0)} \le C\|u\|_{L^p(0,l_0)}.$$

Proof. We first consider the behavior of $(\widetilde{H}u)(s)$ for s near a point s_i where $\vec{\tau}(s)$ has a discontinuity. For simplicity we take $0 < s_i < l_0$. (If $s_i = 0$ we can work with $\int_{-l_0}^{0}$ instead of $\int_0^{l_0}$ (in (10.17)), since u and l are l_0-periodic. Set

$$\vec{\alpha} = \vec{\tau}(s_i + 0) - \vec{\tau}(s_i - 0).$$

Suppose $s_i < s < s_{i+1}$ and consider the portion of the integral $(\widetilde{H}u)(s)$ from $t = s_i$ to $t = s_{i+1}$. Then the integrand satisfies:

$$u(t)\frac{\vec{\tau}(s) \cdot \left(\vec{l}(s) - \vec{l}(t)\right)}{|\vec{l}(s) - \vec{l}(t)|^2} = \frac{u(t)}{s - t} + u(t)F_0\ , \quad F_0 \text{ bounded} \tag{10.19}$$

(since $\tau(\cdot)$ is in $C^1[s_i, s_{i+1}]$).

Next consider the portion of the integral $(\widetilde{H}u)(s)$ for $s_{i-1} < t < s_i$. Introduce the auxiliary $C^{1,1}$ curve $\overrightarrow{l}_*(t)$ in $[s_{i-1}, s_{i+1}]$ defined by

$$\begin{aligned} \overrightarrow{l}_* &= \overrightarrow{l}(t) \quad \text{if} \quad s_i < t < s_{i+1}, \\ \overrightarrow{l}_* &= \overrightarrow{l}(t) + \overrightarrow{\alpha}(t - s_i) \quad \text{if} \quad s_{i-1} < t < s_i. \end{aligned}$$

Then

$$\frac{\overrightarrow{\tau}(s) \cdot \left(\overrightarrow{l}(s) - \overrightarrow{l}(t)\right)}{|\overrightarrow{l}(s) - \overrightarrow{l}(t)|^2} = \frac{\overrightarrow{\tau}(s) \cdot \left[\overrightarrow{\tau}(s)(s-t) - \overrightarrow{\alpha}(t - s_i) + O(s-t)^2\right]}{|\overrightarrow{\tau}(s)(s-t) - \overrightarrow{\alpha}(t - s_i) + O(s-t)^2|^2}.$$

Notice that for $s > s_i$, $\overrightarrow{\tau}(s) = \overrightarrow{\tau}(s_i + 0) + O(s - s_i)$. Since also $\overrightarrow{\tau}(s_i + 0) \cdot \overrightarrow{\tau}(s_i - 0) > -1$, the denominator in the last fraction is $\geq c[(s - s_i)^2 + (t - s_i)^2] \quad (c > 0)$. It now easily follows that, for $s_{i-1} < t < s_i$,

$$\begin{aligned} |u(t)\frac{\overrightarrow{\tau}(s) \cdot \left(\overrightarrow{l}(s) - \overrightarrow{l}(t)\right)}{|\overrightarrow{l}(s) - \overrightarrow{l}(t)|} - \frac{u(t)}{s-t}| &\leq C\frac{|t - s_i| + (s-t)^2}{(s-s_i)^2 + (t-s_i)^2}|u(t)| \\ &\leq C\left(1 + \frac{s_i - t}{(s-s_i)^2 + (t-s_i)^2}\right)|u(t)| \qquad (10.20) \\ &\quad (\text{since } (s-t)^2 \leq 2(s-s_i)^2 + 2(s_i - t)^2) \\ &\leq C\left(1 + \frac{2}{s-t}\right)|u(t)| \end{aligned}$$

since $t < s_i < s$.

We have thus proved that, for $s_i < s < s_{i+1}$,

$$\begin{aligned} &|\int_{s_{i-1}}^{s_{i+1}} u(t)\frac{\overrightarrow{\tau}(s) \cdot \left(\overrightarrow{l}(s) - \overrightarrow{l}(t)\right)}{|\overrightarrow{l}(s) - \overrightarrow{l}(t)|^2}\, dt| \\ &\quad \leq C|\int_{s_{i-1}}^{s_i} \frac{|u(t)|}{t-s}\, dt| + C\int_{s_{i-1}}^{s_{i+1}} |u(t)|\, dt + |\int_{s_{i-1}}^{s_{i+1}} \frac{u(t)}{s-t}\, dt| \\ &\quad = CH\left(|u|\chi_{[s_{i-1}, s_i]}\right)(s) + C\int_{s_{i-1}}^{s_{i+1}} |u(t)|\, dt + \left|H\left(u\chi_{[s_{i-1}, s_{i+1}]}\right)\right|. \end{aligned}$$

A similar estimate holds for s in the interval (s_{i-1}, s_i) and clearly also for s in $(0, s_{i-1})$ or in (s_{i+1}, l_0). Since H is a bounded operator in L^p, the assertion of the lemma follows.

We summarize:

Theorem 10.5. *Let S be given by (10.1)–(10.4). Then there exists a solution of (1.1)–(1.4) of the form (8.7) if there exists a solution I, J of (10.18), where $\widetilde{T}_1, \widetilde{T}_2$ are compact linear integral operators in X_p and $\widetilde{H}$ is a bounded linear operator in X_p, given by (10.17); here p is any number satisfying $1 < p < \infty$.*

Remark 10.1. Theorem 9.1 extends to the case where S satisfies (10.1)–(10.4); thus, the solution having the form (9.2) is unique if the system (10.18) has a unique solution

In order to show that the system (10.18) is a Fredholm system of equations we need to analyze $\widetilde{H}^2$ more carefully. We shall be working with the spaces $X_p, 1 \le p \le 2$; if T is bounded linear operator from X_p to X_p then its norm is denoted by $\|T\|_{L^p}$.

Lemma 10.6. *For any $u \in X_1 \cap X_2$*

$$\widetilde{H}^2 u = -u + Du + Tu \tag{10.21}$$

where D is a compact operator from $L^p(0, l_0)$ into $L^p(0, l_0)$ for $1 \le p \le 2$, and

$$\|Tu\|_{L^1(0,l_0)} \le \frac{1}{4}\ \|u\|_{L^1(0,l_0)}, \tag{10.22}$$

$$\|Tu\|_{L^2(0,l_0)} \le \frac{3+2\sqrt{2}}{4}\ \|u\|_{L^2(0,l_0)}. \tag{10.23}$$

Proof. Using a partition of unity to write

$$u = \Sigma \chi_i u$$

where χ_i is supported $\{s_{i-1} + 2\delta,\ s_{i+1} - 2\delta\}$ for some $\delta > 0$, it is sufficient to concentrate on $\widetilde{H}^2(\chi_i u)$. Setting

$$L_0^p = \{u \in L^p(s_{i-1} + \delta, s_{i+1} - \delta); u(t) = 0 \quad \text{if} \quad t < s_{i-1} + 2\delta \quad \text{or if } t > s_{i+1} - 2\delta\}$$

it is then sufficient to establish (10.21)-(1.23) for u in $L_0^1 \cap L_0^2$. (Here we have taken for simplicity $0 \le s_{i-1},\ \ s_{i+1} \le l_0$; if s_i is the smallest or largest point in Σ then some small modification need to be made, using the l_0-periodicity of the functions $u(t)$.)

To simplify the notation we take

$$s_{i-1} = -1, \ \ s_{i-1} + 2\delta = -\frac{1}{2}, \ \ s_i = 0, \ \ s_{i+1} - 2\delta = \frac{1}{2}, \ \ s_{i+1} = 1. \tag{10.24}$$

We first consider a special case where

$$\vec{l}(s) = \begin{cases} \vec{\alpha} s & \text{if} \quad -1 < s < 0 \\ \vec{\beta} s & \text{if} \quad 0 < s < 1 \end{cases} \tag{10.25}$$

and $|\vec{\alpha}| = |\vec{\beta}| = 1, \quad \vec{\alpha} \ne \vec{\beta}$.

Lemma 10.7. *The assertions of Lemma 10.6 holds for the case where u varies in L_0^p and $\widetilde{H}^2 u$ is considered in $L^p(0, l_0)$, provided (10.24), (10.25) hold.*

Proof. We shall use complex notation

$$\vec{\alpha} = e^{i\theta} \equiv \alpha\ , \quad \vec{\beta} = e^{i\phi} \equiv \beta$$

and set

$$A = \vec{l}(-1), \quad B = \vec{l}(0), \quad C = \vec{l}(1),$$
$$\zeta(s) = \begin{cases} \alpha s & \text{if} \quad -1 < s < 0 \\ \beta s & \text{if} \quad 0 < s < 1. \end{cases}$$

We assume for definiteness that $\overline{AC}$ lies in Ω_1 (i.e., above the curve S), and denote by Γ the boundary of the triangle A, B, C.

Since

$$\vec{\alpha} \cdot \vec{l} = Re\ \alpha\overline{\zeta}, \quad \vec{\beta} \cdot \vec{l} = Re\ \beta\overline{\zeta},$$

we can write

$$\pi \widetilde{H} u(s) = \begin{cases} Re \int\limits_{-1}^{1} \dfrac{\alpha u(t)}{\zeta(s) - \zeta(t)}\, dt\ , & -1 < s < 0, \\ Re \int\limits_{-1}^{1} \dfrac{\beta u(t)}{\zeta(s) - \zeta(t)}\, dt\ , & 0 < s < 1. \end{cases}$$

Define

$$F(z) = \int_{-1}^{1} \frac{u(t)}{z - \zeta(t)}\, dt \quad \text{for} \quad z \in \Omega_1,$$
$$F\big(\zeta(s)\big) = \lim_{\epsilon \to 0+} F\big(\zeta(s) + \epsilon n(s)\big)$$

where

$$n(s) = \begin{cases} i\alpha & \text{if} \quad -1 < s < 0 \\ i\beta & \text{if} \quad 0 < s < 1 \end{cases}$$

is the normal to S pointing into Ω_1.

Notice that for $-1 < s < 0$,

$$\lim_{\epsilon \to 0+} Re \int_{-1}^{0} \frac{\alpha u(t)}{\alpha s + i\alpha\epsilon - \alpha t}\, dt = Re \int_{(-1,0)\setminus J} \frac{\alpha u(t)}{\alpha s - \alpha t}\, dt$$
$$+ \lim_{\epsilon \to 0+} \int_J \frac{s - t}{(s-t)^2 + \epsilon^2}\, [u(t) - u(s)]\, dt + \lim_{\epsilon \to 0+} \int_J \frac{s - t}{(s-t)^2 + \epsilon^2}\, u(s)\, dt$$

where J is a subinterval of $(-1,0)$ symmetric about s. The last integral vanishes by symmetry, whereas the limit of the preceding integral is equal a.e. to

$$\int\limits_J \frac{s-t}{(s-t)^2}[u(t)-u(s)]\,dt.$$

We conclude that

$$\lim_{\substack{z=\zeta(s)+\epsilon n(s)\\ \epsilon\to 0+}} Re\int\limits_{-1}^{0} \frac{\alpha u(t)}{z-\zeta(t)}\,dt = Re\int\limits_{-1}^{0} \frac{\alpha u(t)}{\zeta(s)-\zeta(t)}\,dt\,. \tag{10.26}$$

Similarly

$$\lim_{\substack{z=\zeta(s)+\epsilon n(s)\\ \epsilon\to 0+}} Im\int\limits_{-1}^{0} \frac{\alpha u(t)}{z-\zeta(t)}\,dt = -\pi u(s)\,. \tag{10.27}$$

Using (10.26) we deduce that, for $-1 < s < 0$,

$$Re(\alpha F)\big(\zeta(s)\big) = Re\int\limits_{-1}^{1} \frac{\alpha u(t)}{\zeta(s)-\zeta(t)}\,dt = \pi\widetilde{H}u(s).$$

A similar result holds for $0 < s < 1$; thus

$$\pi\widetilde{H}u(s) = \begin{cases} Re(\alpha F)\big(\zeta(s)\big) & \text{if} \quad -1 < s < 0 \\ Re(\beta F)\big(\zeta(s)\big) & \text{if} \quad 0 < s < 1. \end{cases} \tag{10.28}$$

If $-1 < s < 0$ then by (10.28)

$$\begin{aligned} \pi^2\widetilde{H}^2u(s) &= \pi^2\widetilde{H}(\widetilde{H}u)(s) \\ &= \pi Re\left\{\alpha\int\limits_{-1}^{1}\frac{\widetilde{H}u(t)}{\zeta(s)-\zeta(t)}\,dt\right\} \\ &= \lim_{\substack{z=\zeta(s)+i\alpha\epsilon\\ \epsilon\to 0+}} Re\left\{\alpha\left[\int\limits_{-1}^{0}\frac{Re(\alpha F)\big(\zeta(t)\big)}{z-\zeta(t)}\,dt + \int\limits_{0}^{1}\frac{Re(\beta F)\big(\zeta(t)\big)}{z-\zeta(t)}\,dt\right]\right\} \end{aligned} \tag{10.29}$$

where (10.26) was used in the last equality. Since

$$d\zeta(t) = \begin{cases} a\ dt & \text{if} \quad -1 < t < 0 \\ \beta\ dt & \text{if} \quad 0 < t < 1, \end{cases}$$

we have

(10.30)

$$\alpha\left[\int_{-1}^{0}\frac{Re(\alpha F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt+\int_{0}\frac{Re(\beta F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt\right]$$

$$=\int_{-1}^{0}\frac{\alpha F\big(\zeta(t)\big)}{z-\zeta(t)}\ \alpha dt-\alpha\int_{-1}^{0}\frac{i\ Im(\alpha F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt+\int_{0}^{1}\frac{\alpha Re(\beta F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt$$

$$=\left(\oint_{\Gamma}\ \frac{\alpha F(\zeta)}{z-\zeta}\ d\zeta-\int_{\overline{BC}+\overline{CA}}\ \frac{\alpha F(\zeta)}{z-\zeta}\ d\zeta\right)-\alpha\int_{-1}^{0}\frac{i\ Im(\alpha F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt$$

$$+\int_{0}^{1}\frac{\alpha Re(\beta F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt$$

$$=\left(-2\pi i\alpha F(z)-\int_{\overline{CA}}\ \frac{\alpha F(\zeta)}{z-\zeta}\ d\zeta-\int_{0}^{1}\frac{\alpha F(\zeta)}{z-\zeta(t)}\ (\beta dt)\right)$$

$$-\alpha\int_{-1}^{0}\frac{i\ Im(\alpha F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt+\int_{0}^{1}\frac{\alpha Re(\beta F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt$$

$$=-2\pi i\alpha\big(F(z)+G(z)\big)-\int_{0}^{1}\frac{\alpha i\ Im(\beta F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt-\alpha\int_{-1}^{0}\frac{i\ Im(\alpha F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt$$

where

$$G(z)=\frac{1}{2\pi i}\int_{\overline{CA}}\ \frac{\alpha F(\zeta)}{z-\zeta}\ d\zeta;$$

in the last equation we used the relation $\alpha F=Re(\alpha F)+i\ Im(\alpha F)$. For $-1<s<0$ we clearly have

$$\lim_{\substack{z=\zeta(s)+i\alpha\epsilon\\ \epsilon\to 0+}}\int_{0}^{1}\frac{\alpha i\ Im(\beta F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt=\int_{0}^{1}\frac{\alpha i\ Im(\beta F)\big(\zeta(t)\big)}{\zeta(s)-\zeta(t)}\ dt$$

and, similarly to (10.27),

$$\lim_{\substack{z=\zeta(s)+i\alpha\epsilon\\ \epsilon\to 0+}}\int_{-1}^{0}\frac{\alpha i\ Im(\alpha F)\big(\zeta(t)\big)}{z-\zeta(t)}\ dt=\pi\ Im(\alpha F)\big(\zeta(s)\big)$$

$$+i\int_{-1}^{0}\frac{Im(\alpha F)\big(\zeta(t)\big)}{s-t}\ dt.$$

Substituting (10.30) into (10.29) and using the last two relations, we obtain

$$\begin{aligned}\pi^2\widetilde{H}^2u(s) = Re\Big\{ &-2\pi i\alpha\left[F(\zeta(s))+G(\zeta(s))\right] - \pi\, Im(\alpha F)(\zeta(s)) \\ &-i\int_{-1}^{0}\frac{Im(\alpha F)(\zeta(t))}{s-t}\,dt - \alpha\int_{0}^{1}\frac{i\ Im(\beta F)(\zeta(t))}{\zeta(s)-\zeta(t)}\,dt\Big\} \\ = \pi\ Im(\alpha F)(\zeta(s)) + 2\pi\ Im(\alpha G)(\zeta(s)) &- Re\ \alpha\int_{0}^{1}\frac{i\ Im(\beta F)(\zeta(t))}{\zeta(s)-\zeta(t)}\,dt\ .\end{aligned}\tag{10.31}$$

Now, when $-1 < s < 0$,

$$\alpha F(\zeta(s)) = \lim_{\epsilon\to 0+}\int_{-1}^{0}\frac{\alpha u(t)}{\alpha s + i\alpha\epsilon - \alpha t}\,dt + \int_{0}^{1}\frac{\alpha u(t)}{\alpha s - \beta t}\,dt,$$

and using (10.27) we get

$$\alpha F(\zeta(s)) = -i\pi u(s) + \int_{-1}^{0}\frac{u(t)}{s-t}\,dt + \int_{0}^{1}\frac{u(t)}{s-\frac{\beta}{\alpha}\,t}\,dt\ .\tag{10.32}$$

Similarly, when $0 < s < 1$,

$$\begin{aligned}\beta F(\zeta(s)) &= \lim_{\epsilon\to 0+}\int_{-1}^{0}\frac{\beta u(t)}{\beta s + i\beta\epsilon - \alpha t}\,dt + \int_{0}^{1}\frac{\beta u(t)}{\beta s - \beta t}\ . \\ &= -i\pi u(s) + \int_{0}^{1}\frac{u(t)}{s-t}\,dt + \int_{-1}^{0}\frac{u(t)}{s-\frac{\alpha}{\beta}\,t}\,dt.\end{aligned}\tag{10.33}$$

Using (10.32), (10.33) in (10.31), we get

$$\begin{aligned}\pi^2\widetilde{H}^2u(s) &= \pi\left[-\pi u(s) + Im\int_{0}^{1}\frac{u(t)}{s-\frac{\beta}{\alpha}\,t}\,dt\right] + 2\pi Im(\alpha G)(\zeta(s)) \\ &\quad - Re\left\{\alpha i\int_{0}^{1}\frac{1}{\alpha s-\beta t}\left[-\pi u(t) + Im\int_{-1}^{0}\frac{u(\tau)}{t-\frac{\alpha}{\beta}\,\tau}\,d\tau\right]dt.\right.\end{aligned}\tag{10.34}$$

Noting the cancellation

$$Re \int_0^1 \frac{\alpha i}{\alpha s - \beta t}\, \pi u(t)\, dt + \pi\, Im \int_0^1 \frac{u(t)}{s - \dfrac{\beta}{\alpha}\, \tau}\, dt = 0$$

we get from (10.34)

$$(10.35) \qquad \begin{aligned} \pi^2 \widetilde{H}^2 u(s) &= -\pi^2 u(s) + 2\pi\, Im(\alpha G)\big(\zeta(s)\big) \\ &\quad + \int_0^1 Im \left(\frac{1}{s - \dfrac{\beta}{\alpha}\, t} \right) \int_{-1}^0 \left(Im \frac{1}{t - \dfrac{\alpha}{\beta}\, \tau} \right) u(\tau)\, d\tau\, dt \end{aligned}$$

provided $-1 < s < 0$.

Similarly, if $0 < s < 1$,

$$(10.36) \qquad \begin{aligned} \pi^2 \widetilde{H}^2 u(s) &= -\pi^2 u(s) + 2\pi\, Im(\beta G_1)\big(\zeta(s)\big) \\ &\quad + \int_{-1}^0 Im \left(\frac{1}{s - \dfrac{\alpha}{\beta}\, t} \right) \int_0^1 Im \left(\frac{1}{t - \dfrac{\beta}{\alpha}\, \tau} \right) u(\tau) d\tau\, dt \end{aligned}$$

with a corresponding function G_1.

Noting that

$$Im \frac{1}{s - \dfrac{\beta}{\alpha}\, t}\; Im\, \frac{1}{t - \dfrac{\alpha}{\beta}\, \tau} = -\sin^2(\theta - \phi) \frac{t}{s^2 + t^2}\; \frac{\tau}{t^2 + \tau^2}$$

if $st < 0,\ t\tau < 0$, we can rewrite (10.35), (10.36) in the form

$$(10.37) \qquad \pi^2 \widetilde{H}^2 u = -\pi^2 u + \widetilde{T} u + \widetilde{D} u$$

where

$$\widetilde{D} u(s) = \begin{cases} 2\pi\, Im(\alpha G)\big(\zeta(s)\big) \\ 2\pi\, Im(\beta G_1)\big(\zeta(s)\big), \end{cases}$$

$$(10.38) \quad \widetilde{T} u(s) = -\left\{ \int_{-1}^1 h(s,t) \left(\int_{-1}^1 h(t,\tau) u(\tau)\, d\tau \right) dt \right\} \sin^2(\theta - \phi) = -K^2 u(s),$$

$$Ku(s) = \left[\int_{-1}^1 h(s,t) u(t)\, dt \right] \sin(\theta - \phi),$$

and

$$h(s,t)=\begin{cases}\dfrac{t}{s^2+t^2} & \text{if} \quad st<0\\ 0 & \text{if} \quad st>0.\end{cases}$$

In the definition of $G(z)$, the denominator is uniformly positive in absolute value for all $z=\zeta(s)$, when s varies in the interval

$$s_{i-1}+\delta \le s \le s_{i+1}-\delta, \;\text{ i.e., }\; -\frac{3}{4} \;\le s \le \frac{3}{4}\,.$$

A similar remark applies to G_1. Hence the operator $\widetilde{D}$ is compact from L_0^1 to $L^2(s_{i-1}+\delta, s_{i+1}-\delta)$. The restriction of $\widetilde{D}u$ from L_0^1 to $L^2(\Delta)$, where $\Delta=(0,l_0)\backslash(s_{i-1}+\delta,\; s_{i+1}-\delta)$, is also compact. Indeed, on Δ, $\widetilde{D}u$ coincides with $\pi^2\widetilde{H}^2u$, i.e., $\widetilde{D}u=\pi^2\widetilde{H}(\widetilde{H}\tilde{u})$, and $\widetilde{H}\tilde{u}$ is smooth in Δ (i.e. analytic) We write $\widetilde{H}u=\Sigma\chi_j(\widetilde{H}u)$, apply $\widetilde{H}$ to each $\chi_j(\widetilde{H}u)$, and express $\widetilde{H}(\chi_j\widetilde{H}u)$ in $(s_{j-1}-\delta,\; s_{j+1}+\delta)$ as in (10.37). We then find that $\widetilde{H}^2u$ is a compact operator from L_0^1 into $L^2(\Delta)$.

We have thus proved that $\widetilde{D}$, in (10.37), is a compact operator from L_0^1 into $L^2(0,l_0)$.

Next,

$$\begin{aligned}\|Ku\|_{L^1} &\le \int_{-1}^{1} ds\Big|\int_{-1}^{1} h(s,t)u(t)\;dt\Big|\\ &\le \int_{-1}^{1}\left(\int_{-1}^{1}|h(s,t)|\;ds\right)|u(t)|dt \le \frac{\pi}{2}\;\int_{-1}^{1}|u(t)|\;dt,\end{aligned}$$

which establishes (10.22). To prove (10.23) take first $-1<s<0$. Then

$$\begin{aligned}|Ku(s)| &\le \int_0^1 |u(t)|\frac{t}{s^2+t^2}\;dt \le c_0\int_0^1\frac{|u(t)|}{t-s}\;dt\\ &= c_0\int_{-\infty}^{\infty}\frac{1}{t-s}\left(|(u(t)|\chi_{[0,1]}\right)\;dt\end{aligned}$$

where c_0 is such that

$$\frac{t}{s^2+t^2}\;\le\frac{c_0}{t-s}, \quad \text{i.e.,} \quad c_0=\frac{1+\sqrt{2}}{2}\,.$$

Then

$$\int_{-1}^{0}|Ku(s)|^2\;ds \le c_0^2\pi^2\int_{-1}^{0}\left(H(u\chi_{[0,1]})\right)^2\;dt \le c_0^2\pi^2\;\int_{-\infty}^{\infty}(u\chi_{[0,1]})^2\;dt.$$

A similar estimate holds for $0 < s < 1$ and thus

$$\int_{-1}^{1} |Ku(s)|^2 \, ds \le \frac{3+2\sqrt{2}}{4} \, \pi^2 \int_{-1}^{1} |u(t)|^2 \, dt,$$

and (10.23) follows.

Proof of Lemma 10.6. As before it is sufficient to concentrate on $\widetilde{H}^2$ with u restricted to the space L_0^p and $\widetilde{H}^2$ considered in $L^p(0, l_0)$. Using the notation (10.24) we introduce a curve $\tilde{l}(t)$ consisting of the two tangents to $\vec{l}(t)$ at $t = s_i = 0$. We write $l(t) = \vec{l}(t)$ and denote by $\widetilde{H}_l, \widetilde{H}_{\tilde{l}}$ the operators $\widetilde{H}$ defined with respect to l and $\tilde{l}$, respectively. Set

$$\widetilde{D} = \widetilde{H}_l - \widetilde{H}_{\tilde{l}}.$$

If we can prove that $\widetilde{D}$ is a compact operator from L_0^1 into L^2 then the assertion of Lemma 10.6 will follow from Lemma 10.7. To prove that $\widetilde{D}$ is compact, introduce the function

$$k_l(s,t) = \frac{\vec{\tau} \cdot \big(l(s) - l(t)\big)}{|l(s) - l(t)|^2}$$

and similarly $k_{\tilde{l}}$.

Let $-1 < s < 0$. If $-1 < t < 0$ then

$$k_{\tilde{l}}(s,t) = \frac{1}{s-t}$$

whereas

$$\begin{aligned} k_l(s,t) &= \frac{\vec{\tau}(s) \cdot [\vec{\tau}(s)(s-t) + d_1(s-t)^2]}{|\vec{\tau}(s)(s-t) + d_1(s-t)^2|^2} \\ &= \frac{1}{s-t} + d_2 = k_{\tilde{l}}(s,t) + d_2 \end{aligned} \tag{10.39}$$

where d_j denote uniformly continuous functions of (s,t). On the other hand, if $0 < t < 1$, then

$$k_{\tilde{l}} = \frac{\vec{\alpha} \cdot (\vec{\alpha} s - \vec{\beta} t)}{|\vec{\alpha} s - \vec{\beta} t|^2}$$

and

$$\begin{aligned} k_l &= \frac{(\vec{\alpha} + d_3 s) \cdot (\vec{\alpha} s - \vec{\beta} t + d_3 s^2 + d_4 t^2)}{|\vec{\alpha} s - \vec{\beta} t|^2 + d_5} \\ &= \frac{\vec{\alpha} \cdot (\vec{\alpha} s - \vec{\beta} t) + d_6}{|\vec{\alpha} s - \vec{\beta} t|^2 + d_5} \end{aligned}$$

where $|d_5| \le C(|s|^3 + s^2 t + |s| t^2 + t^3), |d_6| \le C(t^2 + t|s| + s^2)$. Since $|\vec{\alpha} s - \vec{\beta} t|^2 \ge c(s^2 + t^2)$, $c > 0$, we deduce that

$$k_l = k_{\bar{l}} + d_7, \quad d_7 \quad \text{continuous.} \tag{10.40}$$

From (10.39), (10.40) we see that, for $-1 < s < 0$,

$$(\widetilde{D}u)(s) = \int_{-1}^{1} d(s,t)u(t)\, dt$$

where d is a continuous function. The same result can be proved for $0 < s < 1$. Consequently $\widetilde{D}$ is a compact linear operator from L_0^p into L_0^p.

Set

$$\lambda = \frac{(\epsilon_1 - \epsilon_2)^2}{4\epsilon_1 \epsilon_2}\ . \tag{10.41}$$

We return to equations (10.18). Using Lemma 10.6 we can rewrite the second equation in the form

$$[(1+\lambda)E - \lambda T]\, J + DJ + \widetilde{T}_2(I,J) = -\widetilde{J}_0 \tag{10.42}$$

where E is the identity operator. Set

$$B = [(1+\lambda)E - \lambda T]^{-1} = (1+\lambda)^{-1} \sum_{n=0}^{\infty} \left(\frac{\lambda}{1+\lambda}\right)^n T^n; \tag{10.43}$$

this is a bounded linear operator in X_p provided

$$\|T\|_{L^p} < \left|\frac{1+\lambda}{\lambda}\right|. \tag{10.44}$$

when this condition is satisfied we can rewrite (1.18) in the form

$$\begin{aligned} I + \widetilde{T}_1(I,J) &= -I_0, \\ J + BD + B\widetilde{T}_2(I,J) &= -B\widetilde{J}_0, \end{aligned} \tag{10.45}$$

which looks like a Fredholm system.

By Lemma 10.6, (10.44) is satisfied if

$$\frac{1}{4} < \left|\frac{1+\lambda}{\lambda}\right|, \qquad p = 1, \tag{10.46}$$

$$\frac{3 + 2\sqrt{2}}{4} < \left|\frac{1+\lambda}{\lambda}\right|, \quad p = 2; \tag{10.47}$$

Recalling that $\epsilon_j = \epsilon_j' + i\epsilon_j'', \epsilon_j' > 0, \epsilon_j'' \geq 0$, we have

$$\left|\frac{1+\lambda}{\lambda}\right| = \left|\frac{(\epsilon_1+\epsilon_2)^2}{(\epsilon_1-\epsilon_2)^2}\right| = \frac{(\epsilon_1'+\epsilon_2')^2+(\epsilon_1''+\epsilon_2'')^2}{(\epsilon_1'-\epsilon_2')^2+(\epsilon_1''-\epsilon_2'')^2} > 1\,, \tag{10.48}$$

so that (10.46) is always satisfied. Consequently we prefer work in the space X_1. However this requires a careful look at all the terms in $\widetilde{T}_2$, ensuring that they each are a compact operator in X_1. Recall that in the operator $\widetilde{T}_2$ there enter operators which are obtained when we substitute I from (10.15) into (10.16). The "worst" operator in $\widetilde{T}_2$ is

$$c_1 f_1'(s) \int_0^{l_0} (\widetilde{H}u)(t) \left(\frac{\Psi_2}{\epsilon_2} - \frac{\Psi_1}{\epsilon_1}\right) (x(s) - y(t))\ dt,$$

c_1 constant. If $u \in L^1$ then $\widetilde{H}u$ is in $(L^1)_{\text{weak}}$, so that in order to make sense out of the above operator we must write it in the form

$$c_2 f_1'(s) \int u(t')\ dt' \int \frac{\vec{\tau}(t) \cdot (\vec{l}(t) - \vec{l}(t'))}{|\vec{l}(t) - \vec{l}(t')|^2} \log \frac{1}{|x(s) - y(t)|}\ dt$$
$$+T_0 u, \qquad c_2 \quad \text{is another constant,}$$

and T_0 is a compact operator in X_1. The inner integral has the form

$$K(t', s) = \widetilde{H}\left(\log \frac{1}{|x(s) - y(t)|}\right)(t').$$

As in the proof of the previous two lemmas, it is enough to consider what happens in a neighborhood of $s_i = 0$ when $\vec{l}(t)$ consist of two line segments meeting at $t = 0$. Using the same notation as in Lemma 10.7, this reduces to studying the operator K:

$$Ku(s) = \int_{-1}^{1} \log(s - \tau)(T_0 u)(\tau)\ d\tau \tag{10.49}$$

where

$$T_j u(s) = \int_{-1}^{1} h_j(s, t) u(t)\ dt \tag{10.50}$$

and

$$h_0(s, t) = \begin{cases} \dfrac{1}{s - t} & \text{if} \quad st > 0 \\ \dfrac{s}{s^2 + t^2} & \text{if} \quad st < 0. \end{cases} \tag{10.51}$$

Later on we shall also need to examine $T_1 u$ where

$$h_1(s,t) = \begin{cases} \dfrac{t}{s^2+t^2} & \text{if} \quad st < 0 \\ 0 & \text{if} \quad st > 0. \end{cases} \tag{10.52}$$

The operator K does not appear to be compact or even bounded if we consider it in the L^1-norm. We shall therefore resort to a weighted L^1-norm

$$\|u\|_\alpha = \int_{-1}^{1} \frac{|u(t)|}{|t|^\alpha}\, dt\,, \qquad 0 < \alpha < 1.$$

LEMMA 10.8. *The operator $u \to Ku$ from L_0^1 into L^1 is a compact operator provided both u and Ku are taken with the $\| \ \|_\alpha$ norm.*

Here $L^1 = L^1(-1,1)$ and $L_0^1 = \left\{u \in L^1;\ u(t) = 0 \text{ if } |t| > \dfrac{3}{4}\right\}$.

Proof. Write $v = Ku$ and set

$$\tilde{u}(t) = t^{-\alpha} u(t), \quad \tilde{v}(s) = s^{-\alpha} v(s).$$

Then

$$\tilde{v}(s) = \int_{-1}^{1} \tilde{k}(s,t)\tilde{u}(t)\, dt \equiv (\widetilde{K}\tilde{u})(s)$$

where

$$\tilde{k}(s,t) = \frac{t^\alpha}{s^\alpha} \int_{-1}^{1} \log|s-\tau| h_0(\tau,t)\, d\tau\,,$$

and it suffices to show that $\widetilde{K}$ is compact from L_0^1 into L^1.

If $s > 0$ then, for $t > 0$,

$$\tilde{k}(s,t) = \frac{t^\alpha}{s^\alpha} \left[p.v. \int_0^1 \log|s-\tau| \frac{d\tau}{\tau - t} + \int_{-1}^{0} \log|s-\tau| \frac{\tau}{t^2+\tau^2}\, d\tau \right]$$

whereas for $t < 0$

$$\tilde{k}(s,t) = \frac{t^\alpha}{s^\alpha} \left[p.v. \int_{-1}^{0} \log|s-\tau| \frac{d\tau}{\tau - t} + \int_0^1 \log|s-\tau| \frac{\tau}{t^2+\tau^2}\, d\tau \right].$$

A similar representation holds for $s < 0$. Using properties of the Hilbert transform it is easily seen that

$$\begin{aligned}&\tilde{k}(s,t) \quad \text{is a continuous function of } (s,t)\\ &\text{for all } (s,t) \text{ except possibly } s = 0,\ t = 0 \text{ or } s = t.\end{aligned} \tag{10.53}$$

If we can prove that

$$|\tilde{k}(s,t)| \le \frac{Ct^\alpha}{s^\alpha}\left[(\log|t|)^2 + |\log|s|\ \log|t|\right] \tag{10.54}$$

then

$$|\tilde{k}(s,t)| \le \frac{C}{s^\alpha}|\log|s||.$$

Using this estimate and (10.53), we can deduce by a standard argument that $\widetilde{K}$ is a compact operator from L^1_0 into L^1.

To prove (10.54) it suffices to consider the case $s > 0, t > 0$. Then

$$\begin{aligned}\tilde{k}(s,t) &= \frac{t^\alpha}{s^\alpha}\left[p.v.\int_0^1 \log|s-\tau|\frac{d\tau}{\tau - t} + \int_{-1}^0 \log|s-\tau|\frac{\tau}{t^2+\tau^2}\,d\tau\right]\\ &\equiv \frac{t^\alpha}{s^\alpha}(J_1 + J_2)\end{aligned} \tag{10.55}$$

and

$$\begin{aligned}|J_2| &\le \Big|\int_{|\tau|<\frac{s}{2}} + \int_{\frac{s}{2}<|\tau|<2s} + \int_{|\tau|>2s}\Big|\\ &\le C|\log s|\ |\log\ t| + \frac{C}{s}\Big|\int_{-s/2}^{2s} \log|s-\tau|d\tau\Big| - C|\log s|\ \log\ t|\\ &\le C|\log s|\ |\log\ t|.\end{aligned} \tag{10.56}$$

Next

$$J_1 = p.v.\int_0^{2t} \log|s-\tau|\frac{d\tau}{\tau - t} + \int_{2t}^1 \log|s-\tau|\frac{d\tau}{\tau-t} \equiv I_1 + I_2. \tag{10.57}$$

Substituting $\tau = t + \eta$ in I_1 we get

$$I_1 = p.v.\int_{-t}^{t} \frac{\log|s-t-\eta|}{\eta}\,d\eta = \int_0^t \frac{\log|s-t-\eta| - \log|s-t+\eta|}{\eta}\,d\eta$$

so that, upon substituting $\eta = |s-t|\xi$,

$$|I_1| \leq \left| \int_0^\infty \frac{\log|\frac{1+\xi}{1-\xi}|}{\xi}\, d\xi \right| \leq C. \tag{10.58}$$

To estimate I_2 we substitute $\tau = 2t + \eta t$ to get

$$I_2 = \int_0^{\frac{1}{t}-2} \frac{\log|s-2t-\eta t|}{\eta+1}\, d\eta;$$

we may assume that $\frac{1}{t} > 2$. Writing

$$s - 2t - \eta t = t(\gamma - \eta) \quad \text{where} \quad \gamma = \frac{s-2t}{t}$$

we obtain

$$\begin{aligned} |I_2| &\leq \int_0^{\frac{1}{t}-2} \frac{|\log t|}{\eta+1}\, d\eta + \left| \int_0^{\frac{1}{t}-2} \frac{\log|\eta-\gamma|}{\eta+1}\, d\eta \right| \\ &\leq |\log t| \log\frac{1}{t} + \widetilde{I}_2 \end{aligned}$$

where

$$\widetilde{I}_2 = \left(\int_{\substack{0 \\ |\eta-\gamma|<1}}^{\frac{1}{t}-2} + \int_{\substack{0 \\ \eta<\gamma-1}}^{\frac{1}{t}-2} + \int_{\substack{0 \\ \eta>\gamma+1}}^{\frac{1}{t}-2} \right) \frac{|\log|\eta-\gamma||}{\eta+1}\, d\eta.$$

The first term in $\widetilde{I}_2$ is bounded by

$$2\int_0^1 |\log\xi| d\xi$$

and each of the remaining two terms is bounded by

$$\log\frac{1}{t} \int_0^{\frac{1}{t}} \frac{d\eta}{\eta+1} = (\log\ t)^2.$$

It follows that

$$|I_2| \leq C(\log\ t)^2.$$

Combining this with (10.58), (10.57), we see that

$$|J_1| \leq C(\log\ t)^2.$$

Recalling (10.56), (10.55), the assertion (10.54) follows.

Having proved Lemma 10.8 we remark that the compactness of all the other operators in $\widetilde{T}_2$ is proved much more easily. However, since we shall be using the α-norm $\|\ \|_\alpha$, we still must extend also Lemma 10.6 and the estimate (10.22).

The proof of Lemma 10.6, for the α-norm, remains essentially the same; the only difference is that now we have a lightly different estimate on T_1, namely

$$\|T_1 u\|_\alpha \leq \frac{\pi}{2\cos\frac{\pi\alpha}{2}}\ \|u\|_\alpha; \tag{10.59}$$

T_1 is defined by (10.50), (10.52). The proof of this estimate follows from

$$\begin{aligned}\|T_1 u\|_\alpha &\leq \|u\|_\alpha \sup_t \int_{-1}^{1} \frac{|h_1(s,t)|}{|s|^\alpha}\ |t|^\alpha\ ds\\ &\leq \int_0^\infty \frac{d\eta}{\eta^\alpha(1+\eta^2)} \qquad (s = t\eta)\\ &= \frac{\pi}{2\cos\frac{\pi\alpha}{2}}\ .\end{aligned}$$

From (10.59) we conclude that instead of (10.22) there holds:

$$\|T\|_\alpha \leq \frac{1}{4\cos^2\frac{\pi\alpha}{2}}\ \|u\|_\alpha. \tag{10.60}$$

Denote by $X_{1,\alpha}$ the space X_1 where the L^1-norm is taken with weight $\omega(t)$,

$$\omega(t) = \begin{cases} |t-s_i|^{-\alpha} & \text{if} \quad |t-s_i| < \delta\\ 1 & \text{if} \quad |t-s_i| > \delta\end{cases}$$

for some small $\delta > 0$.

Notice that working with the space $X_{1,\alpha}$, the condition (10.44) becomes

$$\|T\|_\alpha \equiv \|T\|_{X_{1,\alpha}} < \left|\frac{1+\lambda}{\lambda}\right|$$

and, in view of (10.60), (10.47), this condition is always satisfied if $4\cos^2\dfrac{\pi\alpha}{2} \geq 1$, i.e., if $0 < \alpha \leq \dfrac{2}{3}$. In this case, then, the system (10.18) is of Fredholm type.

We summarize:

THEOREM 10.9. *The system (10.18) in* $X_{1,\alpha}$ $(\alpha \in (0, \frac{2}{3}])$ *is equivalent to the Fredholm system (10.45) where* BD *and* $B\widetilde{T}_2$ *are compact operators in* $X_{1,\alpha}$.

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REFERENCES

[1] A. BELLOUT AND A. FRIEDMAN, *Scattering by strip grating*, J. Math. Anal. Appl., to appear.

[2] A. BENALDI, *Numerical analysis of the exterior boundary value problem for the time-harmonic Maxwell equations by a boundary finite element method. Part I: The continuous problem 2: The discrete problem*, Math. of Computation, 167 (1984), 29–46 and 47–68.

[3] N.G. DE BRUIJN, *Asymptotic Methods in Analysis*, North-Holland, Amsterdam, 1958.

[4] C.A. COULSON AND A. JEFFREY, *Waves*, Second edition, Longman, London, 1977.

[5] T.K. GAYLORD AND M.G. MOHARAM, *Analysis and applications of optimal diffraction by gratings*, Proceedings IEEE, 73 (1985), 894–937.

[6] I.S. GRADSTEYN AND I.M. RYZHIK, *Tables of Integrals, Series and Products*, Academic Press, New York, 1965.

[7] C. MULLER, *Foundations of Mathematical Theory of Electromagnetic Waves*, Springer–Verlag, Berlin, 1969.

[8] J.C. NEDELEC AND F. STARLING, *Integral equation methods in quasi periodic diffraction problem for the time harmonic Maxwell's equations*, Ecole Polytechnique, Centre de Mathematiques Appliquees, Tech. Report #756, April 1988.

[9] *Electromagnetic Theory of Gratings*, Topics in Current Physics, Vol. 22, edited by R. Petit, Springer–Verlag, Heidelberg, 1980.